Int. J. Anal. Appl. (2023), 21:101

International Journal of Analysis and Applications

Fuzzy $n-$Controlled Metric Space

Salman Furqan[1], Naeem Saleem[1,2,*], Salvatore Sessa[3,*]

[1] *Department of Mathematics, University of Management and Technology, Lahore, Pakistan*
[2] *Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, Pretoria, Medunsa- 0204, South Africa*
[3] *Dipartimento di Architettura, Università Dinapoli Federico II, Via Toledo 403, 80121 Napoli, Italy*

** Corresponding author:* naeem.saleem2@gmail.com,*sessa@unina.it*

Abstract. This manuscript consists of the idea of $n-$controlled metric space in fuzzy set theory to generalize a number of fuzzy metric spaces in the literature, for example, pentagonal, hexagonal, triple, and double controlled metric spaces and many other spaces in fuzzy environment. Various examples are given to explain definitions and results. We define open ball, convergence of a sequence and a Cauchy sequence in the context of fuzzy $n-$controlled metric space. We also prove, by means of an example, that a fuzzy $n-$controlled metric space is not Hausdorff. At the end of the article, an application is given to prove the uniqueness of the solution to fractional differential equations.

1. Introduction

The applications of fixed point theory is the key to prove the uniqueness of the solution of a scientific problem with the help of Banach fixed point theorem [1]. Researchers have implemented this famous theorem in other directions ([2–5]) and obtained interesting results. There are many generalizations of [1]. For example, Edelstein [6], generalized the Banach theorem in 1961. Kannan, [7] proved Banach's theorems without using the completeness of the metric and continuity of the contraction, however, he obtained the same conclusion but different sufficient conditions. Similar results were proved by Chatterjea [8]. In 1974, Ćirić [9], utilized the quasi contractive mappings that generalizes [1]. He also introduced multi-valued quasi contractions. Samet et. al [10] introduced a very interesting contraction, called $\alpha-\psi-$contraction, that enhanced and generalized numerous

Received: Jul. 26, 2023.

2020 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. fuzzy metric space; controlled metric space; fixed point; fractional differential equation.

https://doi.org/10.28924/2291-8639-21-2023-101

ISSN: 2291-8639

results in the literature. In 2014, Jleli et. al [11] gave the generalized version of [1] by introducing the function θ that satisfies certain properties.

Since all the above generalizations of [1] need to be continuous mappings, so Suzuki [12] gave the idea of Suzuki type mappings in which the contraction need not be continuous. Using $F-$contractions, which is given by Wardowski [13], and Suzuki contraction, the authors in [14], introduced generalized Suzuki $F-$contractions. Same authors have discussed the notion of Suzuki-type $(\alpha, \beta, \gamma_g)-$generalized proximal contractions and proved some results. Recently, Saleem et al. [15], gave the idea of modified $F-$contractions, generalized Suzuki $F-$contractions and proved some interesting results.

In 1965, Zadeh [16] generalized the definition of a crisp set by defining the fuzzy set that gives more efficient and accurate results. As fuzzy set addresses the uncertainty and give more accuracy compared to crisp set, researcher have used fuzzy sets in almost every branch of mathematics, see ([17–19]). Metric space in a fuzzy environment is the most studied topic. The first definition of metric space using fuzzy sets was given by Kramosil et. al [20] which is considered as the generalization of statistical metric spaces defined by Menger [21]. But in their definition, they did not discuss any topological aspects. The convergence of a sequence in fuzzy metric spaces was defined by Grabiec [22]. By discussing Cauchyness he proved the fuzzy version of the Banach theorem. As topological properties of metric spaces play a vital role so, George and Veeramani [23] generalized the definition given in [22] by discussing topology and proved that it is Hausdorff.

Branciari [24] introduced generalized metric space which is known as rectangular metric space or $b-$Branciari space. He proved Banach-Caccippoli type fixed point results. In [25], the author has introduced a fuzzy version of $b-$metric space and generalized some spaces. The authors in [26] utilized the function α to generalize the notion of [25] by introducing an extended version of a fuzzy $b-$metric space and proved interesting results. Sezen [27] first used a controlled function to define the concept of controlled spaces in fuzzy sets theory. She utilized the sense of [20] and prove Banach fixed point results. Saleem et al. [28] used two functions α and β and defined double controlled metric in a fuzzy environment which generalizes the results in [27]. Chugh et al. [29] gave the fuzzy version of [24] by giving the concept of rectangular fuzzy metric space. The notion of a rectangular $b-$metric space in fuzzy set theory is given by [30] to generalize the notion given in [29]. Recently, the concept of an extended rectangular metric space in a fuzzy environment is given by Saleem et al. [31] that generalize the results of [30] and [29]. They also proved that this space is not Hausdorff. The authors in [32] utilized three functions f, g, h and gave the notions of fuzzy triple controlled metric spaces. They also showed, with the help of an example, that this space is not Hausdorff. The ideas of extended hexagonal $b-$metric and pentagonal controlled metric spaces in the fuzzy environments were given by Zubair et al. [33] and Hussain et al. [34] respectively and proved some fixed point results. In [35], the authors have introduced graphical fuzzy metric spaces and proved interesting results.

Since fractional calculus gives more accurate and efficient results as compared to ordinary calculus, which involves integer order derivatives, scientists have been utilizing fractional calculus in many disciplines. It is helpful to make mathematical models of certain phenomena, like epidemic models, bird flu models, influenza types of hepatitis, SARS, HIV, dengue, malaria and many others. The concept of the fractional derivative is very old when Leibnitz and L'Hopital talked about half-order derivatives. In answer to their question, Lacroix [36] claimed that $\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = 2\sqrt{\frac{x}{\pi}}$. Abel [37] was the first mathematician who utilized fractional calculus as an application. He applied fractional calculus to the tautochrone problem. This result attracted Liouville [38], who applied his results to potential theory. Later Reimann involves a definite integral in the definition of a fractional derivative which was applicable to power series having non-integer exponents. There are some other definitions of a fractional order derivative, see ([39–41]). But among all the above definitions, the modern definition of a fractional derivative is due to Caputo [42]. The main drawback of previous definitions is that the fractional differential equation requires a strange set of initial conditions. Caputo utilized the more classical initial conditions compared to Reimann-Liouville fractional derivative which is frequently used in applications.

In this paper we define $n-$controlled metric space in fuzzy set theory that generalizes almost all the metric spaces discussed above. We prove some fixed point results and elaborate our results with examples. We use the sense of [23] to define this space. We will use $\alpha - \phi-$contractive mapping in our main results that generalize some existing fixed point theorems in the literature. Each result and definition is supported by examples, further, we prove that this newly defined space is not Hausdorff.

2. Preliminaries

Definition 2.1 ([43]). *A binary operation* $* : I \times I \to I, (I = [0, 1])$ *is known as continuous triangular norm* (CTN)*, if for all* $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4 \in [0, 1]$, $*$ *satisfy:*

(∗1) $*(\mathfrak{s}_1, \mathfrak{s}_2) = *(\mathfrak{s}_2, \mathfrak{s}_1)$;

(∗2) $*(\mathfrak{s}_1, *(\mathfrak{s}_2, \mathfrak{s}_3)) = *(*(\mathfrak{s}_1, \mathfrak{s}_2), \mathfrak{s}_3)$;

(∗3) $*$ *is continuous;*

(∗4) $*(\mathfrak{s}_1, 1) = \mathfrak{s}_1$ *for every* $\mathfrak{s}_1 \in [0, 1]$;

(∗5) $*(\mathfrak{s}_1, \mathfrak{s}_3) \leq *(\mathfrak{s}_2, \mathfrak{s}_4)$ *whenever* $\mathfrak{s}_1 \leq \mathfrak{s}_2$, $\mathfrak{s}_3 \leq \mathfrak{s}_4$.

Definition 2.2 ([23]). *Let* $\Omega \neq \emptyset$*, a fuzzy set* $F_m : \Omega \times \Omega \times (0, \infty)$ *is called fuzzy metric on* Ω *with* $*$ *as a* (CTN)*, if for all* $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \in \Omega$*, the following conditions holds:*

(FM1) $F_m(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) > 0$;

(FM2) $F_m(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = 1$ *for all* $t > 0$*, if and only if* $\mathfrak{s}_1 = \mathfrak{s}_2$;

(FM3) $F_m(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = F_m(\mathfrak{s}_2, \mathfrak{s}_1, \mathfrak{t})$;

(FM4) $F_m(\mathfrak{s}_1, \mathfrak{s}_3, \mathfrak{t}_1 + \mathfrak{t}_2) \geq F_m(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}_1) * F_m(\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{t}_2)$ *for all* $\mathfrak{t}_1, \mathfrak{t}_2 > 0$;

(FM5) $F_m(\mathfrak{s}_1, \mathfrak{s}_2, \cdot) : (0, \infty) \longrightarrow [0, 1]$ *is continuous.*

The triplet $(\Omega, F_m, *)$ *is called a fuzzy metric space.*

Definition 2.3 ([28]). *Let* $\Omega \neq \emptyset$, $\alpha, \beta : \Omega \times \Omega \to [1, \infty)$ *are two non-comparable functions. Then a fuzzy set* $F_D : \Omega \times \Omega \times (0, \infty) \longrightarrow [0, 1]$ *is fuzzy double controlled metric with* $*$ *as* (CTN)*, if for all* $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \in \Omega$*, the following conditions holds:*

(F_D1) $F_D(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) > 0$;

(F_D2) $F_D(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = 1$ *for all* $\mathfrak{t} > 0$, *if and only if* $\mathfrak{s}_1 = \mathfrak{s}_2$;

(F_D3) $F_D(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = F_D(\mathfrak{s}_2, \mathfrak{s}_1, \mathfrak{t})$;

(F_D4) $F_D(\mathfrak{s}_1, \mathfrak{s}_3, \mathfrak{t}_1 + \mathfrak{t}_2) \geq F_D\left(\mathfrak{s}_1, \mathfrak{s}_2, \frac{\mathfrak{t}_1}{\alpha(\mathfrak{s}_1, \mathfrak{s}_2)}\right) * F_D\left(\mathfrak{s}_2, \mathfrak{s}_3, \frac{\mathfrak{t}_2}{\beta(\mathfrak{s}_2, \mathfrak{s}_3)}\right)$ *for all* $\mathfrak{t}_1, \mathfrak{t}_2 > 0$;

(F_D5) $F_D(\mathfrak{s}_1, \mathfrak{s}_2, \cdot) : (0, \infty) \longrightarrow [0, 1]$ *is continuous.*

Then $(\Omega, F_D, *)$ *is called a fuzzy double controlled metric space.*

Example 2.1 ([28]). *Let* $\Omega = \{1, 2, 3\}$ *and* $\alpha, \beta : \Omega \times \Omega \longrightarrow [1, \infty)$ *be two non-comparable continuous functions given by* $\alpha(\mathfrak{s}_1, \mathfrak{s}_2) = \mathfrak{s}_1 + \mathfrak{s}_2 + 1$ *and* $\beta(\mathfrak{s}_2, \mathfrak{s}_3) = {\mathfrak{s}_2}^2 + {\mathfrak{s}_3}^2 - 1$. *Define* $M : \Omega \times \Omega \times (0, \infty) \longrightarrow [0, 1]$ *as*

$$F_D(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = \frac{\min\{\mathfrak{s}_1, \mathfrak{s}_2\} + \mathfrak{t}}{\max\{\mathfrak{s}_1, \mathfrak{s}_2\} + \mathfrak{t}}.$$

Then $(\Omega, M, *)$ *is fuzzy double controlled metric type space with product t-norm.*

Definition 2.4 ([32]). *Let* $\Omega \neq \emptyset$ *and consider three functions* $f, g, h : \Omega \times \Omega \to [1, \infty)$. *A fuzzy set* $F_T : \Omega \times \Omega \times (0, \infty)$ *is fuzzy triple controlled metric with* (CTN) $*$*, if for* $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega$ *and all distinct* $\mathfrak{s}_3, \mathfrak{s}_4 \in \Omega \setminus \{\mathfrak{s}_1, \mathfrak{s}_2\}$*, the following conditions are satisfied:*

(F_T1) $F_T(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) > 0$;

(F_T2) $F_T(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = 1$ *for all* $\mathfrak{t} > 0$ *if and only if* $\mathfrak{s}_1 = \mathfrak{s}_2$;

(F_T3) $F_T(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = F_T(\mathfrak{s}_2, \mathfrak{s}_1, \mathfrak{t})$;

(F_T4) $F_T(\mathfrak{s}_1, \mathfrak{s}_4, \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3) \geq F_T(\mathfrak{s}_1, \mathfrak{s}_2, \frac{t}{f(\mathfrak{s}_1, \mathfrak{s}_2)}) * F_T(\mathfrak{s}_2, \mathfrak{s}_3, \frac{\mathfrak{t}_1}{g(\mathfrak{s}_2, \mathfrak{s}_3)}) * F_T(\mathfrak{s}_3, \mathfrak{s}_4, \frac{\mathfrak{t}_2}{h(\mathfrak{s}_3, \mathfrak{s}_4)})$, *for all* $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3 > 0$;

(F_T5) $F_T(\mathfrak{s}_1, \mathfrak{s}_2, \cdot) : (0, \infty) \to [0, 1]$ *is continuous.*

Then $(\Omega, F_T, *)$ *is called a fuzzy triple controlled metric space.*

Example 2.2 ([32]). *Let* $\Omega = [0, 1]$ *and* $F_T : \Omega \times \Omega \times (0, \infty] \to [0, 1]$ *be defined as* $F_T(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = e^{-\frac{|\mathfrak{s}_1 - \mathfrak{s}_2|}{\mathfrak{t}}}$ *for all* $\mathfrak{t} > 0$, *further let* $f, g, h : \Omega \times \Omega \to [0, \infty]$ *be continuous functions defined by* $f(\mathfrak{s}_1, \mathfrak{s}_2) = \mathfrak{s}_1 + \mathfrak{s}_2 + 1$, $g(\mathfrak{s}_2, \mathfrak{s}_3) = {\mathfrak{s}_2}^2 + \mathfrak{s}_3 + 1$ *and* $h(\mathfrak{s}_3, \mathfrak{s}_4) = {\mathfrak{s}_3}^2 + {\mathfrak{s}_4}^2 + 1$. *Then* $(\Omega, F_T, *)$ *is a fuzzy triple controlled metric space.*

Definition 2.5 ([34]). *Let* $\Omega \neq \emptyset$, $Q_i : \Omega \times \Omega \to [1, \infty)$, $(1 \leq i \leq 5)$ *be given functions, then a fuzzy set* F_Q *on* $\Omega \times \Omega \times [0, \infty)$ *is fuzzy pentagonal controlled metric with a* (CTN) $*$*, if for any distinct* $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5, \mathfrak{s}_6 \in \Omega$*, the following conditions are satisfied:*

(F_Q1) $F_Q(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) > 0$;

(F_Q2) $F_Q(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = 1$ *for all* $t > 0$ *if and only if* $\mathfrak{s}_1 = \mathfrak{s}_2$;

(F_Q3) $F_Q(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = F_Q(\mathfrak{s}_2, \mathfrak{s}_1, \mathfrak{t})$;

(F_Q4) $F_Q(\mathfrak{s}_1, \mathfrak{s}_6, \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + \mathfrak{t}_4 + \mathfrak{t}_5) \geq F_Q(\mathfrak{s}_1, \mathfrak{s}_2, \frac{\mathfrak{t}_1}{Q_1(\mathfrak{s}_1,\mathfrak{s}_2)}) * F_Q(\mathfrak{s}_2, \mathfrak{s}_3, \frac{\mathfrak{t}_2}{Q_2(\mathfrak{s}_2,\mathfrak{s}_3)})$
$* F_Q(\mathfrak{s}_3, \mathfrak{s}_4, \frac{\mathfrak{t}_3}{Q_3(\mathfrak{s}_3,\mathfrak{s}_4)}) * F_Q(\mathfrak{s}_4, \mathfrak{s}_5, \frac{\mathfrak{t}_4}{Q_4(\mathfrak{s}_4,\mathfrak{s}_5)}) * F_Q(\mathfrak{s}_5, \mathfrak{s}_6, \frac{\mathfrak{t}_5}{Q_5(\mathfrak{s}_5,\mathfrak{s}_6)})$,
for all $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5 > 0$;

(F_Q5) $F_Q(\mathfrak{t}, \nu, \cdot) : (0, \infty) \to [0, 1]$ *is continuous and* $\lim_{t\to\infty} F_Q(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = 1$.

Then $(\Omega, F_Q, *)$ *is called a fuzzy pentagonal controlled metric space.*

Example 2.3 ([34]). *Let* $\Omega = \{1, 2, 3, 4, 5, 6\}$ *and define* $F_Q : \Omega \times \Omega \times [0, \infty) \longrightarrow [0, 1]$ *as* $F_Q(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = \frac{\mathfrak{t}}{\mathfrak{t}+|\mathfrak{s}_1-\mathfrak{s}_2|^6}$ *with controlled functions* $Q_1(\mathfrak{s}_1, \mathfrak{s}_2) = 1 + \mathfrak{s}_1 + \mathfrak{s}_2, Q_2(\mathfrak{s}_1, \mathfrak{s}_2) = 1 + {\mathfrak{s}_1}^2 + {\mathfrak{s}_2}^2, Q_3(\mathfrak{s}_1, \mathfrak{s}_2) = 1 + \frac{\mathfrak{s}_1}{\mathfrak{s}_2}, Q_4(\mathfrak{s}_1, \mathfrak{s}_2) = 1 + \frac{\mathfrak{s}_2}{\mathfrak{s}_1}, Q_5(\mathfrak{s}_1, \mathfrak{s}_2) = 1 + {\mathfrak{s}_2}^2{\mathfrak{s}_1}^2$. *Then* $(\Omega, F_Q, *)$ *is fuzzy pentagonal controlled metric space with product* $t-$*norm.*

3. Main Results

This section contains definitions, examples and theorems related to fuzzy $n-$controlled metric space. We will also deduce some important remarks that prove generalizations of many metric spaces in fuzzy set theory. We will define open ball and will prove that the newly defined space is not Hausdorff. Each result is elaborated with the help of examples. Now we give the definition of a fuzzy $n-$controlled metric space in the sense of [23]:

Definition 3.1. *Let* $\Omega \neq \emptyset$ *and* $\alpha_i : \Omega \times \Omega \longrightarrow [1, \infty)(1 \leq i \leq n)$ *be* n *non-comparable functions. A fuzzy set* F^c_n *on* $\Omega \times \Omega \times (0, \infty)$*, together with a* (CTN) $*$*, is called a fuzzy* $n-$*controlled metric, if* F^c_n *satisfies:*

($M1$) $F^c_n(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) > 0$;

($M2$) $F^c_n(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = 1$ *for all* $\mathfrak{t} > 0$, *if and only if* $\mathfrak{s}_1 = \mathfrak{s}_2$;

($M3$) $F^c_n(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = F^c_n(\mathfrak{s}_2, \mathfrak{s}_1, \mathfrak{t})$;

($M4$) $F^c_n(\mathfrak{s}_1, \mathfrak{s}_{n+1}, \mathfrak{t}_1 + \mathfrak{t}_2 + \ldots + \mathfrak{t}_\mathfrak{n}) \geq F^c_n(\mathfrak{s}_1, \mathfrak{s}_2, \frac{\mathfrak{t}_1}{\alpha_1(\mathfrak{s}_1,\mathfrak{s}_2)}) * F^c_n(\mathfrak{s}_2, \mathfrak{s}_3, \frac{\mathfrak{t}_2}{\alpha_2(\mathfrak{s}_2,\mathfrak{s}_3)}) * \ldots$
$* F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \frac{\mathfrak{t}_\mathfrak{n}}{\alpha_n(\mathfrak{s}_n,\mathfrak{s}_{n+1})})$, *for all* $\mathfrak{t}_\mathfrak{n} > 0$;

($M5$) $F^c_n(\mathfrak{s}_1, \mathfrak{s}_2, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

for all distinct $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \ldots, \mathfrak{s}_{n+1} \in \Omega$. *The quadruple* $(\Omega, F^c_n, \alpha_n, *)$ *is called a fuzzy* $n-$*controlled metric space* (FnCMS).

From definition (3.1), we have the following remarks.

Remark 3.1. **(i)** *If we restrict ourselves to six distinct elements* $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5, \mathfrak{s}_6$, *then a* (FnCMS) *reduces to fuzzy pentagonal controlled metric space [34] with controlled functions* $\alpha_i, 1 \leq i \leq 5$.

(ii) *If we restrict ourselves to four distinct elements* $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$, *then a* (FnCMS) *reduces to fuzzy triple controlled metric space [32] with controlled functions* $\alpha_i, 1 \leq i \leq 3$.

(iii) *If we restrict ourselves to three distinct elements* $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$*, then a* (FnCMS) *reduces to the definition in [28] with controlled functions* $\alpha_i, 1 \leq i \leq 2$.

Keeping the above remarks in mind, we can deduce extended fuzzy $b-$rectangular, fuzzy $b-$rectangular, fuzzy rectangular, and some other metric spaces in fuzzy set theory.
Now we will give an example that justifies definition 3.1. We will restrict ourselves to finite n.

Example 3.1. *Consider* $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$ *and* $\alpha_i : \Omega \times \Omega \longrightarrow [1, \infty)(1 \leq i \leq 6)$ *be defined as* $\alpha_1 = 1 + \mathfrak{s}_1 + \mathfrak{s}_2$, $\alpha_2 = 1 + {\mathfrak{s}_1}^2 + \mathfrak{s}_2$, $\alpha_3 = 1 + \mathfrak{s}_1 + {\mathfrak{s}_2}^2$, $\alpha_4 = 1 + {\mathfrak{s}_1}^2 + {\mathfrak{s}_2}^2$, $\alpha_5 = 1 + \mathfrak{s}_1 + {\mathfrak{s}_2}^3$, $\alpha_6 = 1 + {\mathfrak{s}_1}^3 + {\mathfrak{s}_2}^3$. *Now define* $F_n^c : \Omega \times \Omega \times (0, \infty) \to [0, 1]$ *as:*

$$F_n^c(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = \frac{\min\{\mathfrak{s}_1, \mathfrak{s}_2\} + \mathfrak{t}}{\max\{\mathfrak{s}_1, \mathfrak{s}_2\} + \mathfrak{t}}.$$

Then with product $t-$*norm* $(\Omega, F_n^c, \alpha_i, *)$ *is a* (FnCMS)*. Here we will prove only (M4) as (M1)-(M3) and (M5) are easy to prove. Let* $\mathfrak{s}_1 = 1$, $\mathfrak{s}_2 = 7$*, then*

$$\begin{aligned} F_n^c(1, 7, \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + \mathfrak{t}_4 + \mathfrak{t}_5 + \mathfrak{t}_6) &= \frac{\min\{1, 7\} + \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + \mathfrak{t}_4 + \mathfrak{t}_5 + \mathfrak{t}_6}{\max\{1, 7\} + \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + \mathfrak{t}_4 + \mathfrak{t}_5 + \mathfrak{t}_6} \\ &= \frac{1 + \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + \mathfrak{t}_4 + \mathfrak{t}_5 + \mathfrak{t}_6}{7 + \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + \mathfrak{t}_4 + \mathfrak{t}_5 + \mathfrak{t}_6}. \end{aligned}$$

Now

$$F_n^c(1, 2, \frac{\mathfrak{t}_1}{\alpha_1(1, 2)}) = \frac{\min\{1, 2\} + \frac{\mathfrak{t}_2}{\alpha_1(1, 2)}}{\max\{1, 2\} + \frac{\mathfrak{t}_1}{\alpha_1(1, 2)}} = \frac{1 + \frac{\mathfrak{t}_1}{4}}{2 + \frac{\mathfrak{t}_1}{4}} = \frac{4 + \mathfrak{t}_1}{8 + \mathfrak{t}_1},$$

$$F_n^c(2, 3, \frac{\mathfrak{t}_2}{\alpha_2(2, 3)}) = \frac{\min\{2, 3\} + \frac{\mathfrak{t}_2}{\alpha_2(2, 3)}}{\max\{2, 3\} + \frac{\mathfrak{t}_2}{\alpha_2(2, 3)}} = \frac{2 + \frac{\mathfrak{t}_2}{8}}{3 + \frac{\mathfrak{t}_2}{8}} = \frac{16 + \mathfrak{t}_2}{24 + \mathfrak{t}_2},$$

$$F_n^c(3, 4, \frac{\mathfrak{t}_3}{\alpha_3(3, 4)}) = \frac{\min\{3, 4\} + \frac{\mathfrak{t}_3}{\alpha_3(3, 4)}}{\max\{3, 4\} + \frac{\mathfrak{t}_3}{\alpha_3(3, 4)}} = \frac{3 + \frac{\mathfrak{t}_3}{20}}{4 + \frac{\mathfrak{t}_3}{20}} = \frac{60 + t_3}{80 + \mathfrak{t}_3},$$

$$F_n^c(4, 5, \frac{\mathfrak{t}_4}{\alpha_4(4, 5)}) = \frac{\min\{4, 5\} + \frac{\mathfrak{t}_4}{\alpha_4(4, 5)}}{\max\{4, 5\} + \frac{\mathfrak{t}_4}{\alpha_4(4, 5)}} = \frac{4 + \frac{\mathfrak{t}_4}{42}}{5 + \frac{\mathfrak{t}_4}{42}} = \frac{168 + \mathfrak{t}_4}{210 + \mathfrak{t}_4},$$

$$F_n^c(5, 6, \frac{\mathfrak{t}_5}{\alpha_5(5, 6)}) = \frac{\min\{5, 6\} + \frac{\mathfrak{t}_5}{\alpha_5(5, 6)}}{\max\{5, 6\} + \frac{\mathfrak{t}_5}{\alpha_5(5, 6)}} = \frac{5 + \frac{\mathfrak{t}_5}{222}}{6 + \frac{\mathfrak{t}_5}{222}} = \frac{1110 + \mathfrak{t}_5}{1332 + \mathfrak{t}_5},$$

$$\mathcal{F}^c_n(6,7,\frac{\mathfrak{t}_6}{\alpha_6(6,7)}) = \frac{\min\{6,7\}+\frac{\mathfrak{t}_6}{\alpha_6(6,7)}}{\max\{6,7\}+\frac{\mathfrak{t}_6}{\alpha_6(6,7)}} = \frac{6+\frac{\mathfrak{t}_6}{560}}{7+\frac{\mathfrak{t}_6}{560}} = \frac{3360+\mathfrak{t}_6}{3920+\mathfrak{t}_6}.$$

Clearly,

$$\mathcal{F}^c_n(1,7,\mathfrak{t}_1+\mathfrak{t}_2+\mathfrak{t}_3+\mathfrak{t}_4+\mathfrak{t}_5+\mathfrak{t}_6) \geq \mathcal{F}^c_n(1,2,\frac{\mathfrak{t}_1}{\alpha_1(1,2)}) * \mathcal{F}^c_n(2,3,\frac{\mathfrak{t}_2}{\alpha_2(2,3)}) * \mathcal{F}^c_n(3,4,\frac{\mathfrak{t}_3}{\alpha_3(3,4)}) * \mathcal{F}^c_n(4,5,\frac{\mathfrak{t}_4}{\alpha_4(4,5)}) * \mathcal{F}^c_n(5,6,\frac{\mathfrak{t}_5}{\alpha_5(5,6)}) * \mathcal{F}^c_n(6,7,\frac{\mathfrak{t}_6}{\alpha_6(6,7)}).$$

Similarly, we can prove in other cases. Hence $(\Omega, \mathcal{F}^c_n, \alpha_i, *)$ *is* (FnCMS) *for* $n = 6$*. In the same steps, we can prove higher values of* n*.*

Definition 3.2. *Let* $\{\mathfrak{s}_n\}$ *be a sequence in* (FnCMS) $(\Omega, \mathcal{F}^c_n, \alpha_n, *)$*. Then:*

(1) $\{\mathfrak{s}_n\}$ *is convergent sequence, if for any* $\mathfrak{t} > 0$*, there exists* $\mathfrak{s} \in \Omega$ *satisfy*

$$\lim_{n\to\infty} \mathcal{F}^c_n(\mathfrak{s}_n, \mathfrak{s}, \mathfrak{t}) = 1,$$

(2) $\{\mathfrak{s}_n\}$ *is Cauchy sequence if for all* $\mathfrak{t} > 0,\ p > 0,$

$$\lim_{n\to\infty} \mathcal{F}^c_n(\mathfrak{s}_{n+p}, \mathfrak{s}_n, \mathfrak{t}) = 1.$$

A *(FnCMS)* $(\Omega, \mathcal{F}^c_n, \alpha_n, *)$ is called complete *(FnCMS)*, if every Cauchy sequence $\{\mathfrak{s}_n\}$ converges to some $\mathfrak{s} \in \Omega$.

Definition 3.3. *Let* $(\Omega, \mathcal{F}^c_n, \alpha_n, *)$ *be a* (FnCMS)*, then the open ball* $B(\mathfrak{s}, \mathfrak{r}, \mathfrak{t})$*, is given by*

$$B(\mathfrak{s}, \mathfrak{r}, \mathfrak{t}) = \{\mathfrak{v} \in \Omega : \mathcal{F}^c_n(\mathfrak{s}, \mathfrak{v}, t) > 1 - \mathfrak{r}\},$$

where $\mathfrak{s}$ *is the center and* $\mathfrak{r}$ *is the radius of the ball.*

In next example, we will prove a *(FnCMS)* need not to be Hausdorff.

Example 3.2. *Take the* (FnCMS) *of example (3.1) and define* $B_1(1, 0.4, 5)$ *with center* $\mathfrak{s}_1 = 1$*, radius* $\mathfrak{r}_1 = 0.4$ *and* $\mathfrak{t}_1 = 5$ *as*

$$B_1(1, 0.4, 5) = \{\mathfrak{s} \in \Omega : \mathcal{F}^c_n(1, \mathfrak{s}, 5) > 0.6\}.$$

Let $1 \in \Omega$*, then* $\mathcal{F}^c_n(1,1,5) = \frac{\min\{1,1\}+5}{\max\{1,1\}+5} = \frac{1+5}{1+5} = 1$*, so* $1 \in B_1(1, 0.4, 5)$*.*

Let $2 \in \Omega$*, then* $\mathcal{F}^c_n(1,2,5) = \frac{\min\{1,2\}+5}{\max\{1,2\}+5} = \frac{1+5}{2+5} = 0.8571$*, so* $2 \in B_1(1, 0.4, 5)$*.*

Let $3 \in \Omega$*, then* $\mathcal{F}^c_n(1,3,5) = \frac{\min\{1,3\}+5}{\max\{1,3\}+5} = \frac{1+5}{3+5} = 0.75$*, so* $3 \in B_1(1, 0.4, 5)$*.*

Let $4 \in \Omega$*, then* $\mathcal{F}^c_n(1,4,5) = \frac{\min\{1,4\}+5}{\max\{1,4\}+5} = \frac{1+5}{4+5} = 0.6666$*, so* $4 \in B_1(1, 0.4, 5)$*.*

Let $5 \in \Omega$*, then* $\mathcal{F}^c_n(1,5,5) = \frac{\min\{1,5\}+5}{\max\{1,5\}+5} = \frac{1+5}{5+5} = 0.6$*, so* $5 \notin B_1(1, 0.4, 5)$*.*

Let $6 \in \Omega$, *then* $F_n^c(1,6,5) = \dfrac{\min\{1,6\}+5}{\max\{1,6\}+5} = \dfrac{1+5}{6+5} = 0.5454$, *so* $6 \notin B_1(1,0.4,5)$.
Let $7 \in \Omega$, *then* $F_n^c(1,7,5) = \dfrac{\min\{1,7\}+5}{\max\{1,7\}+5} = \dfrac{1+5}{7+5} = 0.5$, *so* $7 \notin B_1(1,0.4,5)$.
Thus $B_1(1,0.4,5) = \{1,2,3,4\}$. *Now define* $B_2(2,0.2,5)$, *the open ball with center* $\mathfrak{s}_2 = 2$, *radius* $\mathfrak{r}_2 = 0.2$ *and* $\mathfrak{t}_2 = 5$. *Then*

$$B_2(2,0.2,5) = \{\mathfrak{s} \in \Omega : F_n^c(1,\mathfrak{s},5) > 0.8\}.$$

Let $1 \in \Omega$, *then* $F_n^c(2,1,5) = \dfrac{\min\{2,1\}+5}{\max\{2,1\}+5} = \dfrac{1+5}{2+5} = 0.8571$, *so* $1 \in B_2(2,0.2,5)$.
Let $2 \in \Omega$, *then* $F_n^c(2,2,5) = \dfrac{\min\{2,2\}+5}{\max\{2,2\}+5} = \dfrac{2+5}{2+5} = 1$, *so* $2 \in B_2(2,0.2,5)$.
Let $3 \in \Omega$, *then* $F_n^c(2,3,5) = \dfrac{\min\{2,3\}+5}{\max\{2,3\}+5} = \dfrac{2+5}{3+5} = 0.875$, *so* $3 \in B_2(2,0.2,5)$.
Let $4 \in \Omega$, *then* $F_n^c(2,4,5) = \dfrac{\min\{2,4\}+5}{\max\{2,4\}+5} = \dfrac{2+5}{4+5} = 0.7777$, *so* $4 \notin B_2(2,0.2,5)$.
Let $5 \in \Omega$, *then* $F_n^c(2,5,5) = \dfrac{\min\{2,5\}+5}{\max\{2,5\}+5} = \dfrac{2+5}{5+5} = 0.7$, *so* $5 \notin B_2(2,0.2,5)$.
Let $6 \in \Omega$, *then* $F_n^c(2,6,5) = \dfrac{\min\{2,6\}+5}{\max\{2,6\}+5} = \dfrac{2+5}{6+5} = 0.6363$, *so* $6 \notin B_2(2,0.2,5)$.
Let $7 \in \Omega$, *then* $F_n^c(2,7,5) = \dfrac{\min\{2,7\}+5}{\max\{2,7\}+5} = \dfrac{2+5}{7+5} = 0.5833$, *so* $7 \notin B_2(2,0.2,5)$.
Thus $B_2(2,0.2,5) = \{1,2,3\}$. *Clearly* $B_1(1,0.4,5) \cap B_2(2,0.2,5) \neq \emptyset$. *Hence a* (FnCMS) *need not to be Hausdorff.*

Remark 3.2. *In the light of remark (3.1), a pentagonal, hexagonal, triple controlled, double controlled, b−extended and controlled rectangular, b−rectangular metric space and some other fuzzy metric spaces are also not Hausdorff.*

Denote Φ, the family of all functions $\phi : [0,\infty) \to [0,\infty)$ which are nondecreasing and having properties:

(i) $\phi(\mathfrak{t}) < \mathfrak{t}$,
(ii) $\lim\limits_{n\to\infty} \phi^n(\mathfrak{t}) = 0$,

for all $\mathfrak{t} > 0$, where ϕ^n denotes the n−th iteration of ϕ.

Definition 3.4. *Let* $(\Omega, F_n^c, \alpha_n, *)$ *be a* (FnCMS), $\alpha : \Omega \times \Omega \times (0,\infty) \longrightarrow [0,\infty)$ *and* $T : \Omega \longrightarrow \Omega$ *be two mappings. Then* T *is called an* α*−admissible, if for all* $t > 0$, $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega$,

$$\alpha(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) \geq 1 \Longrightarrow \alpha(T\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t}) \geq 1.$$

Example 3.3. *Let* $\Omega = [0,\infty)$, *define* $\alpha : \Omega \times \Omega \times [0,\infty)$ *by*

$$\alpha(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = \begin{cases} e^{\frac{t\mathfrak{s}_2}{\mathfrak{s}_1}}, \ if \ \mathfrak{s}_1 \geq \mathfrak{s}_2, \mathfrak{s}_1 \neq 0 \\ 0, if \ \mathfrak{s}_1 < \mathfrak{s}_2 \end{cases}$$

and $T\mathfrak{s} = 5\mathfrak{s}$. *Then clearly* T *is* α*−admissible.*

Definition 3.5. *Let* $(\Omega, F_n^c, \alpha_n, *)$ *be a* (FnCMS)*. Then the mapping* $T : \Omega \longrightarrow \Omega$ *is called a generalized* $\alpha - \phi-$*fuzzy contractive mapping if for two functions* $\phi \in \Phi$ *and* $\alpha : \Omega \times \Omega \times (0, \infty) \longrightarrow [0, \infty)$*, we have*

$$\alpha(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})\left(\frac{1}{F(T\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} - 1\right) \leq \phi\left(\frac{1}{F^*(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})} - 1\right), \tag{3.1}$$

for all $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega$ *and* $\mathfrak{t} > 0$*, where*

$$F^*(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = \min\{F_n^c(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}), F_n^c(\mathfrak{s}_1, T\mathfrak{s}_1, \mathfrak{t}), F_n^c(\mathfrak{s}_2, T\mathfrak{s}_2, \mathfrak{t}), \frac{2F_n^c(\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})F_n^c(\mathfrak{s}_2, T\mathfrak{s}_1, \mathfrak{t})}{F_n^c(\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t}) + F_n^c(\mathfrak{s}_2, T\mathfrak{s}_1, \mathfrak{t})}\}.$$

We now prove Banach fixed point theorem by using generalized $\alpha - \phi-$fuzzy contraction.

Theorem 3.1. *Let* $(\Omega, F_n^c, \alpha_n, *)$ *be a complete* (FnCMS) *with* $\alpha_n : \Omega \times \Omega \to [1, \infty)$ *be n non-comparable functions and* $T : \Omega \longrightarrow \Omega$ *be a generalized* $\alpha - \phi-$*fuzzy contractive mapping satisfying:*
(i) T *is* $\alpha-$*admissible,*
(ii) for all $t > 0$*, there exists* $\mathfrak{s}_0 \in \Omega$ *satisfying* $\alpha(\mathfrak{s}_0, T\mathfrak{s}_0, t) \geq 1$*,*
(iii) T *is continuous,*
then, T *has a fixed point.*

Proof. Let $\mathfrak{s}_0 \in \Omega_0$ be arbitrary point and consider for all $n \in \mathbb{N}$, the sequence $\{\mathfrak{s}_n\}$ in Ω by the formula $\mathfrak{s}_n = T\mathfrak{s}_{n-1}$. Assume for all $n \in \mathbb{N}$, $\mathfrak{s}_n \neq \mathfrak{s}_{n-1}$. Since T is $\alpha-$admissible and $\alpha(\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{t}) = \alpha(\mathfrak{s}_0, T\mathfrak{s}_0, \mathfrak{t}) \geq 1$, so for any $\mathfrak{t} > 0$, we have $\alpha(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = \alpha(T\mathfrak{s}_0, T\mathfrak{s}_1, \mathfrak{t}) \geq 1$. Ultimately, $\alpha(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}) \geq 1$. Now using (3.1),

$$\begin{aligned} \frac{1}{F_n^c(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})} - 1 &= \frac{1}{F_n^c(T\mathfrak{s}_{n-1}, T\mathfrak{s}_n, \mathfrak{t})} - 1 \\ &\leq \alpha(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t})\left(\frac{1}{F_n^c(T\mathfrak{s}_{n-1}, T\mathfrak{s}_n, \mathfrak{t})} - 1\right. \\ &\leq \phi\left(\frac{1}{F^*(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t})} - 1\right), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} F^*(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}) = \min\Big\{&F_n^c(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}), F_n^c(\mathfrak{s}_{n-1}, T\mathfrak{s}_{n-1}, \mathfrak{t}), \\ &F_n^c(\mathfrak{s}_n, T\mathfrak{s}_n, \mathfrak{t}), \frac{2F_n^c(\mathfrak{s}_{n-1}, T\mathfrak{s}_n, \mathfrak{t})F_n^c(T\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t})}{F_n^c(T\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}) + F_n^c(\mathfrak{s}_{n-1}, T\mathfrak{s}_n, \mathfrak{t})}\Big\}. \end{aligned}$$

On simplifying, we have

$$F^*(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}) = \min\{F_n^c(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}), F_n^c(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t}), \frac{2F_n^c(\mathfrak{s}_{n-1}, \mathfrak{s}_{n+1}, \mathfrak{t})}{1 + F_n^c(\mathfrak{s}_{n-1}, \mathfrak{s}_{n+1}, \mathfrak{t})}\}. \tag{3.3}$$

Now consider,

$$\begin{aligned} \frac{2F_n^c(\mathfrak{s}_{n-1}, \mathfrak{s}_{n+1}, \mathfrak{t})}{1 + F_n^c(\mathfrak{s}_{n-1}, \mathfrak{s}_{n+1}, \mathfrak{t})} &= \frac{2}{\frac{1}{F_n^c(\mathfrak{s}_{n-1}, \mathfrak{s}_{n+1}, \mathfrak{t})} + 1} \\ &\geq \frac{2}{\frac{1}{F_n^c(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t})} + \frac{1}{F_n^c(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})}} \\ &\geq min\{F_n^c(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}), F_n^c(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})\}. \end{aligned} \tag{3.4}$$

Using in (3.3), we have $F^*(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}) = min\{F^c_n(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}), F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})\}$. Substituting this value in (3.2), we get

$$\frac{1}{F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})} - 1 \leq \phi\Big(\frac{1}{min\{F^c_n(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}), F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})\}} - 1\Big). \tag{3.5}$$

Now, if $min\{F^c_n(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t}), F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})\} = F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})$, then as $\phi(\mathfrak{t}) < \mathfrak{t}$, we have

$$\frac{1}{F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})} - 1 \leq \phi\Big(\frac{1}{F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})} - 1\Big) < \frac{1}{F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})} - 1,$$

which is a contradiction. So, $min\{F^c_n(\mathfrak{s}_{n-1}, \mathfrak{s}_n, t), F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})\} = F^c_n(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t})$ and

$$\frac{1}{F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})} - 1 \leq \phi\Big(\frac{1}{F^c_n(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t})} - 1\Big) < \frac{1}{F^c_n(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t})} - 1. \tag{3.6}$$

Hence $F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t}) > F^c_n(\mathfrak{s}_{n-1}, \mathfrak{s}_n, \mathfrak{t})$. So, the sequence $\{F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})\}$ is strictly increasing in $[0, 1]$, for all $\mathfrak{t} > 0$. Let for all $t > 0$, $S^*(\mathfrak{t}) = \lim_{n\longrightarrow\infty} F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t})$. We claim that $\mathcal{S}^*(\mathfrak{t}) = 1$. On contrary, assume $\mathcal{S}^*(\mathfrak{t}_o) < 1$, for some $\mathfrak{t}_o > 0$. Taking limit on both sides of (3.6), we have

$$\frac{1}{\mathcal{S}^*(\mathfrak{t}_o)} - 1 \leq \phi\Big(\frac{1}{\mathcal{S}^*(\mathfrak{t}_o)} - 1\Big) < \frac{1}{\mathcal{S}^*(\mathfrak{t}_o)} - 1,$$

a contradiction. Thus, we have

$$\lim_{n\longrightarrow\infty} F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t}) = 1, \mathfrak{t} > 0. \tag{3.7}$$

To prove Cauchyness of $\{\mathfrak{s}_n\}$, consider the cases as:

Case-1. When $p = 2\mathfrak{q} + 1$(odd), then by writing $\mathfrak{t} = \frac{(2\mathfrak{q}+1)\mathfrak{t}}{2\mathfrak{q}+1} = \frac{\mathfrak{t}}{2\mathfrak{q}+1} + \frac{\mathfrak{t}}{2\mathfrak{q}+1} + \ldots + \frac{\mathfrak{t}}{2\mathfrak{q}+1}$, we have

$$\begin{aligned} F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+2\mathfrak{q}+1}, \mathfrak{t}) &\geq F^c_n\Big(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_1(\mathfrak{s}_n, \mathfrak{s}_{n+1})}\Big) * F^c_n\Big(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})}\Big) \\ &* F^c_n\Big(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})}\Big) * \ldots * F^c_n\Big(\mathfrak{s}_{n+2\mathfrak{q}}, \mathfrak{s}_{n+2\mathfrak{q}+1}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_n(\mathfrak{s}_{n+2\mathfrak{q}}, \mathfrak{s}_{n+2\mathfrak{q}+1})}\Big). \end{aligned}$$

Using (3.7) and applying limit $n \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+2\mathfrak{q}+1}, \mathfrak{t}) \geq 1 * 1 \ldots * 1 = 1.$$

Case-2. When $p = 2\mathfrak{q}$(even), then by writing $\mathfrak{t} = \frac{(2\mathfrak{q})\mathfrak{t}}{2\mathfrak{q}} = \frac{\mathfrak{t}}{2\mathfrak{q}} + \frac{\mathfrak{t}}{2\mathfrak{q}} + \ldots + \frac{\mathfrak{t}}{2\mathfrak{q}}$, we have

$$\begin{aligned} F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+2\mathfrak{q}}, \mathfrak{t}) &\geq F^c_n\Big(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_1(\mathfrak{s}_n, \mathfrak{s}_{n+1})}\Big) * F^c_n\Big(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})}\Big) \\ &* F^c_n\Big(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})}\Big) * \ldots * F^c_n\Big(\mathfrak{s}_{n+2\mathfrak{q}-1}, \mathfrak{s}_{n+2\mathfrak{q}}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_n(\mathfrak{s}_{n+2\mathfrak{q}-1}, \mathfrak{s}_{n+2\mathfrak{q}})}\Big). \end{aligned}$$

Using (3.7) and applying limit $n \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+2\mathfrak{q}}, \mathfrak{t}) \geq 1 * 1 * 1 \ldots * 1 = 1.$$

Hence in either case, $\lim_{n\to\infty} F_n^c(\mathfrak{s}_n, \mathfrak{s}_{n+p}, \mathfrak{t}) = 1$, showing Cauchyness of $\{\mathfrak{s}_n\}$ and converges to $\mathfrak{s} \in \Omega$, so

$$\lim_{n\to\infty} F_n^c(\mathfrak{s}_n, \mathfrak{s}, \mathfrak{t}) = 1.$$

Now as T is continuous, we get $T\mathfrak{s}_n \longrightarrow T\mathfrak{s}$, for all $\mathfrak{t} > 0$. Now, we have $\lim_{n\longrightarrow\infty} F_n^c(\mathfrak{s}_{n+1}, T\mathfrak{s}, \mathfrak{t}) = \lim_{n\longrightarrow\infty} F_n^c(T\mathfrak{s}_n, T\mathfrak{s}, \mathfrak{t}) = 1$, for all $\mathfrak{t} > 0$, that is $\mathfrak{s}_n \longrightarrow T\mathfrak{s}$. The uniqueness of the limit implies that $T\mathfrak{s} = \mathfrak{s}$. i-e $\mathfrak{s}$ is the fixed point of T. □

Following example elaborates Theorem (3.1).

Example 3.4. *Let*

$$\Omega_1 = \{\frac{\mathfrak{p}}{\mathfrak{q}} : \mathfrak{p} = 0, 1, 3, 9, ..., \mathfrak{q} = 1, 4, ..., 3k + 1, ...\},$$

$$\Omega_2 = \{\frac{\mathfrak{p}}{\mathfrak{q}} : \mathfrak{p} = 1, 3, 9, ..., \mathfrak{q} = 2, 5, ..., 3k + 2, ...\},$$

$$\Omega_3 = \{2k : k \in \mathbb{N}\},$$

and $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. *Let* $\mathfrak{t}_1 * \mathfrak{t}_1 = \mathfrak{t}_1\mathfrak{t}_2$ *for all* $\mathfrak{t}_1, \mathfrak{t}_2 \in [0, 1]$ *and* $F_n^c(\mathfrak{s}_1, \mathfrak{s}_2, t) = \frac{t}{t+|\mathfrak{s}_1-\mathfrak{s}_2|^n}$ *for all* $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega$ *with* $\alpha_n : \Omega \times \Omega \to [1, \infty)$ *and* $t > 0$. *Define* $T : \Omega \longrightarrow \Omega$ *by*

$$T\mathfrak{s} = \begin{cases} \frac{3\mathfrak{s}}{11}, & \mathfrak{s} \in \Omega_1, \\ \frac{\mathfrak{s}}{8}, & \mathfrak{s} \in \Omega_2, \\ 2\mathfrak{s}, & \mathfrak{s} \in \Omega_3, \end{cases}$$

and $\alpha : \Omega^2 \times (0, \infty) \to [0, \infty)$ *by*

$$\alpha(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = \begin{cases} 1, & \mathfrak{s}_1, \mathfrak{s}_2 \in \Omega_1 \cup \Omega_2, \\ 0, & otherwise. \end{cases}$$

If $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega_1$, *then*

$$(\frac{1}{F_n^c(T\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} - 1) = \frac{|\frac{3\mathfrak{s}_1}{11} - \frac{3\mathfrak{s}_2}{11}|^n}{\mathfrak{t}} = (\frac{3}{11})^n \frac{|\mathfrak{s}_1 - \mathfrak{s}_2|^n}{\mathfrak{t}} = (\frac{3}{11})^n (\frac{1}{F_n^c(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})} - 1)$$
$$\leq (\frac{6}{11})^n (\frac{1}{F^*(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})} - 1).$$

If $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega_2$, *then*

$$(\frac{1}{F_n^c(T\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} - 1) = \frac{|\frac{\mathfrak{s}_1}{8} - \frac{\mathfrak{s}_2}{8}|^n}{\mathfrak{t}} = (\frac{1}{8})^n \frac{|\mathfrak{s}_1 - \mathfrak{s}_2|^n}{\mathfrak{t}} = (\frac{1}{8})^n (\frac{1}{F_n^c(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})} - 1)$$
$$\leq (\frac{6}{11})^n (\frac{1}{F^*(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})} - 1).$$

If $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega_3$, *then* $\alpha(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = 0$, *and (3.1) trivially holds.*
Now, if $\mathfrak{s}_1 \in \Omega_1$ *and* $\mathfrak{s}_2 \in \Omega_2$, *then*

$$(\frac{1}{F_n^c(T\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} - 1) = \frac{|\frac{3\mathfrak{s}_1}{11} - \frac{\mathfrak{s}_2}{8}|^n}{\mathfrak{t}} = (\frac{3}{11})^n \frac{|\mathfrak{s}_1 - \frac{11}{24}\mathfrak{s}_2|^n}{\mathfrak{t}}.$$

So, if $\mathfrak{s}_1 > \frac{11}{24}\mathfrak{s}_2$, *then*

$$\begin{aligned}(\frac{1}{F_n^c(T\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} - 1) &= (\frac{3}{11})^n \frac{|\mathfrak{s}_1 - \frac{11}{24}\mathfrak{s}_2|^n}{\mathfrak{t}} \leq (\frac{3}{11})^n \frac{|\mathfrak{s}_1 - \frac{1}{8}\mathfrak{s}_2|^n}{\mathfrak{t}} \\ &\leq (\frac{6}{11})^n [\frac{1}{2}(\frac{1}{F_n^c(\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} - 1)] \\ &= (\frac{6}{11})^n [\frac{1}{2}(\frac{1}{F_n^c(\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} + 1) - 1] \\ &\leq (\frac{6}{11})^n [\frac{1}{2}(\frac{1}{F_n^c(\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} + \frac{1}{F_n^c(\mathfrak{s}_2, T\mathfrak{s}_1, \mathfrak{t})}) - 1] \\ &= (\frac{6}{11})^n \Big[\frac{1}{\frac{2F_n^c(\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})F_n^c(\mathfrak{s}_2, T\mathfrak{s}_1, \mathfrak{t})}{F_n^c(\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t}) + F_n^c(\mathfrak{s}_2, T\mathfrak{s}_1, \mathfrak{t})}} - 1\Big] \\ &\leq (\frac{6}{11})^n (\frac{1}{F^*(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})} - 1)\end{aligned}$$

and if $\mathfrak{s}_1 < \frac{11}{24}\mathfrak{s}_2$, *then*

$$\begin{aligned}(\frac{1}{F_n^c(T\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} - 1) &= (\frac{3}{11})^n \frac{|\frac{11}{24}\mathfrak{s}_2 - \mathfrak{s}_1|^n}{\mathfrak{t}} \leq (\frac{3}{11})^n \frac{|\mathfrak{s}_2 - \mathfrak{s}_1|^n}{\mathfrak{t}} \\ &= (\frac{3}{11})^n (\frac{1}{F_n^c(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})} - 1) \leq (\frac{6}{11})^n (\frac{1}{F^*(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})} - 1).\end{aligned}$$

We see that $(\frac{1}{F_n^c(T\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} - 1) \leq (\frac{6}{11})^n (\frac{1}{F^*(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})} - 1)$ *for all* $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega_1 \cup \Omega_2$. *So, by definition of* α, *we get* $\alpha(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})(\frac{1}{F_n^c(T\mathfrak{s}_1, T\mathfrak{s}_2, \mathfrak{t})} - 1) \leq (\frac{6}{11})^n (\frac{1}{F^*(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t})} - 1)$ *for all* $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega$. *Thus, T is a generalized* α-ϕ-*fuzzy contractive mapping with* $\phi(\mathfrak{t}) = (\frac{6}{11})^n \mathfrak{t}$. *Also, for* $\mathfrak{s}_o = 1$, *we have* $\alpha(\mathfrak{s}_o, T\mathfrak{s}_o, \mathfrak{t}) = \alpha(1, \frac{3}{11}, \mathfrak{t}) = 1$. *It is easy to check that T is* $\alpha-$*admissible and the condition (iii) in Theorem (3.1) holds. So, by Theorem (3.1), T has a fixed point, i-e* $\mathfrak{s} = 0$.

Theorem 3.2. *Let* $\alpha_n : \Omega \times \Omega \to [1, \infty)$ *and* $(\Omega, F_n^c, \alpha_n, *)$ *be a complete* (FnCMS) *with*

$$\lim_{\mathfrak{t} \to \infty} F_n^c(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = 1. \tag{3.8}$$

Also let T be a self-mapping on Ω *satisfying:*

$$F_n^c(T\mathfrak{s}_1, T\mathfrak{s}_2, K\mathfrak{t}) \geq F_n^c(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}), \tag{3.9}$$

for all $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega$. *Then the mapping T has a unique fixed point in* Ω.

Proof. Let $\mathfrak{s}_0 \in \Omega$ and the sequence $T\mathfrak{s}_n = T^{n+1}\mathfrak{s}_0 = \mathfrak{s}_{n+1}$. After routine steps, we have

$$F_n^c(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{t}) \geq F_n^c(\mathfrak{s}_0, \mathfrak{s}_1, \frac{\mathfrak{t}}{K^n}). \tag{3.10}$$

Consider the sequence $\{\mathfrak{s}_n\}$ in Ω, then

Case-1. When $p = 2\mathfrak{q}+1$(odd), then by writing $\mathfrak{t} = \frac{(2\mathfrak{q}+1)\mathfrak{t}}{2\mathfrak{q}+1} = \frac{\mathfrak{t}}{2\mathfrak{q}+1} + \frac{\mathfrak{t}}{2\mathfrak{q}+1} + \ldots + \frac{\mathfrak{t}}{2\mathfrak{q}+1}$, we have

$$\begin{aligned} &F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+2\mathfrak{q}+1}, \mathfrak{t}) \\ &\geq F^c_n\Big(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_1(\mathfrak{s}_n, \mathfrak{s}_{n+1})}\Big) * F^c_n\Big(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})}\Big) \\ &* F^c_n\Big(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})}\Big) * \ldots * F^c_n\Big(\mathfrak{s}_{n+2\mathfrak{q}}, \mathfrak{s}_{n+2\mathfrak{q}+1}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_n(\mathfrak{s}_{n+2\mathfrak{q}}, \mathfrak{s}_{n+2\mathfrak{q}+1})}\Big), \end{aligned}$$

using (3.10), we have

$$\begin{aligned} &F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+2\mathfrak{q}+1}, \mathfrak{t}) \\ &\geq F^c_n\Big(\mathfrak{s}_0, \mathfrak{s}_1, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_1(\mathfrak{s}_n, \mathfrak{s}_{n+1})K^n}\Big) * F^c_n\Big(\mathfrak{s}_0, \mathfrak{s}_1, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})K^{n+1}}\Big) \\ &* F^c_n\Big(\mathfrak{s}_0, \mathfrak{s}_1, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})K^{n+2}}\Big) * \ldots * F^c_n\Big(\mathfrak{s}_{n+2\mathfrak{q}}, \mathfrak{s}_{n+2\mathfrak{q}+1}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}+1}}{\alpha_n(\mathfrak{s}_{n+2\mathfrak{q}}, \mathfrak{s}_{n+2\mathfrak{q}+1})K^{n+2\mathfrak{q}}}\Big), \end{aligned}$$

applying limit $n \longrightarrow \infty$, we have

$$\lim_{n \longrightarrow \infty} F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+2\mathfrak{q}+1}, \mathfrak{t}) \geq 1 * 1 * 1 \ldots * 1 = 1.$$

Case-2. When $p = 2\mathfrak{q}$(even), then by writing $\mathfrak{t} = \frac{(2\mathfrak{q})\mathfrak{t}}{2\mathfrak{q}} = \frac{\mathfrak{t}}{2\mathfrak{q}} + \frac{\mathfrak{t}}{2\mathfrak{q}} + \ldots + \frac{\mathfrak{t}}{2\mathfrak{q}}$, we have

$$\begin{aligned} &F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+2\mathfrak{q}}, \mathfrak{t}) \\ &\geq F^c_n\Big(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_1(\mathfrak{s}_n, \mathfrak{s}_{n+1})}\Big) * F^c_n\Big(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})}\Big) \\ &* F^c_n\Big(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})}\Big) * \ldots * F^c_n\Big(\mathfrak{s}_{n+2\mathfrak{q}-1}, \mathfrak{s}_{n+2\mathfrak{q}}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_n(\mathfrak{s}_{n+2\mathfrak{q}-1}, \mathfrak{s}_{n+2\mathfrak{q}})}\Big), \end{aligned}$$

using (3.10), we have

$$\begin{aligned} &F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+2\mathfrak{q}}, \mathfrak{t}) \\ &\geq F^c_n\Big(\mathfrak{s}_0, \mathfrak{s}_1, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_1(\mathfrak{s}_n, \mathfrak{s}_{n+1})K^n}\Big) * F^c_n\Big(\mathfrak{s}_0, \mathfrak{s}_1, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})K^{n+1}}\Big) \\ &* F^c_n\Big(\mathfrak{s}_0, \mathfrak{s}_1, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})K^{n+2}}\Big) * \ldots * F^c_n\Big(\mathfrak{s}_{n+2\mathfrak{q}-1}, \mathfrak{s}_{n+2\mathfrak{q}}, \frac{\frac{\mathfrak{t}}{2\mathfrak{q}}}{\alpha_n(\mathfrak{s}_{n+2\mathfrak{q}-1}, \mathfrak{s}_{n+2\mathfrak{q}})K^{n+2\mathfrak{q}-1}}\Big), \end{aligned}$$

applying limit $n \longrightarrow \infty$, we have

$$\lim_{n \longrightarrow \infty} F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+2\mathfrak{q}+1}, \mathfrak{t}) \geq 1 * 1 * 1 \ldots * 1 = 1.$$

Thus in both cases, we have

$$\lim_{n \to \infty} F^c_n(\mathfrak{s}_n, \mathfrak{s}_{n+p}, \mathfrak{t}) = 1,$$

showing $\{\mathfrak{s}_n\}$ is Cauchy in Ω and converges in Ω, so

$$\lim_{n\to\infty} F_n^c(\mathfrak{s}_n, \mathfrak{s}, \mathfrak{t}) = 1.$$

Next to show that $\mathfrak{s}$ is the fixed point of T. Here again arises two cases:

Case-1. When $n = 2\mathfrak{q} + 1$ is odd, then by writing $\mathfrak{t} = \frac{(2\mathfrak{q}+1)\mathfrak{t}}{2\mathfrak{q}+1} = \frac{\mathfrak{t}}{2\mathfrak{q}+1} + \cdots \frac{\mathfrak{t}}{2\mathfrak{q}+1} + \frac{\mathfrak{t}}{2\mathfrak{q}+1}$, we have

$$\begin{aligned}
F_n^c(\mathfrak{s}, T\mathfrak{s}, \mathfrak{t}) &\geq F_n^c\Big(\mathfrak{s}, \mathfrak{s}_n, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_1(\mathfrak{s}, \mathfrak{s}_n)}\Big) * F_n^c\Big(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2}, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})}\Big) \\
&* F_n^c\Big(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3}, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})}\Big) * \cdots \\
&* F_n^c\Big(\mathfrak{s}_{2\mathfrak{q}+1}, T\mathfrak{s}, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_n(\mathfrak{s}_{2\mathfrak{q}+1}, T\mathfrak{s})}\Big) \\
&\geq F_n^c\Big(\mathfrak{s}, \mathfrak{s}_n, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_1(\mathfrak{s}, \mathfrak{s}_n)}\Big) * F_n^c\Big(T\mathfrak{s}_n, T\mathfrak{s}_{n+1}, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})}\Big) \\
&* F_n^c\Big(T\mathfrak{s}_{n+1}, T\mathfrak{s}_{n+2}, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})}\Big) * \cdots \\
&* F_n^c\Big(T\mathfrak{s}_{2\mathfrak{q}}, T\mathfrak{s}, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_n(\mathfrak{s}_{2\mathfrak{q}+1}, T\mathfrak{s})}\Big) \\
&\geq F_n^c\Big(\mathfrak{s}, \mathfrak{s}_n, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_1(\mathfrak{s}, \mathfrak{s}_n)}\Big) * F_n^c\Big(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})K}\Big) \\
&* F_n^c\Big(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2}, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})K}\Big) * \cdots \\
&* F_n^c\Big(\mathfrak{s}_{2\mathfrak{q}}, \mathfrak{s}, \frac{\mathfrak{t}}{(2\mathfrak{q}+1)\alpha_n(\mathfrak{s}_{2\mathfrak{q}+1}, T\mathfrak{s})K}\Big) \\
&\longrightarrow 1 * 1 * 1 = 1,
\end{aligned}$$

as $n \to \infty$.

Case-2. When $n = 2\mathfrak{q}$ is even, then by writing $\mathfrak{t} = \frac{(2\mathfrak{q})\mathfrak{t}}{2\mathfrak{q}} = \frac{\mathfrak{t}}{2\mathfrak{q}} + \cdots \frac{\mathfrak{t}}{2\mathfrak{q}} + \frac{\mathfrak{t}}{2\mathfrak{q}}$, we have

$$\begin{aligned}
F_n^c(\mathfrak{s}, T\mathfrak{s}, \mathfrak{t}) &\geq F_n^c\Big(\mathfrak{s}, \mathfrak{s}_n, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_1(\mathfrak{s}, \mathfrak{s}_n)}\Big) * F_n^c\Big(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2}, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})}\Big) \\
&* F_n^c\Big(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3}, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})}\Big) * \cdots \\
&* F_n^c\Big(\mathfrak{s}_{2\mathfrak{q}}, T\mathfrak{s}, \frac{\mathfrak{t}}{(2s)\alpha_n(\mathfrak{s}_{2\mathfrak{q}}, T\mathfrak{s})}\Big) \\
&\geq F_n^c\Big(\mathfrak{s}, \mathfrak{s}_n, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_1(\mathfrak{s}, \mathfrak{s}_n)}\Big) * F_n^c\Big(T\mathfrak{s}_n, T\mathfrak{s}_{n+1}, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})}\Big) \\
&* F_n^c\Big(T\mathfrak{s}_{n+1}, T\mathfrak{s}_{n+2}, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})}\Big) * \cdots \\
&* F_n^c\Big(T\mathfrak{s}_{2\mathfrak{q}-1}, T\mathfrak{s}, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_n(\mathfrak{s}_{2\mathfrak{q}}, T\mathfrak{s})}\Big) \\
&\geq F_n^c\Big(\mathfrak{s}, \mathfrak{s}_n, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_1(\mathfrak{s}, \mathfrak{s}_n)}\Big) * F_n^c\Big(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_2(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2})K}\Big)
\end{aligned}$$

$$
\begin{aligned}
&* F^c_n\Big(\mathfrak{s}_{n+1}, \mathfrak{s}_{n+2}, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_3(\mathfrak{s}_{n+2}, \mathfrak{s}_{n+3})K}\Big) * \cdots \\
&* F^c_n\Big(\mathfrak{s}_{2\mathfrak{q}-1}, \mathfrak{s}, \frac{\mathfrak{t}}{(2\mathfrak{q})\alpha_n(\mathfrak{s}_{2\mathfrak{q}}, T\mathfrak{s})K}\Big) \\
&\longrightarrow 1 * 1 * 1 = 1,
\end{aligned}
$$

as $n \to \infty$, hence in either case, $\mathfrak{s}$ is the fixed point of T.

Uniqueness: Assume $T\mathfrak{s}' = \mathfrak{s}'$ for any other $\mathfrak{s}' \in \Omega$, then

$$
F^c_n(\mathfrak{s}, \mathfrak{s}', \mathfrak{t}) = F^c_n(T\mathfrak{s}, T\mathfrak{s}', \mathfrak{t}) \geq F^c_n(\mathfrak{s}, \mathfrak{s}', \frac{\mathfrak{t}}{K}),
$$

which shows the uniqueness of $\mathfrak{s}$. □

Example 3.5. *Let* $\Omega = [0, 1]$ *and* $\alpha_i : \Omega \times \Omega \longrightarrow [1, \frac{1}{K})(1 \leq i \leq n)$*. Define a* (FnCMS) $(\Omega, F^c_n, \alpha_n, *)$ *as*

$$
F^c_n(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}) = \exp^{-\dfrac{(\mathfrak{s}_1 - \mathfrak{s}_2)^n}{\mathfrak{t}}},
$$

with product $t-$*norm. Further let* $T : \Omega \longrightarrow \Omega$ *be defined as* $T\mathfrak{s} = 1 - \frac{\mathfrak{s}}{3}$*.*
Now

$$
\begin{aligned}
F^c_n(T\mathfrak{s}_1, T\mathfrak{s}_2, K\mathfrak{t}) &= \exp^{-\dfrac{(T\mathfrak{s}_1 - T\mathfrak{s}_2)^n}{K\mathfrak{t}}}, \\
&= \exp^{-\dfrac{(1 - \frac{\mathfrak{s}_1}{3} - 1 + \frac{\mathfrak{s}_2}{3})^n}{K\mathfrak{t}}}, \\
&= \exp^{-\dfrac{(\mathfrak{s}_1 - \mathfrak{s}_2)^n}{3^n K\mathfrak{t}}}, \\
&\geq \exp^{-\dfrac{(\mathfrak{s}_1 - \mathfrak{s}_n)^n}{\mathfrak{t}}}, \\
&= F^c_n(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}).
\end{aligned}
$$

By Theorem (3.2), T *has a unique fixed point, here* $\mathfrak{s} = \frac{3}{4}$*.*

4. Application to Fractional Differential Equations

Fractional calculus has brought many significant improvements in scientific research. It deals with the variable derivative that gives more accuracy and helps to make models of mathematical problems. Whereas an ordinary derivative was not so good in this regard because it deals with integer order derivatives. The main idea of fractional derivatives and integrals is usually associated with Liouville. However, mathematicians had already studied derivatives containing fractional order. Fractional calculus was the subject of Leibnitz's study. Later, Euler also made a contribution to it. Liouville, Reimann, Abel, Litnikov, Hadamard, Weyl, and many other mathematicians from past and present have made significant improvements in the study of fractional calculus and now it is a symbolic topic in mathematics. This section is devoted to prove the uniqueness of the solution of the following fractional differential equation consisting of Caputo fractional derivative

$$D^{\delta}_{0+}\nu(\xi) + g(\xi, \nu(\xi)) = 0,\ 0 < \xi < 1, \tag{4.1}$$

where, $1 < \delta \leq 2,\ \nu(0) + \nu'(0) = 0,\ \nu(1) + \nu'(1) = 0$ are the boundary conditions with $g : [0, 1] \times [0, \infty) \longrightarrow [0, \infty)$ being continuous. Define a complete *(FnCMS)* $(\Omega, F^{c}_{n}, \alpha_n, *)$ on $\Omega = C([0, 1], \mathbb{R})$ as

$$F^{c}_{n}(\nu, \mu, t) = \exp -\frac{\sup_{\xi\in[0,1]} |\nu(\xi) - \mu(\xi)|^n}{t}$$

for all $\nu, \mu \in \Omega,\ t > 0$, where $t_1 * t_2 = t_1 t_2$. Note that $\nu \in \Omega$ solves (4.1) whenever $\nu \in \Omega$ is the solution of

$$\begin{aligned}\nu(\xi) = &\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\xi)g(\zeta, \nu(\zeta))d\zeta + \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\xi)g(\zeta, \nu(\zeta))d\zeta\\ &+ \frac{1}{\Gamma(\delta)}\int_0^{\xi} (\xi-\zeta)^{\delta-1}g(\zeta, \nu(\zeta))d\zeta\end{aligned}$$

Theorem 4.1. *Consider the operator $H : \Omega \longrightarrow \Omega$ as:*

$$\begin{aligned}H\nu(\xi) = &\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\xi)g(\zeta, \nu(\zeta))d\zeta + \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\xi)g(\zeta, \nu(\zeta))d\zeta\\ &+ \frac{1}{\Gamma(\delta)}\int_0^{\xi} (\xi-\zeta)^{\delta-1}g(\zeta, \nu(\zeta))d\zeta.\end{aligned}$$

suppose the conditions:

(i) *for all $\nu, \mu \in \Omega,\ g : [0, 1] \times [0, \infty) \longrightarrow [0, \infty)$, satisfies*

$$|g(\zeta, \nu(\zeta)) - g(\zeta, \mu(\zeta))| \leq K^{\frac{1}{n}}|\nu(\zeta) - \mu(\zeta)|,$$

(ii)

$$\sup_{\xi\in(0,1)}\left|\frac{1-\xi}{\Gamma(\delta+1)} + \frac{1-\xi}{\Gamma(\delta)} + \frac{\xi^{\delta}}{\Gamma(\delta+1)}\right|^n = \eta < 1,$$

holds. Then equation (4.1) has a unique solution.

Proof. Let $\nu, \mu \in \Omega$ and consider

$$\begin{aligned}\Big|H\nu(\xi) - H\mu(\xi)\Big|^n = &\Big|\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\xi)(g(\zeta, \nu(\zeta) - g(\zeta, \mu(\zeta))))d\zeta\\ &+ \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\xi)(g(\zeta, \nu(\zeta) - g(\zeta, \mu(\zeta))))d\zeta\\ &+ \frac{1}{\Gamma(\delta)}\int_0^{\xi} (\xi-\zeta)^{\delta-1}(g(\zeta, \nu(\zeta)) - g(\zeta, \mu(\zeta)))d\zeta\Big|^n\end{aligned}$$

$$\leq \Big(\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\xi)\Big|g(\zeta,\nu(\zeta)-g(\zeta,\mu(\zeta)))\Big|d\zeta$$
$$+\frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\xi)\Big|g(\zeta,\nu(\zeta)-g(\zeta,\mu(\zeta)))\Big|d\zeta$$
$$+\frac{1}{\Gamma(\delta)}\int_0^\xi (\xi-\zeta)^{\delta-1}\Big|g(\zeta,\nu(\zeta))-g(\zeta,\mu(\zeta))\Big|d\zeta\Big)^n$$
$$\leq \Big(\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\xi)K^{\frac{1}{n}}|\nu(\zeta)-\mu(\zeta)|d\zeta$$
$$+\frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\xi)K^{\frac{1}{n}}|\nu(\zeta)-\mu(\zeta)|d\zeta$$
$$+\frac{1}{\Gamma(\delta)}\int_0^\xi (\xi-\zeta)^{\delta-1}K^{\frac{1}{n}}|\nu(\zeta)-\mu(\zeta)|d\zeta\Big)^n$$
$$=K|\nu(\xi)-\mu(\xi)|^n\Big(\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\xi)d\zeta$$
$$+\frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\xi)d\zeta+\frac{1}{\Gamma(\delta)}\int_0^\xi (\xi-\zeta)^{\delta-1}d\zeta\Big)^n$$
$$=K|\nu(\xi)-\mu(\xi)|^n\Big(\frac{1-\xi}{\Gamma(\delta+1)}+\frac{1-\xi}{\Gamma(\delta)}+\frac{\xi^\delta}{\Gamma(\delta+1)}\Big)^n$$
$$\leq K|\nu(\xi)-\mu(\xi)|^n\sup_{\xi\in(0,1)}\Big(\frac{1-\xi}{\Gamma(\delta+1)}+\frac{1-\xi}{\Gamma(\delta)}+\frac{\xi^\delta}{\Gamma(\delta+1)}\Big)^n$$
$$=\eta.K|\nu(\xi)-\mu(\xi)|^n$$
$$\leq K|\nu(\xi)-\mu(\xi)|^n,$$

so, we have

$$\Big|H\nu(\xi)-H\mu(\xi)\Big|^n\leq K|\nu(\xi)-\mu(\xi)|^n,$$

i-e

$$-\frac{\sup_{\xi\in[0,1]}\Big|H\nu(\xi)-H\mu(\xi)\Big|^n}{Kt}\geq-\frac{\sup_{\xi\in[0,1]}|\nu(\xi)-\mu(\xi)|^n}{t},$$

$$\exp\Big(-\frac{\sup_{\xi\in[0,1]}\Big|H\nu(\xi)-H\mu(\xi)\Big|^n}{Kt}\Big)\geq\exp\Big(-\frac{\sup_{\xi\in[0,1]}|\nu(\xi)-\mu(\xi)|^n}{t}\Big),$$

thus, we have

$$F_n^c(H\nu(\xi),H\mu(\xi),Kt)\geq F_n^c(\nu(\xi),\mu(\xi),t),$$

from Theorem 3.2, the equation (4.1) has a unique solution. □

5. Conclusion

We introduced the concept of *(FnCMS)* that extends numerous metric spaces in fuzzy literature. Examples are given that justify and support definitions and main results. Utilizing generalized $\alpha-\phi-$contraction we have to find fixed points. The Banach fixed point in the settings of *(FnCMS)* has also been proved. We discussed the topological properties and proved a *(FnCMS)* is not Hausdorff. Finally, an application is provided for a fractional differential equation that shows the uniqueness of the solution. Our new concepts can further be employed in different directions in the literature. Significantly, one can apply these results to the system of fractional differential equations [44] and to the generalized stochastic functional equation emerging in the psychological theory of learning [45].

References

[1] S. Banach, Sur les Opérations dans les Ensembles Abstraits et Leur Application aux Équations Intégrales, Fund. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[2] S. Czerwik, Contraction Mappings in *b*-Metric Spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11. https://dml.cz/handle/10338.dmlcz/120469.

[3] V. Gregori, A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets Syst. 125 (2002), 245–252. https://doi.org/10.1016/s0165-0114(00)00088-9.

[4] N. Saleem, B. Ali, M. Abbas, et al. Fixed Points of Suzuki Type Generalized Multivalued Mappings in Fuzzy Metric Spaces With Applications, Fixed Point Theory Appl. 2015 (2015), 36. https://doi.org/10.1186/s13663-015-0284-7.

[5] D. Miheţ, Fuzzy-Contractive Mappings in Non-Archimedean Fuzzy Metric Spaces, Fuzzy Sets Syst. 159 (2008), 739–744. https://doi.org/10.1016/j.fss.2007.07.006.

[6] M. Edelstein, An Extension of Banach's Contraction Principle, Proc. Amer. Math. Soc. 12 (1961), 7–10. https://doi.org/10.1090/s0002-9939-1961-0120625-6.

[7] R. Kannan, Some Results on Fixed Points—II, Amer. Math. Mon. 76 (1969), 405–408. https://doi.org/10.1080/00029890.1969.12000228.

[8] S.K. Chatterjea, Fixed Point Theorems, C.R. Acad., Bulgare Sci. 25 (1972), 727–730.

[9] Lj.B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974), 267–273. https://doi.org/10.1090/s0002-9939-1974-0356011-2.

[10] B. Samet, C. Vetro, P. Vetro, Fixed Point Theorems for $\alpha-\psi$-Contractive Type Mappings, Nonlinear Anal.: Theory Meth. Appl. 75 (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014.

[11] M. Jleli, B. Samet, A New Generalization of the Banach Contraction Principle, J. Inequal. Appl. 2014 (2014), 38. https://doi.org/10.1186/1029-242x-2014-38.

[12] T. Suzuki, A Generalized Banach Contraction Principle That Characterizes Metric Completeness, Proc. Amer. Math. Soc. 136 (2007), 1861-1870. https://doi.org/10.1090/s0002-9939-07-09055-7.

[13] D. Wardowski, Fixed Points of a New Type of Contractive Mappings in Complete Metric Spaces, Fixed Point Theory Appl. 2012 (2012), 94. https://doi.org/10.1186/1687-1812-2012-94.

[14] N. Saleem, M. Abbas, Z. Raza, Fixed Fuzzy Point Results of Generalized Suzuki Type F-Contraction Mappings in Ordered Metric Spaces, Georgian Math. J. 27 (2017), 307–320. https://doi.org/10.1515/gmj-2017-0048.

[15] N. Saleem, I. Habib, M.D. la Sen, Some New Results on Coincidence Points for Multivalued Suzuki-Type Mappings in Fairly Complete Spaces, Computation. 8 (2020), 17. https://doi.org/10.3390/computation8010017.

[16] L.A. Zadeh, Fuzzy Sets, Inform. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.

[17] J.J. Buckley, T. Feuring, Introduction to Fuzzy Partial Differential Equations, Fuzzy Sets Syst. 105 (1999), 241–248. https://doi.org/10.1016/s0165-0114(98)00323-6.

[18] O. Kaleva, Fuzzy Differential Equations, Fuzzy Sets Syst. 24 (1987), 301–317. https://doi.org/10.1016/0165-011428872990029-7.

[19] M.L. Puri, D.A. Ralescu, Differentials of Fuzzy Functions, J. Math. Anal. Appl. 91 (1983), 552–558. https://doi.org/10.1016/0022-247x(83)90169-5.

[20] I. Kramosil, J. Michálek, Fuzzy Metrics and Statistical Metric Spaces, Kybernetika. 11 (1975), 336–344. http://dml.cz/dmlcz/125556.

[21] K. Menger, Statistical Metrics, Proc. Natl. Acad. Sci. U.S.A. 28 (1942), 535–537. https://doi.org/10.1073/pnas.28.12.535.

[22] M. Grabiec, Fixed Points in Fuzzy Metric Spaces, Fuzzy Sets Syst. 27 (1988), 385–389. https://doi.org/10.1016/0165-0114(88)90064-4.

[23] A. George, P. Veeramani, On Some Results in Fuzzy Metric Spaces, Fuzzy Sets Syst. 64 (1994), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7.

[24] A. Branciari, A Fixed Point Theorem of Banach-Caccioppoli-type on a Class of Generalized Metric Spaces, Publ. Math. Debr. 57 (2000), 31–37.

[25] S. Nădăban, Fuzzy *b*-Metric Spaces, Int. J. Comput. Commun. 11 (2016), 273–281. https://doi.org/10.15837/IJCCC.2016.2.2443.

[26] F. Mehmood, R. Ali, C. Ionescu, et al. Extended Fuzzy $b-$metric Spaces, J. Math. Anal. 8 (2017), 124–131.

[27] M.S. Sezen, Controlled Fuzzy Metric Spaces and Some Related Fixed Point Results, Numer. Meth. Part. Differ. Equ. 37 (2020), 583–593. https://doi.org/10.1002/num.22541.

[28] N. Saleem, H. Işık, S. Furqan, C. Park, Fuzzy Double Controlled Metric Spaces and Related Results, J. Intell. Fuzzy Syst. 40 (2021), 9977–9985. https://doi.org/10.3233/JIFS-202594.

[29] R. Chugh, S. Kumar, Weakly Compatible Maps in Generalized Fuzzy Metric Spaces, J. Anal. 10 (2002), 65–74.

[30] F. Mehmood, R. Ali, N. Hussain, Contractions in Fuzzy Rectangular *b*-metric Spaces with Application, J. Intell. Fuzzy Syst. 37 (2019), 1275–1285. https://doi.org/10.3233/JIFS-182719/.

[31] N. Saleem, S. Furqan, M. Abbas, F. Jarad, Extended Rectangular Fuzzy $b-$Metric Space with Application, AIMS Math. 7 (2022), 16208-16230.

[32] S. Furqan, H. Işık, N. Saleem, Fuzzy Triple Controlled Metric Spaces and Related Fixed Point Results, J. Funct. Spaces. 2021 (2021), 9936992. https://doi.org/10.1155/2021/9936992.

[33] S.T. Zubair, K. Gopalan, T. Abdeljawad, et al. On Fuzzy Extended Hexagonal *b*-Metric Spaces with Applications to Nonlinear Fractional Differential Equations, Symmetry. 13 (2021), 2032. https://doi.org/10.3390/sym13112032.

[34] A. Hussain, U. Ishtiaq, K. Ahmed, H. Al-Sulami, On Pentagonal Controlled Fuzzy Metric Spaces with an Application to Dynamic Market Equilibrium, J. Funct. Spaces. 2022 (2022), 5301293. https://doi.org/10.1155/2022/5301293.

[35] N. Saleem, U. Ishtiaq, L. Guran, et al. On Graphical Fuzzy Metric Spaces with Application to Fractional Differential Equations, Fractal Fract. 6 (2022), 238. https://doi.org/10.3390/fractalfract6050238.

[36] S.F. Lacroix, Traité du Calcul Différentiel et du Calcul intégral, Duprat, Paris, 1797.

[37] N.H. Abel, Auflösung einer mechanischen Aufgabe, J. Reine Angew. Math. 1 (1826), 153–157. https://doi.org/10.1515/crll.1826.1.153.

[38] J. Liouville, Mémoire sur le Théorème des Fonctions Complémentaires, J. Reine Angew. Math. 11 (1834), 1–19. https://doi.org/10.1515/crll.1834.11.1.

[39] N.Y. Sonin, On Differentiation With Arbitrary Index, Moscow Matem. Sbornik, 6 (1869), 1–38.

[40] A.V. Letnikov, Theory of Differentiation With an Arbitrary Index, Math. Sb. 3 (1868), 1–66.

[41] H. Laurent, Sur le Calcul Des dérivées à Indices Quelconques, Nouvelles Annales de Mathématiques: Journal des Candidats Aux écoles Polytechnique et Normale. 3 (1884), 240–252.

[42] M. Caputo, Linear Models of Dissipation whose Q is almost Frequency Independent–II, Geophys. J. Int. 13 (1967), 529–539. https://doi.org/10.1111/j.1365-246x.1967.tb02303.x.

[43] B. Schweizer, A. Sklar, Statistical Metric Spaces, Pac. J. Math. 10 (1960), 313–334. https://doi.org/10.2140/pjm.1960.10.313.

[44] Z.A. Khan, I. Ahmad, K. Shah, Applications of Fixed Point Theory to Investigate a System of Fractional Order Differential Equations, J. Funct. Spaces. 2021 (2021), 1399764. https://doi.org/10.1155/2021/1399764.

[45] A. Turab, W. Ali, C. Park, A Unified Fixed Point Approach to Study the Existence and Uniqueness of Solutions to the Generalized Stochastic Functional Equation Emerging in the Psychological Theory of Learning, AIMS Math. 7 (2022), 5291–5304. https://doi.org/10.3934/math.2022294.