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Almost Pseudo Symmetric Kähler Manifolds Admitting Conformal Ricci-Yamabe Metric

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Abstract. The novelty of the paper is to investigate the nature of conformal Ricci-Yamabe soliton on almost pseudo symmetric, almost pseudo Bochner symmetric, almost pseudo Ricci symmetric and almost pseudo Bochner Ricci symmetric Kähler manifolds. Also, we explore the harmonic aspects of conformal η-Ricci-Yamabe soliton on Kähler spcetime manifolds with a harmonic potential function f and deduce the necessary and sufficient conditions for the 1-form η, which is the g-dual of the vector field ξ on such spacetime to be a solution of Schrödinger-Ricci equation.

1. Introduction

It is well-known that the symmetric spaces play a crucial role in differential geometry. Let $(\mathfrak{M}^n, g)$ be an n-dimensional Riemannian manifold with the metric g and the Levi-Civita connection ∇. An $(\mathfrak{M}^n, g)$ is called locally symmetric if $\nabla\mathcal{R}$=0; where $\mathcal{R}$ is the Riemannian curvature tensor of $(\mathfrak{M}^n, g)$. The class of Riemannian symmetric manifolds is a natural generalization of the class of manifolds of constant curvature. The notion of locally symmetric manifolds has been studied by many authors in several ways to a different extent such as conformally symmetric manifolds [7], semi-symmetric

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manifolds [24], pseudo symmetric manifolds [5, 13], weakly symmetric manifolds [25] and almost pseudo concircularly symmetric manifolds [11] etc.

A non-flat $(\mathfrak{M}^n, g)$ is said to be an almost pseudo symmetric (briefly, $A(PS)$) manifold [11] if its curvature tensor satisfies the condition

$$\begin{aligned}(\nabla_{\mathcal{E}}\mathcal{R})(\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{W}) &= [\mathcal{A}(\mathcal{E})+\mathcal{B}(\mathcal{E})]\mathcal{R}(\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{W})+\mathcal{A}(\mathcal{F})\mathcal{R}(\mathcal{E},\mathcal{G},\mathcal{H},\mathcal{W}) \\ &+ \mathcal{A}(\mathcal{G})\mathcal{R}(\mathcal{F},\mathcal{E},\mathcal{H},\mathcal{W})+\mathcal{A}(\mathcal{H})\mathcal{R}(\mathcal{F},\mathcal{G},\mathcal{E},\mathcal{W}) \\ &+ \mathcal{A}(\mathcal{W})\mathcal{R}(\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{E}),\end{aligned} \tag{1.1}$$

where $\mathcal{A}$, $\mathcal{B}$ are two non-zero 1-forms defined by

$$g(\mathcal{E},\varrho)=\mathcal{A}(\mathcal{E}), \quad g(\mathcal{E},\rho)=\mathcal{B}(\mathcal{E}), \tag{1.2}$$

for all vector fields $\mathcal{E}$, ∇ denotes the operator of covariant differentiation with respect to the metric g. The 1-forms $\mathcal{A}$ and $\mathcal{B}$ are called the associated 1-forms. If $\mathcal{A}=\mathcal{B}$, then an $A(PS)$ manifold reduces to a pseudo symmetric manifold, introduced by Chaki [5]. If $\mathcal{A}=\mathcal{B}=0$, then the manifold reduces to a symmetric manifold in the sense of Cartan [9].

A non-flat $(\mathfrak{M}^n, g)$ whose Ricci tensor $\mathcal{S}$ of type $(0,2)$ satisfies the condition

$$(\nabla_{\mathcal{E}}\mathcal{S})(\mathcal{F},\mathcal{G})=[\mathcal{A}(\mathcal{E})+\mathcal{B}(\mathcal{E})]\mathcal{S}(\mathcal{F},\mathcal{G})+\mathcal{A}(\mathcal{F})\mathcal{S}(\mathcal{E},\mathcal{G})+\mathcal{A}(\mathcal{G})\mathcal{S}(\mathcal{F},\mathcal{E}) \tag{1.3}$$

is known as almost pseudo Ricci symmetric (briefly, $A(PRS)$) manifold. Since the notion of pseudo symmetry in the sense of Chaki [5] is different from that of Deszcz [10] and to be noted that an $A(PRS)$ manifold is not a particular case of a weakly symmetric manifold introduced by Tamassy and Binh [25]. Tamassy, De and Binh [26] derived a lot of results on weakly symmetric and weakly Ricci symmetric Kähler manifolds in 2000. About a decade ago, the authors Narain and Yadav studied weak concircular symmetries of Lorentzian concircular structure manifolds [3]. In 2010, Shaikh et al. [23] studied and contributed some remarkable results on quasi-conformlly flat almost pseudo Ricci symmetric manifolds.

The present paper is organized as follows: After preliminaries in section 2 we recall the fundamental results of Kähler manifolds and conformal η-Ricci-Yamabe solitons (briefly, CERYS). In section 3 we study an A(PS) Kähler manifold admitting CRYS and deduce some appreciable results. Section 4, concern with the investigation of an A(PBS) Kähler manifold admitting CRYS. In section 5, we also study A(PRS) Kähler manifolds admitting CRYS. Next, in section 6 we consider A(PBRS) Kähler manifolds admitting CRYS and prove that if the Bochner Ricci tensor $\mathcal{Z}_B$ is of codazzi type then the manifold is an Einstein manifold. Also, the harmonic aspects of CERYS on Kähler spcetime manifold with a harmonic potential function f are also studied in section 7. Finally, we light up some applications of CERYS, which is based on Theorem 7.1 in section 8.

2. Kähler Manifolds

A Kähler manifold is an $n(=2m)$-dimensional manifold, with a complex structure $\mathcal{J}$ and a positive definite metric g satisfying the conditions

$$\mathcal{J}^2 = -\mathcal{I}, \quad g(\mathcal{J}\mathcal{E}, \mathcal{J}\mathcal{F}) = g(\mathcal{E}, \mathcal{F}), \quad (\nabla_{\mathcal{E}}\mathcal{J})(\mathcal{F}) = 0, \tag{2.1}$$

where ∇ denotes the Levi-Civita connection.

In a Kähler manifold we have [2]:

$$\begin{cases} (i)\ \mathcal{R}(\mathcal{J}\mathcal{E}, \mathcal{J}\mathcal{F}) = \mathcal{R}(\mathcal{E}, \mathcal{F}), \\ (ii)\ \mathcal{S}(\mathcal{E}, \mathcal{F}) = \mathcal{S}(\mathcal{J}\mathcal{E}, \mathcal{J}\mathcal{F}), \\ (iii)\ \mathcal{S}(\mathcal{J}\mathcal{E}, \mathcal{F}) = -\mathcal{S}(\mathcal{E}, \mathcal{J}\mathcal{F}). \end{cases} \tag{2.2}$$

A Kähler manifold is called an almost pseudo Bochner symmetric (briefly, $A(PBS)$) manifold if its Bochner curvature tensor $\mathcal{B}_c$ of type $(0,4)$ is not zero and satisfies the condition

$$\begin{aligned} (\nabla_{\mathcal{E}}\mathcal{B}_c)(\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{W}) &= [\mathcal{A}(\mathcal{E}) + \mathcal{B}(\mathcal{E})]\mathcal{B}_c(\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{W}) + \mathcal{A}(\mathcal{F})\mathcal{R}(\mathcal{E}, \mathcal{G}, \mathcal{H}, \mathcal{W}) \\ &+ \mathcal{A}(\mathcal{G})\mathcal{B}_c(\mathcal{F}, \mathcal{E}, \mathcal{H}, \mathcal{W}) + \mathcal{A}(\mathcal{H})\mathcal{R}(\mathcal{F}, \mathcal{G}, \mathcal{E}, \mathcal{W}) \\ &+ \mathcal{A}(\mathcal{W})\mathcal{B}_c(\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{E}), \end{aligned} \tag{2.3}$$

where $\mathcal{A}$, $\mathcal{B}$ are two non zero 1-forms and Bochner curvature tensor $\mathcal{B}_c$ of type $(0,4)$ is defined as

$$\begin{aligned} \mathcal{B}_c(\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}) &= \mathcal{R}(\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}) - \frac{1}{2n+4}[g(\mathcal{F}, \mathcal{G})\mathcal{S}(\mathcal{E}, \mathcal{H}) - g(\mathcal{F}, \mathcal{H})\mathcal{S}(\mathcal{E}, \mathcal{G}) \\ &+ g(\mathcal{J}\mathcal{F}, \mathcal{G})\mathcal{S}(\mathcal{J}\mathcal{E}, \mathcal{H}) - g(\mathcal{J}\mathcal{F}, \mathcal{H})\mathcal{S}(\mathcal{J}\mathcal{E}, \mathcal{G}) + g(\mathcal{E}, \mathcal{H})\mathcal{S}(\mathcal{F}, \mathcal{G}) \\ &- g(\mathcal{E}, \mathcal{G})\mathcal{S}(\mathcal{F}, \mathcal{H}) + g(\mathcal{J}\mathcal{E}, \mathcal{H})\mathcal{S}(\mathcal{J}\mathcal{F}, \mathcal{G}) - g(\mathcal{J}\mathcal{E}, \mathcal{G})\mathcal{S}(\mathcal{J}\mathcal{F}, \mathcal{H}) \\ &- 2\mathcal{S}(\mathcal{F}\mathcal{J}\mathcal{E})g(\mathcal{J}\mathcal{G}, \mathcal{H}) - 2\mathcal{S}(\mathcal{J}\mathcal{G}, \mathcal{H})g(\mathcal{J}\mathcal{E}, \mathcal{F})] \\ &+ \frac{\tau}{(2n+2)(2n+4)}[g(\mathcal{F}, \mathcal{G})g(\mathcal{E}, \mathcal{H}) - g(\mathcal{E}, \mathcal{G})g(\mathcal{F}, \mathcal{H}) \\ &+ g(\mathcal{J}\mathcal{F}, \mathcal{G})g(\mathcal{J}\mathcal{E}, \mathcal{H}) - g(\mathcal{J}\mathcal{E}, \mathcal{G})g(\mathcal{J}\mathcal{F}, \mathcal{H}) - 2g(\mathcal{J}\mathcal{E}, \mathcal{F})g(\mathcal{J}\mathcal{G}, \mathcal{H})]. \end{aligned} \tag{2.4}$$

A Kähler manifold is called almost pseudo Bochner Ricci symmetric (briefly, $A(PBRS)$) manifold if its Bochner Ricci tensor $\mathcal{Z}_B$ of type $(0,2)$ is not identically zero and satisfies the condition

$$(\nabla_{\mathcal{E}}\mathcal{Z}_B)(\mathcal{F}, \mathcal{G}) = [\mathcal{A}(\mathcal{E}) + \mathcal{B}(\mathcal{E})]\mathcal{Z}_B(\mathcal{F}, \mathcal{G}) + \mathcal{A}(\mathcal{F})\mathcal{Z}_B(\mathcal{E}, \mathcal{G}) + \mathcal{A}(\mathcal{G})\mathcal{Z}_B(\mathcal{F}, \mathcal{E}), \tag{2.5}$$

where $\mathcal{A}$, $\mathcal{B}$ are nowhere vanishing 1-forms and $\mathcal{Z}_B$ is given by

$$\mathcal{Z}_B(\mathcal{F}, \mathcal{G}) = \frac{n}{2n+4}[\mathcal{S}(\mathcal{F}, \mathcal{G}) - \frac{\tau}{2(n+1)}g(\mathcal{F}, \mathcal{G})]. \tag{2.6}$$

The idea of conformal Ricci flow as a generalization of classical Ricci flow on an $(\mathfrak{M}^n, g)$ is defined by [14]

$$\frac{\partial g}{\partial t} = -2(\mathcal{S} + \frac{g}{n}) - pg, \quad \tau(g) = -1, \tag{2.7}$$

where p defines a time dependent non-dynamical scalar field (also called the conformal pressure), g is the Riemannian metric, τ and $\mathcal{S}$ represent the scalar curvature and the Ricci tensor of $\mathfrak{M}^n$, respectively. The term -pg plays a role of constraint force to maintain τ in the above equation.

The conformal Ricci soliton on $(\mathfrak{M}^n, g)$ is defined by [4]:

$$\mathfrak{L}_{\mathcal{E}} g + 2\mathcal{S} = \left\{\frac{1}{n}(pn+2) - 2\mu\right\} g, \tag{2.8}$$

where $\mathfrak{L}_{\mathcal{E}}$ represents the Lie derivative operator along the smooth vector field $\mathcal{E}$ on $\mathfrak{M}^n$ and $\mu \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers).

A new class of geometric flows called Ricci-Yamabe flow of type (κ, l), which is scalar combination of Ricci and Yamabe flows and is defined by [15]

$$\frac{\partial}{\partial t} g(t) = -2\kappa\mathcal{S}(g(t)) - l\tau(t)g(t), \quad g(0) = g_0, \tag{2.9}$$

for some scalars κ and l.

An $(\mathfrak{M}^n, g)$ is said to have Ricci-Yamabe solitons (RYS) if [12, 20, 21, 28]:

$$\mathfrak{L}_{\mathcal{E}} g + 2\kappa\mathcal{S} + (2\mu - l\tau)g = 0, \tag{2.10}$$

where $l, \kappa, \mu \in \mathbb{R}$.

Recently, the authors in [19, 27] studied conformal Ricci-Yamabe soliton (CRYS) and is defined on $(\mathfrak{M}^n, g)$ by

$$\mathfrak{L}_{\mathcal{E}} g + 2\kappa\mathcal{S} + \left\{2\mu - l\tau - \frac{1}{n}(pn+2)\right\} g = 0. \tag{2.11}$$

An $(\mathfrak{M}^n, g)$ is said to have CERYS if [16]

$$\mathfrak{L}_{\mathcal{E}} g + 2\kappa\mathcal{S} + \left\{2\mu - l\tau - \frac{1}{n}(pn+2)\right\} g + 2\lambda\eta\otimes\eta = 0, \tag{2.12}$$

where $l, \kappa, \mu, \lambda \in \mathbb{R}$ and η is a 1-form on $(\mathfrak{M}^n, g)$. If $\mathcal{E}$ is the gradient of a smooth function f on $\mathfrak{M}^n$, then the equation (2.12) is called the gradient conformal η-Ricci-Yamabe soliton (briefly, gradient CERYS) and it turns to

$$\nabla^2 f + \kappa\mathcal{S} + \left\{\mu - \frac{l\tau}{2} - \frac{1}{2}(p + \frac{2}{n})\right\} g + \lambda\eta\otimes\eta = 0, \tag{2.13}$$

where $\nabla^2 f$ is said to be the Hessian of f. A CRYS (or gradient CRYS) is said to be shrinking, steady or expanding if $\mu < 0$, $= 0$ or > 0, respectively. Thus, a CERYS (or gradient CERYS) reduces to (i) conformal η-Ricci soliton if $\kappa = 1$, $l = 0$, (ii) conformal η-Yamabe soliton if $\kappa = 0$, $l = 1$, and (iii) conformal η-Einstein soliton if $\kappa = 1$, $l = -1$.

3. A(PS) Kähler manifolds admitting CRYS

Let $(\mathfrak{M}^n, g)$ be an A(PS) Kähler manifold admitting CRYS. By taking the covariant derivative of (2.2)(i), we have

$$(\nabla_{\mathcal{E}}\mathcal{R})(\mathcal{J}\mathcal{F}, \mathcal{J}\mathcal{G}, \mathcal{H}, \mathcal{W}) = (\nabla_{\mathcal{E}}\mathcal{R})(\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{W}). \tag{3.1}$$

In view of (1.1) and (2.2)(i), equation (3.1) takes the form

$$\begin{aligned}\mathcal{A}(\mathcal{F})\mathcal{R}(\mathcal{E}, \mathcal{G}, \mathcal{H}, \mathcal{W}) \quad &+ \mathcal{A}(\mathcal{G})\mathcal{R}(\mathcal{F}, \mathcal{E}, \mathcal{H}, \mathcal{W}) \\ &= \mathcal{A}(\mathcal{J}\mathcal{F})\mathcal{R}(\mathcal{E}, \mathcal{J}\mathcal{G}, \mathcal{H}, \mathcal{W}) + \mathcal{A}(\mathcal{J}\mathcal{G})\mathcal{R}(\mathcal{J}\mathcal{F}, \mathcal{E}, \mathcal{H}, \mathcal{W}),\end{aligned} \tag{3.2}$$

which by contracting gives

$$\mathcal{S}(\mathcal{E}, \mathcal{W}) = 0. \tag{3.3}$$

Therefore, we can state

Theorem 3.1. *Every A(PS) Kähler manifold is Ricci flat.*

With reference to [5], we state

Corollary 3.1. *Every A(PS) Kähler manifold can not be conformally flat.*

Using (3.3) in (2.11), we have

$$(\mathfrak{L}_{\mathcal{E}}g)(\mathcal{F}, \mathcal{G}) + \left\{2\mu - l\tau - \frac{1}{n}(pn+2)\right\} g(\mathcal{F}, \mathcal{G}) = 0. \tag{3.4}$$

Replacing $\mathcal{F} = \mathcal{G} = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and summing over $i (1 \leq i \leq n)$, we find

$$div(\mathcal{E}) + \left\{n\mu - \frac{nl\tau}{2} - \frac{1}{2}(pn+2)\right\} = 0. \tag{3.5}$$

If we assume that $\mathcal{E}$ is solenoidal, then $div(\mathcal{E})$=0. Thus (3.5) reduces

$$\mu = \frac{1}{2n}[l\tau n + (pn+2)]. \tag{3.6}$$

Next, we assume that the vector field $\mathcal{E}$ is of gradient type, i.e., $\mathcal{E}$=$grad(f)$, where f is a smooth function on $\mathfrak{M}^n$. Thus we can state:

Corollary 3.2. *If the metric g of an n-dimensional A(PS) Kähler manifold be a CRYS $(g, \mathcal{E}, \kappa, l, \mu)$, where $\mathcal{E}$ is the gradient of a smooth function f, then*

$$\nabla^2 f = -\left\{n\mu - \frac{nl\tau}{2} - \frac{1}{2}(pn+2)\right\}.$$

Corollary 3.3. *If an A(PS) Kähler manifold admits a CRYS and $\mathcal{E}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $p \gtreqqless -(l\tau + \frac{2}{n})$.*

Corollary 3.4. *If an A(PS) Kähler manifold admits a CRYS and the vector field $\mathcal{E}$ is solenoidal, then we have*

Types of soliton	*Conditions for the solitons to be expanding, steady and shrinking*
conformal Ricci soliton	$p \gtreqless -\frac{2}{n}$
conformal Yamabe soliton	$p \gtreqless -(\tau + \frac{2}{n})$
conformal Einstein soliton	$p \gtreqless (l\tau - \frac{2}{n})$

A vector field $\mathcal{E}$ is said to be a conformally Killing if and only if the following relation holds:

$$(\mathfrak{L}_{\mathcal{E}} g)(\mathcal{F}, \mathcal{G}) = 2\psi g(\mathcal{F}, \mathcal{G}), \tag{3.7}$$

where ψ is some function. Moreover, if ψ is not constant then $\mathcal{E}$ is said to be proper. Also when ψ is constant, then $\mathcal{E}$ is called homothetic vector field; and if $\psi(\neq 0)$, then $\mathcal{E}$ is said to be proper homothetic vector field. If ψ=0, then $\mathcal{E}$ is called Killing vector field. Thus, from (3.4) and (3.6), we notice that the vector field $\mathcal{E}$ is killing vector field. Now we state the result:

Theorem 3.2. *Let $(g, \mathcal{E}, \kappa, l, \mu)$ be a CRYS in an A(PS) Kähler manifold, then the vector field $\mathcal{E}$ is killing.*

4. A(PBS) Kähler manifolds admitting CRYS

Let $(\mathfrak{M}^n, g)$ be an A(PBS) Kähler manifold. Then in view of (2.1),(2.2)(i),(2.3) and (2.4), we have

$$(\nabla_{\mathcal{E}} \mathcal{B}_c)(\mathcal{J}\mathcal{F}, \mathcal{J}\mathcal{G}, \mathcal{H}, \mathcal{W}) = (\nabla_{\mathcal{E}} \mathcal{B}_c)(\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{W}). \tag{4.1}$$

Using (2.3) in (4.1), we get

$$\begin{aligned} &\mathcal{A}(\mathcal{F})\mathcal{B}_c(\mathcal{E}, \mathcal{G}, \mathcal{H}, \mathcal{W}) + \mathcal{A}(\mathcal{G})\mathcal{B}_c(\mathcal{F}, \mathcal{E}, \mathcal{H}, \mathcal{W}) \\ &= \mathcal{A}(\mathcal{J}\mathcal{F})\mathcal{B}_c(\mathcal{E}, \mathcal{J}\mathcal{G}, \mathcal{H}, \mathcal{W}) + \mathcal{B}_c(\mathcal{J}\mathcal{G})\mathcal{R}(\mathcal{J}\mathcal{F}, \mathcal{E}, \mathcal{H}, \mathcal{W}). \end{aligned} \tag{4.2}$$

After suitable contraction, we get

$$\mathcal{Z}_B(\mathcal{E}, \mathcal{W}) = 0. \tag{4.3}$$

By virtue of (4.3), equation (2.6) have the form

$$\mathcal{S}(\mathcal{E}, \mathcal{W}) = \frac{\tau}{2(n+1)} g(\mathcal{E}, \mathcal{W}). \tag{4.4}$$

Thus, we can state

Theorem 4.1. *An A(PBS) Kähler manifold is an Einstein manifold.*

Now, by using (4.4) in (2.11), we have

$$(\mathfrak{L}_{\mathcal{E}} g)(\mathcal{F}, \mathcal{G}) + \left\{ \frac{\kappa\tau}{(n+1)} + 2\mu - l\tau - \frac{1}{n}(pn+2) \right\} g(\mathcal{F}, \mathcal{G}) = 0. \tag{4.5}$$

By putting $\mathcal{F}=\mathcal{G}=e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and summing over i $(1\leq i\leq n)$, we obtain

$$div(\mathcal{E})+\left\{\frac{\tau n\kappa}{2(n+1)}+n\mu-\frac{nl\tau}{2}-\frac{1}{2}(pn+2)\right\}=0. \tag{4.6}$$

If we suppose that $\mathcal{E}$ is solenoidal, then $div(\mathcal{E})$=0. Thus the equation (4.6) gives

$$\mu=\frac{1}{2n}[l\tau n+(pn+2)-\frac{\tau n\kappa}{(n+1)}]. \tag{4.7}$$

Again, we suppose that the vector field $\mathcal{E}$ is of gradient type, i.e, $\mathcal{E}$=$grad(f)$, where f is a smooth function on $\mathfrak{M}^n$. Then, we state the following:

Corollary 4.1. *If the metric g of an A(PBS) Kähler manifold be a CRYS $(g,\mathcal{E},\kappa,l,\mu)$, where $\mathcal{E}$ is the gradient of a smooth function f. Then*

$$\nabla^2 f=-\left\{\frac{\tau n\kappa}{2(n+1)}+n\mu-\frac{nl\tau}{2}-\frac{1}{2}(pn+2)\right\}.$$

Corollary 4.2. *If an A(PBS) Kähler manifold admits a CRYS and $\mathcal{E}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $p \gtreqless -(l\tau+\frac{2}{n}-\frac{\tau\kappa}{n+1})$.*

Corollary 4.3. *If an A(PBS) Kähler manifold admits a RYS and $\mathcal{E}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $(n+1)l \gtreqless \kappa$.*

Corollary 4.4. *If an A(PBS) Kähler manifold admits a CRYS and the vector field $\mathcal{E}$ is solenoidal, then*

Types of soliton	*Conditions for the solitons to be expanding, steady and shrinking*
conformal Ricci soliton	$p \gtreqless -(\frac{2}{n}-\frac{\tau}{n+1})$
conformal Yamabe soliton	$p \gtreqless -(\tau+\frac{2}{n})$
conformal Einstein soliton	$p \gtreqless (l\tau-\frac{2}{n}+\frac{\tau\kappa}{n+1})$

Now, let the Bochner Ricci tensor $\mathcal{Z}_B$ is of codazzi type, that is,

$$(\nabla_{\mathcal{E}}\mathcal{Z}_B)(\mathcal{F},\mathcal{G})=(\nabla_{\mathcal{E}}\mathcal{Z}_B)(\mathcal{F},\mathcal{G}). \tag{4.8}$$

In view of (1.3), (4.8) yields

$$(\nabla_{\mathcal{E}}\mathcal{Z}_B)(\mathcal{F},\mathcal{G})=[\mathcal{A}(\mathcal{E})+\mathcal{B}(\mathcal{E})]\mathcal{Z}_B(\mathcal{F},\mathcal{G})+\mathcal{A}(\mathcal{F})\mathcal{Z}_B(\mathcal{E},\mathcal{G})+\mathcal{A}(\mathcal{G})\mathcal{Z}_B(\mathcal{E},\mathcal{F}). \tag{4.9}$$

On the other hand, from (2.2)(ii) we have

$$(\nabla_{\mathcal{E}}\mathcal{Z}_B)(\mathcal{J}\mathcal{F},\mathcal{J}\mathcal{G})=(\nabla_{\mathcal{E}}\mathcal{Z}_B)(\mathcal{F},\mathcal{G}). \tag{4.10}$$

If we put

$$(\nabla_{\mathcal{E}}\mathcal{Z}_B)(\mathcal{F},\mathcal{G})=\mathbb{T}(\mathcal{E},\mathcal{F},\mathcal{G}),$$

where $\mathbb{T}$ is symmetric in pairs $(\mathcal{E},\mathcal{F})$,$(\mathcal{F},\mathcal{G})$ and $(\mathcal{G},\mathcal{E})$. Since $\mathcal{Z}_B$ is of codazzi type, then in view of (4.10), one can find $\mathbb{T}(\mathcal{E},\mathcal{F},\mathcal{G})$=0. Thus from (4.9) we get

$$[\mathcal{A}(\mathcal{E})+\mathcal{B}(\mathcal{E})]\mathcal{Z}_B(\mathcal{F},\mathcal{G})+\mathcal{A}(\mathcal{F})\mathcal{Z}_B(\mathcal{E},\mathcal{G})+\mathcal{A}(\mathcal{G})\mathcal{Z}_B(\mathcal{E},\mathcal{F})=0, \tag{4.11}$$

which by contracting gives

$$\mathcal{Z}_B(\mathcal{F},\mathcal{G})=0. \tag{4.12}$$

With the help of (4.12), equation (2.6) can be written as

$$\mathcal{S}(\mathcal{F},\mathcal{G})=\frac{\tau}{2(n+1)}g(\mathcal{F},\mathcal{G}). \tag{4.13}$$

Thus, we can state

Theorem 4.2. *If an A(PBS) Kähler manifold admits Codazzi type Bochner Ricci tensor, then the manifold is an Einstein manifold.*

Also, using (4.13) in (2.8), we obtain

$$(\mathfrak{L}_{\mathcal{E}}g)(\mathcal{F},\mathcal{G})+\left\{\frac{\kappa\tau}{(n+1)}+2\mu-l\tau-\frac{1}{n}(pn+2)\right\}g(\mathcal{F},\mathcal{G})=0. \tag{4.14}$$

Replacing $\mathcal{F}=\mathcal{G}=e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and summing over $i(1\leq i\leq n)$, we obtain

$$div(\mathcal{E})+\left\{\frac{\tau n\kappa}{2(n+1)}+n\mu-\frac{nl\tau}{2}-\frac{1}{2}(pn+2)\right\}=0. \tag{4.15}$$

If $\mathcal{E}$ is solenoidal, then $div(\mathcal{E})$=0. Therefore, equation (4.15) gives

$$\mu=\frac{1}{2n}[l\tau n+(pn+2)-\frac{\tau n\kappa}{(n+1)}]. \tag{4.16}$$

Thus, we have results

Corollary 4.5. *If an A(PBS) Kähler manifold with Codazzi type Bochner Ricci tensor admits a CRYS and $\mathcal{E}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $p \gtreqless -(l\tau+\frac{2}{n}-\frac{\tau\kappa}{n+1})$.*

Corollary 4.6. *If the metric g of an A(PBS) Kähler manifold with Codazzi type Bochner Ricci tensor be a CRYS $(g,\mathcal{E},\kappa,l,\mu)$, where $\mathcal{E}$ is the gradient of a smooth function f. Then we have*

$$\nabla^2 f=-\left\{\frac{\tau n\kappa}{2(n+1)}+n\mu-\frac{nl\tau}{2}-\frac{1}{2}(pn+2)\right\}.$$

Corollary 4.7. *If an A(PBS) Kähler manifold with Codazzi type Bochner Ricci tensor admits a RYS and $\mathcal{E}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $(n+1)l \gtreqless \kappa$.*

Corollary 4.8. *If an A(PBS) Kähler manifold with Codazzi type Bochner Ricci tensor admits a CRYS, then the vector field $\mathcal{E}$ is killing.*

Corollary 4.9. *If an A(PBS) Kähler manifold with Codazzi type Bochner Ricci tensor admits a CRYS and the vector field $\mathcal{E}$ is solenoidal, then*

Types of soliton	*Conditions for the solitons to be expanding, steady and shrinking*
conformal Ricci soliton	$p \gtreqless -(\frac{2}{n}-\frac{\tau}{n+1})$
conformal Yamabe soliton	$p \gtreqless -(\tau+\frac{2}{n})$
conformal Einstein soliton	$p \gtreqless (l\tau-\frac{2}{n}+\frac{\tau\kappa}{n+1})$

5. A(PRS) Kähler manifolds admitting CRYS

We suppose that $(\mathfrak{M}^n, g)$ be an A(PRS) Kähler manifold. Then from (1.3) and (2.2)(ii), we get

$$(\nabla_{\mathcal{E}}\mathcal{S})(\mathcal{J}\mathcal{F}, \mathcal{J}\mathcal{G}) = (\nabla_{\mathcal{E}}\mathcal{S})(\mathcal{F}, \mathcal{G}). \tag{5.1}$$

In view of (1.3), equation (5.1) can be written as

$$\mathcal{A}(\mathcal{J}\mathcal{F})\mathcal{S}(\mathcal{E}, \mathcal{J}\mathcal{G}) + \mathcal{A}(\mathcal{J}\mathcal{G})\mathcal{S}(\mathcal{J}\mathcal{F}, \mathcal{E}) = \mathcal{A}(\mathcal{F})\mathcal{S}(\mathcal{E}, \mathcal{G}) + \mathcal{A}(\mathcal{G})\mathcal{S}(\mathcal{F}, \mathcal{E}), \tag{5.2}$$

which by contracting gives

$$\mathcal{S}(\mathcal{E}, \mathcal{G}) = 0. \tag{5.3}$$

Therefore, we can state

Theorem 5.1. *An A(PRS) Kähler manifold is Ricci flat.*

Likewise section 3, we state the following:

Corollary 5.1. *Every A(PRS) Kähler manifold can not be conformally flat.*

Corollary 5.2. *If the metric g of an A(PRS) Kähler manifold be a CRYS $(g, \mathcal{E}, \kappa, l, \mu)$, where $\mathcal{E}$ is the gradient of a smooth function f, then we have*

$$\nabla^2 f = -\left\{n\mu - \frac{nl\tau}{2} - \frac{1}{2}(pn+2)\right\}.$$

Corollary 5.3. *If an A(PRS) Kähler manifold admits a CRYS and $\mathcal{E}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $p \gtreqqless -(l\tau + \frac{2}{n})$.*

Corollary 5.4. *If an A(PRS) Kähler manifold admits a CRYS and the vector field $\mathcal{E}$ is solenoidal, then we have*

Types of soliton	*Conditions for the solitons to be expanding, steady and shrinking*
conformal Ricci soliton	$p \gtreqqless -\frac{2}{n}$
conformal Yamabe soliton	$p \gtreqqless -(\tau + \frac{2}{n})$
conformal Einstein soliton	$p \gtreqqless (l\tau - \frac{2}{n})$

Theorem 5.2. *Let $(g, \mathcal{E}, \kappa, l, \mu)$ be a CRYS in an A(PRS) Kähler manifold, then the vector field $\mathcal{E}$ is killing.*

Next, we suppose that the Ricci tensor $\mathcal{S}$ is of codazzi type [1, 22], that is,

$$(\nabla_{\mathcal{E}}\mathcal{S})(\mathcal{F}, \mathcal{G}) = (\nabla_{\mathcal{E}}\mathcal{S})(\mathcal{F}, \mathcal{G}). \tag{5.4}$$

In view of (1.3), equation (5.4) can be written as

$$(\nabla_{\mathcal{E}}\mathcal{S})(\mathcal{F}, \mathcal{G}) = [\mathcal{A}(\mathcal{E}) + \mathcal{B}(\mathcal{E})]\mathcal{S}(\mathcal{F}, \mathcal{G}) + \mathcal{A}(\mathcal{F})\mathcal{S}(\mathcal{E}, \mathcal{G}) + \mathcal{A}(\mathcal{G})\mathcal{S}(\mathcal{E}, \mathcal{F}). \tag{5.5}$$

Also from (2.2)(ii), we have

$$(\nabla_{\mathcal{E}}\mathcal{S})(\mathcal{J}\mathcal{F},\mathcal{J}\mathcal{G}) = (\nabla_{\mathcal{E}}\mathcal{S})(\mathcal{F},\mathcal{G}). \tag{5.6}$$

In particular, if we put

$$(\nabla_{\mathcal{E}}\mathcal{S})(\mathcal{F},\mathcal{G}) = \mathbb{T}(\mathcal{E},\mathcal{F},\mathcal{G}),$$

where $\mathbb{T}$ is symmetric in pairs $(\mathcal{E},\mathcal{F})$,$(\mathcal{F},\mathcal{G})$ and $(\mathcal{G},\mathcal{E})$. Since $\mathcal{S}$ is of codazzi type, then in view of (5.6), one can find $\mathbb{T}(\mathcal{E},\mathcal{F},\mathcal{G})$=0. Thus from (5.5), we get

$$[\mathcal{A}(\mathcal{E})+\mathcal{B}(\mathcal{E})]\mathcal{S}(\mathcal{F},\mathcal{G})+\mathcal{A}(\mathcal{F})\mathcal{S}(\mathcal{E},\mathcal{G})+\mathcal{A}(\mathcal{G})\mathcal{S}(\mathcal{E},\mathcal{F}) = 0, \tag{5.7}$$

which by contracting over $\mathcal{F}$ and $\mathcal{E}$ gives

$$\mathcal{S}(\mathcal{F},\mathcal{G}) = 0. \tag{5.8}$$

Thus we have the following result:

Theorem 5.3. *An A(PRS) Kähler manifold admitting Codazzi type Ricci tensor is Ricci flat.*

Also, using (5.8) in (2.8), we obtain

$$(\mathfrak{L}_{\mathcal{E}}g)(\mathcal{F},\mathcal{G}) + \left\{2\mu - l\tau - \frac{1}{n}(pn+2)\right\} g(\mathcal{F},\mathcal{G}) = 0. \tag{5.9}$$

Replacing $\mathcal{F}=\mathcal{G}=e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and summing over $i(1\le i\le n)$, we obtain

$$div(\mathcal{E}) + \left\{n\mu - \frac{nl\tau}{2} - \frac{1}{2}(pn+2)\right\} = 0. \tag{5.10}$$

If $\mathcal{E}$ is solenoidal, then $div(\mathcal{E})$=0. Therefore, equation (5.10) gives

$$\mu = \frac{1}{2n}[l\tau n + (pn+2)]. \tag{5.11}$$

Let $\mathcal{E}$=$grad(f)$, where f is a smooth function on $\mathfrak{M}^n$. Then from equation (5.11), we get

$$\nabla^2 f = -\left\{\mu - \frac{l\tau}{2} - \frac{1}{2n}(pn+2)\right\}. \tag{5.12}$$

Therefore, we can state:

Theorem 5.4. *If an A(PRS) Kähler manifold with Codazzi type Ricci tensor admits a CRYS and $\mathcal{E}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $p \gtreqless -(l\tau+\frac{2}{n})$.*

Corollary 5.5. *If the metric g of an A(PRS) Kähler manifold with Codazzi type Ricci tensor admits a CRYS $(g,\mathcal{E},\kappa,l,\mu)$, where $\mathcal{E}$ is the gradient of a smooth function f, then we have*

$$\nabla^2 f = -\left\{\mu - \frac{l\tau}{2} - \frac{1}{2n}(pn+2)\right\}.$$

Corollary 5.6. *If an A(PRS) Kähler manifold with Codazzi type Ricci tensor admits a CRYS and the vector field $\mathcal{E}$ is solenoidal, then we have*

Types of soliton	*Conditions for the solitons to be expanding, steady and shrinking*
conformal Ricci soliton	$p \gtreqless -\frac{2}{n}$
conformal Yamabe soliton	$p \gtreqless -(\tau + \frac{2}{n})$
conformal Einstein soliton	$p \gtreqless (l\tau - \frac{2}{n})$

Corollary 5.7. *If an A(PRS) Kähler manifold with Codazzi type Ricci tensor admits a CRYS, then the vector field $\mathcal{E}$ is killing.*

6. A(PBRS) Kähler manifolds admitting CRYS

Let $(\mathfrak{M}^n, g)$ be an A(PBRS) Kähler manifold, then we obtain

$$\mathcal{Z}_B(\mathcal{J}\mathcal{F}, \mathcal{J}\mathcal{G}) = \mathcal{Z}_B(\mathcal{F}, \mathcal{G}). \tag{6.1}$$

After taking the covariant derivative of (6.1), we get

$$(\nabla_\mathcal{E}\mathcal{Z}_B)(\mathcal{J}\mathcal{F}, \mathcal{J}\mathcal{G}) = (\nabla_\mathcal{E}\mathcal{Z}_B)(\mathcal{F}, \mathcal{G}). \tag{6.2}$$

Using (6.2) in (2.3), we yields

$$\mathcal{A}(\mathcal{J}\mathcal{F}) + \mathcal{Z}_B(\mathcal{J}\mathcal{E}, \mathcal{G}) + \mathcal{A}(\mathcal{J}\mathcal{G})\mathcal{Z}_B(\mathcal{J}\mathcal{F}, \mathcal{E}) = \mathcal{A}(\mathcal{F})\mathcal{Z}_B(\mathcal{E}, \mathcal{G}) + \mathcal{A}(\mathcal{G})\mathcal{Z}_B(\mathcal{F}, \mathcal{E}), \tag{6.3}$$

which by a suitable contaction leads to

$$\mathcal{Z}_B(\mathcal{E}, \mathcal{G}) = 0. \tag{6.4}$$

With the help of (6.4), equation (2.6) takes the form

$$\mathcal{S}(\mathcal{E}, \mathcal{G}) = \frac{\tau}{2(n+1)} g(\mathcal{E}, \mathcal{G}). \tag{6.5}$$

We conclude the result as follows:

Theorem 6.1. *An A(PBRS) Kähler manifold is an Einstein manifold.*

Likewise section 4, we state the followings:

Corollary 6.1. *If the metric g of an A(PBRS) Kähler manifold be a CRYS $(g, \mathcal{E}, \kappa, l, \mu)$, where $\mathcal{E}$ is the gradient of a smooth function f. Then we have*

$$\nabla^2 f = -\left\{\frac{\tau n\kappa}{2(n+1)} + n\mu - \frac{nl\tau}{2} - \frac{1}{2}(pn+2)\right\}.$$

Corollary 6.2. *If an A(PBRS) Kähler manifold admits a CRYS and $\mathcal{E}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $p \gtreqless -(l\tau + \frac{2}{n} - \frac{\tau\kappa}{n+1})$.*

Corollary 6.3. *If an A(PBRS) Kähler manifold admits a RYS and $\mathcal{E}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $(n+1)l \gtreqless \kappa$.*

Corollary 6.4. *If an A(PBRS) Kähler manifold admits a CRYS, then the vector field $\mathcal{E}$ is killing.*

Corollary 6.5. *If an A(PBRS) Kähler manifold admits a CRYS and the vector field $\mathcal{E}$ is solenoidal, then*

Types of soliton	*Conditions for the solitons to be expanding, steady and shrinking*
conformal Ricci soliton	$p \gtreqqless -(\frac{2}{n} - \frac{\tau}{n+1})$
conformal Yamabe soliton	$p \gtreqqless -(\tau + \frac{2}{n})$
conformal Einstein soliton	$p \gtreqqless (l\tau - \frac{2}{n} + \frac{\tau\kappa}{n+1})$

7. Harmonic Aspect of CERYS on Kähler spacetime manifolds

A 4-dimensional Kähler manifold has general relativistic perfect fluid spacetime such type of manifold is called Kählerian spacetime manifold. Let η be a g-dual 1-form of the given vector field ξ, considering $g(\mathcal{E},\xi) = \eta(\mathcal{E})$ and $g(\xi,\xi) = -1$. Then ξ is called a solution of the Schrödinger-Ricci equation if it satisfies

$$div(\mathfrak{L}_\xi g) = 0, \tag{7.1}$$

where $(\mathfrak{L}_\xi g)$ is Lie derivative for the vector field ξ. In [8], Chow et al. studied the divergence of the Lie derivative and has the form

$$div(\mathfrak{L}_\xi g) = (\Gamma + \mathcal{S})(\xi) + d(div(\xi)), \tag{7.2}$$

where Γ denote the Laplace-Hodge operator with respect to the metric g and $\mathcal{S}$ is the Ricci tensor. Now we recall the notion of CERYS

$$\mathfrak{L}_\xi g + 2\kappa\mathcal{S} + \left\{2\mu - l\tau - (p + \frac{1}{2})\right\} g + 2\lambda\eta\otimes\eta = 0. \tag{7.3}$$

Taking trace of equation (7.3), we get

$$div(\xi) + (\kappa - 2l)\tau + 4\mu - 2(p + \frac{1}{2}) + \lambda|\xi|^2 = 0. \tag{7.4}$$

Also, by direct calculation, we obtain

$$div(\eta\otimes\eta) = div(\xi)\eta + \nabla_\xi\eta. \tag{7.5}$$

By taking the divergence of (7.3) and using (7.5) we have

$$div(\mathfrak{L}_\xi g) + (\kappa - 2l)d(\tau) + 2\lambda[div(\xi)\eta + \nabla_\xi\eta] = 0. \tag{7.6}$$

For the Schrödinger-Ricci soliton, we mention that a 1-form π is a solution of the Schrödinger-Ricci equation if

$$(\Gamma + \mathcal{S})(\pi) + d(div(\pi)) = 0. \tag{7.7}$$

Hence, we have next results.

Theorem 7.1. *A conformal η-Ricci-Yamabe soliton of type (κ, l) in a Kählerian spacetimes $(\mathfrak{M}^4, g)$ with η the g-dual of the vector field ξ. Then η is the solution of the Schrödinger-Ricci equation if and only if*

$$d(\tau) = \frac{2\lambda}{(\kappa - 2l)}[\{(\kappa - 2l)\tau + 4\mu - (p + \frac{1}{2}) + \lambda|\xi|^2\}\eta + \nabla_\xi \eta]. \tag{7.8}$$

Proof. With the help of (7.4), (7.7) and using the fact that $2div(\mathcal{S})=(\kappa-2l)d(\tau)$, it follows that η is a solution of the Schrödinger-Ricci equation if and only if (7.8) holds. □

8. Some applications

As an application, we obtain the following results which is based on Theorem 7.1 for the conformal η-Ricci-Yamabe soliton, conformal η-Yamabe soliton, and conformal η-Einstein soliton ($\kappa = 1$, $l = 0$, $\kappa = 0$, $l = 1$, and $\kappa = 1$, $l = -1$) (cf. [6], [17], [18]).

Corollary 8.1. *A conformal η-Ricci soliton in a Kählerian spacetime $(\mathfrak{M}^4, g)$ with η the g-dual of the vector field ξ. Then η is the solution of the Schrödinger-Ricci equation if and only if*

$$d(\tau) = 2\lambda[\tau + 4\mu - 2(p + \frac{1}{2}) + \lambda|\xi|^2\}\eta + \nabla_\xi \eta].$$

Corollary 8.2. *A conformal η-Yamabe soliton in a Kählerian spacetime $(\mathfrak{M}^4, g)$ with η the g-dual of the vector field ξ. Then η is the solution of the Schrödinger-Ricci equation if and only if*

$$d(\tau) = \lambda[2\tau - 4\mu + 2(p + \frac{1}{2}) - \lambda|\xi|^2\}\eta - \nabla_\xi \eta].$$

Corollary 8.3. *A conformal η-Einstein soliton in a Kählerian spacetime $(\mathfrak{M}^4, g)$ with η the g-dual of the vector field ξ. Then η is the solution of the Schrödinger-Ricci equation if and only if*

$$d(\tau) = \frac{2\lambda}{3}[3\tau + 4\mu - 2(p + \frac{1}{2}) + \lambda|\xi|^2\}\eta + \nabla_\xi \eta].$$

9. Conclusions

The conformal η-Ricci-Yamabe soliton on Riemannian manifolds (or, pseudo-Riemannian manifolds) is of great importance in the area of differential geometry, especially in Riemannian geometry and in special relativistic physics as well. In essence, Ricci-Yamabe flow is the most prominent flagship of modern physics. The conformal η-Ricci-Yamabe soliton is a brand-new concept that deals geometric and physical applications with mathematical physics, general relativity, quantum cosmology, quantum gravity, and black hole theory in addition to the differentiable manifold field. As far as our knowledge goes, the properties of conformal η-Ricci-Yamabe soliton in an almost pseudo symmetric Kähler manifolds have been studied in this paper.

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