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Almost α-$\star$-Continuity for Multifunctions

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Abstract. This paper is concerned with the concepts of upper and lower almost α-$\star$-continuous multifunctions. Moreover, several characterizations of upper and lower almost α-$\star$-continuous multifunctions are investigated. In particular, the relationships between α-$\star$-continuity and almost α-$\star$-continuity are established.

1. Introduction

It is well known that multifunctions play a very important role in the theory of classical point set topology. Several different forms of continuous multifunctions have been introduced and studied over the years. Many authors have researched and investigated several stronger and weaker forms of continuous multifunctions. Njåstad [19] introduced a weak form of open sets called α-sets. Noiri [20] and Popa [23] investigated a class of functions called almost α-continuous or almost feebly continuous. Popa and Noiri [22] extended the concept of almost α-continuous functions to multifunctions and introduced the notion of upper (lower) almost α-continuous multifunctions. Moreover, Popa and Noiri [21] obtained some characterizations of upper (lower) almost α-continuous multifunctions. Boonpok et al. [9] introduced and studied the concepts of upper and lower almost (τ_1, τ_2)-precontinuous multifunctions. Viriyapong and Boonpok [25] investigated some characterizations of upper and lower almost $(\tau_1, \tau_2)\alpha$-continuous multifunctions. The concept of ideals in topological spaces has been studied by Kuratowski [18] and Vaidyanathaswamy [24] which is one of the important areas of research in the branch of mathematics. Janković and Hamlett [17] introduced the concept of $\mathscr{I}$-open sets in ideal topological spaces. Abd El-Monsef et al. [1] further investigated $\mathscr{I}$-open sets and $\mathscr{I}$-continuous

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functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açikgöz et al. [3] studied the concepts of α-$\mathscr{I}$-continuous functions and α-$\mathscr{I}$-open functions in ideal topological spaces and obtained several characterizations of these functions. Hatir and Noiri [16] introduced the notions of semi-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. Furthermore, Hatir and Noiri [14] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuity. Hatir et al. [13] introduced and investigated the notions of strong β-$\mathscr{I}$-open sets and strongly β-$\mathscr{I}$-continuous functions. In [8], the author introduced and investigated the concepts of upper and lower almost $\star$-continuous multifunctions. Several characterizations of almost $\alpha(\star)$-continuous multifunctions and almost $\beta(\star)$-continuous multifunctions were studied in [7] and [6], respectively. In this paper, we introduce the concepts of upper and lower almost α-$\star$-continuous multifunctions. Moreover, several characterizations of upper and lower almost α-$\star$-continuous multifunctions are investigated.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [18], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [17] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$.

A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi***-*$\mathscr{I}$*-open* [12] (resp. *semi-*$\mathscr{I}$*-open* [16]) if $A \subseteq \mathrm{Cl}(\mathrm{Int}^\star(A))$ (resp. $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(A))$). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi***-*$\mathscr{I}$*-closed* [12] (resp. semi-$\mathscr{I}$-closed [16]). For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}$-closed (resp. semi*-$\mathscr{I}$-closed) sets containing A is called the *semi-*$\mathscr{I}$*-closure* [10] (resp. *semi***-*$\mathscr{I}$*-closure* [10]) of A and is denoted by $s\mathrm{Cl}_{\mathscr{I}}(A)$ (resp. $s^\star\mathrm{Cl}_{\mathscr{I}}(A)$). The union of all semi-$\mathscr{I}$-open (resp. semi*-$\mathscr{I}$-open) sets contained in A is called the *semi-*$\mathscr{I}$*-interior* (resp. *semi***-*$\mathscr{I}$*-interior*) of A and is denoted by $s\mathrm{Int}_{\mathscr{I}}(A)$ (resp. $s^\star\mathrm{Int}_{\mathscr{I}}(A)$).

Lemma 2.1. [7] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *If A is an open set, then $s^\star Cl_{\mathscr{I}}(A) = Int(Cl^\star(A))$.*
(2) *If A is a $\star$-open set, then $sCl_{\mathscr{I}}(A) = Int^\star(Cl(A))$.*

Recall that a subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *α-$\star$-closed* [2] if

$$\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A))) \subseteq A.$$

The complement of an α-$\star$-closed set is said to be *α-$\star$-open*.

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all α-$\star$-closed sets containing A is called the *α-$\star$-closure* [5] of A and is denoted by $\star\alpha\mathrm{Cl}(A)$. The *$\alpha$-$\star$-interior* [5] of A is defined by the union of all α-$\star$-open sets contained in A and is denoted by $\star\alpha\mathrm{Int}(A)$.

Lemma 2.2. [5] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *$\star\alpha Cl(A)$ is α-$\star$-closed.*
(2) *A is α-$\star$-closed if and only if $A = \star\alpha Cl(A)$.*

Lemma 2.3. [5] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) *A is α-$\star$-open in X;*
(2) *$G \subseteq A \subseteq Int^\star(Cl(G))$ for some $\star$-open set G;*
(3) *$G \subseteq A \subseteq sCl_{\mathscr{I}}(G)$ for some $\star$-open set G;*
(4) *$A \subseteq sCl_{\mathscr{I}}(Int^\star(A))$.*

Lemma 2.4. [5] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *A is α-$\star$-closed in X if and only if $sInt_{\mathscr{I}}(Cl^\star(A)) \subseteq A$.*
(2) *$sInt_{\mathscr{I}}(Cl^\star(A)) = Cl^\star(Int(Cl^\star(A)))$.*
(3) *$\star\alpha Cl(A) = A \cup Cl^\star(Int(Cl^\star(A)))$.*
(4) *$\star\alpha Int(A) = A \cap Int^\star(Cl(Int^\star(A)))$.*

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [4] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X \mid F(x) \subseteq B\}$$

and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower almost α-$\star$-continuous multifunctions

In this section, we introduce the notions of upper and lower almost α-$\star$-continuous multifunctions. Moreover, several characterizations of upper and lower almost α-$\star$-continuous multifunctions are discussed.

Definition 3.1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

1. *upper almost α-$\star$-continuous at a point $x \in X$ if for each $\star$-open set V of Y such that $F(x) \subseteq V$, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq Int^{\star}(Cl(V))$;*
2. *lower almost α-$\star$-continuous at a point $x \in X$ if for each $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an α-$\star$-open set U of X containing x such that $F(z) \cap Int^{\star}(Cl(V)) \neq \emptyset$ for every $z \in U$;*
3. *upper (resp. lower) almost α-$\star$-continuous if F is upper (resp. lower) almost α-$\star$-continuous at each point of X.*

Theorem 3.1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

1. *F is upper almost α-$\star$-continuous at $x \in X$;*
2. *for any $\star$-open set V of Y containing $F(x)$, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq sCl_{\mathscr{J}}(V)$;*
3. *$x \in \star\alpha Int(F^{+}(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set V of Y containing $F(x)$;*
4. *$x \in Int^{\star}(Cl(Int^{\star}(F^{+}(sCl_{\mathscr{J}}(V)))))$ for every $\star$-open set V of Y containing $F(x)$.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\star$-open set of Y containing $F(x)$. Then, there exists an α-$\star$-open set U containing x such that $F(U) \subseteq \mathrm{Int}^{\star}(\mathrm{Cl}(V))$ and by Lemma 2.1(2), we have $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}}(V)$.

(2) $\Rightarrow$ (3): Let V be any $\star$-open set of Y containing $F(x)$. By (2), there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}}(V)$. Thus, $U \subseteq F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V))$ and hence $x \in \star\alpha\mathrm{Int}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))$.

(3) $\Rightarrow$ (4): Let V be any $\star$-open set of Y containing $F(x)$. By (3), we have $x \in \star\alpha\mathrm{Int}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))$ and by Lemma 2.4(4), $x \in \mathrm{Int}^{\star}(\mathrm{Cl}(\mathrm{Int}^{\star}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))))$.

(4) $\Rightarrow$ (1): Let V be any $\star$-open set of Y containing $F(x)$. Then by (4),

$$x \in \mathrm{Int}^{\star}(\mathrm{Cl}(\mathrm{Int}^{\star}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))))$$

and by Lemma 2.4(4), $x \in \star\alpha\mathrm{Int}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))$. Therefore, there exists an α-$\star$-open set U of X containing x such that $U \subseteq F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V))$; hence $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}}(V)$. Since V is $\star$-open, by Lemma 2.1(2),

$$F(U) \subseteq \mathrm{Int}^{\star}(\mathrm{Cl}(V)).$$

This shows that F is upper almost α-$\star$-continuous at x. □

Theorem 3.2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower almost α-$\star$-continuous at $x \in X$;*
(2) *for any $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an α-$\star$-open set U of X containing x such that $F(z) \cap sCl_{\mathscr{J}}(V) \neq \emptyset$ for each $z \in U$;*
(3) *$x \in \star\alpha Int(F^-(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$;*
(4) *$x \in Int^\star(Cl(Int^\star(F^-(sCl_{\mathscr{J}}(V)))))$ for every $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 3.1. □

Definition 3.2. [7] *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) *$R^\star$-$\mathscr{I}$-open if $A = Int^\star(Cl(A))$;*
(2) *$R^\star$-$\mathscr{I}$-closed if its complement is $R^\star$-$\mathscr{I}$-open.*

Theorem 3.3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is upper almost α-$\star$-continuous;*
(2) *for each $x \in X$ and each $\star$-open set V of Y containing $F(x)$, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq sCl_{\mathscr{J}}(V)$;*
(3) *for each $x \in X$ and each $R^\star$-$\mathscr{J}$-open set V of Y containing $F(x)$, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq V$;*
(4) *$F^+(V)$ is α-$\star$-open in X for every $R^\star$-$\mathscr{J}$-open set V of Y;*
(5) *$F^-(K)$ is α-$\star$-closed in X for every $R^\star$-$\mathscr{J}$-closed set K of Y;*
(6) *$F^+(V) \subseteq \star\alpha Int(F^+(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set V of Y;*
(7) *$\star\alpha Cl(F^-(sInt_{\mathscr{J}}(K))) \subseteq F^-(K)$ for every $\star$-closed set K of Y;*
(8) *$\star\alpha Cl(F^-(Cl^\star(Int(K)))) \subseteq F^-(K)$ for every $\star$-closed set K of Y;*
(9) *$\star\alpha Cl(F^-(Cl^\star(Int(Cl^\star(B))))) \subseteq F^-(Cl^\star(B))$ for every subset B of Y;*
(10) *$Cl^\star(Int(Cl^\star(F^-(Cl(Int^\star(K)))))) \subseteq F^-(K)$ for every $\star$-closed set K of Y;*
(11) *$Cl^\star(Int(Cl^\star(F^-(sInt_{\mathscr{J}}(K))))) \subseteq F^-(K)$ for every $\star$-closed set K of Y;*
(12) *$F^+(V) \subseteq Int^\star(Cl(Int^\star(F^+(sCl_{\mathscr{J}}(V)))))$ for every $\star$-open set V of Y.*

Proof. (1) ⇒ (2): The proof follows from Theorem 3.1.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Let V be any $R^\star$-$\mathscr{J}$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by (3), there exists an α-$\star$-open set U_x of X containing x such that $F(U_x) \subseteq V$. Therefore, we have $x \in U_x \subseteq F^+(V)$ and so $F^+(V) = \cup_{x \in F^+(V)} U_x$ is α-$\star$-open in X.

(4) ⇒ (5): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(5) ⇒ (6): Let V be any $\star$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V \subseteq s\mathrm{Cl}_{\mathscr{J}}(V)$. Thus, $x \in F^+(s\mathrm{Cl}_{\mathscr{J}}(V)) = X - F^-(Y - s\mathrm{Cl}_{\mathscr{J}}(V))$. Since $Y - s\mathrm{Cl}_{\mathscr{J}}(V)$ is $R^\star$-$\mathscr{J}$-closed in Y and by (5),

$$F^-(Y - s\mathrm{Cl}_{\mathscr{J}}(V))$$

is α-$\star$-closed in X. Thus, $F^+(s\text{Cl}_{\mathscr{J}}(V))$ is α-$\star$-open in X and hence $x \in \star\alpha\text{Int}(F^+(s\text{Cl}_{\mathscr{J}}(V)))$. This shows that $F^+(V) \subseteq \star\alpha\text{Int}(F^+(s\text{Cl}_{\mathscr{J}}(V)))$.

(6) $\Rightarrow$ (7): Let K be any $\star$-closed set of Y. Then, $Y - K$ is $\star$-open and by (6),

$$\begin{aligned} X - F^-(K) &= F^+(Y - K) \\ &\subseteq \star\alpha\text{Int}(F^+(s\text{Cl}_{\mathscr{J}}(Y - K))) \\ &= \star\alpha\text{Int}(F^+(Y - s\text{Int}_{\mathscr{J}}(K))) \\ &= \star\alpha\text{Int}(X - F^-(s\text{Int}_{\mathscr{J}}(K))) \\ &= X - \star\alpha\text{Cl}(F^-(s\text{Int}_{\mathscr{J}}(K))). \end{aligned}$$

Thus, $\star\alpha\text{Cl}(F^-(s\text{Int}_{\mathscr{J}}(K))) \subseteq F^-(K)$.

(7) $\Rightarrow$ (8): The proof is obvious since $s\text{Int}_{\mathscr{J}}(K) = \text{Cl}^\star(\text{Int}(K))$ for every $\star$-closed set K of Y.

(8) $\Rightarrow$ (9): The proof is obvious.

(9) $\Rightarrow$ (10): It follows from Lemma 2.4(3) that $\text{Cl}^\star(\text{Int}(\text{Cl}^\star(B))) \subseteq \star\alpha\text{Cl}(B)$ for every subset B of Y. Thus, for every $\star$-closed set K of Y, we have

$$\begin{aligned} \text{Cl}^\star(\text{Int}(\text{Cl}^\star(F^-(\text{Cl}^\star(\text{Int}(K)))))) &\subseteq \star\alpha\text{Cl}(F^-(\text{Cl}^\star(\text{Int}(K)))) \\ &= \star\alpha\text{Cl}(F^-(\text{Cl}^\star(\text{Int}(\text{Cl}^\star(K))))) \\ &\subseteq F^-(\text{Cl}^\star(K)) \\ &= F^-(K). \end{aligned}$$

(10) $\Rightarrow$ (11): The proof is obvious since $s\text{Int}_{\mathscr{J}}(K) = \text{Cl}^\star(\text{Int}(K))$ for every $\star$-closed set K of Y.

(11) $\Rightarrow$ (12): Let V be any $\star$-open set of Y. Then $Y - V$ is $\star$-closed in Y and by (11),

$$\text{Cl}^\star(\text{Int}(\text{Cl}^\star(F^-(s\text{Int}_{\mathscr{J}}(Y - V))))) \subseteq F^-(Y - V) = X - F^+(V).$$

Moreover, we have

$$\begin{aligned} \text{Cl}^\star(\text{Int}(\text{Cl}^\star(F^-(s\text{Int}_{\mathscr{J}}(Y - V))))) &= \text{Cl}^\star(\text{Int}(\text{Cl}^\star(F^-(Y - s\text{Cl}_{\mathscr{J}}(V))))) \\ &= \text{Cl}^\star(\text{Int}(\text{Cl}^\star(X - F^+(s\text{Cl}_{\mathscr{J}}(V))))) \\ &= X - \text{Int}^\star(\text{Cl}(\text{Int}^\star(F^+(s\text{Cl}_{\mathscr{J}}(V))))). \end{aligned}$$

Thus, $F^+(V) \subseteq \text{Int}^\star(\text{Cl}(\text{Int}^\star(F^+(s\text{Cl}_{\mathscr{J}}(V)))))$.

(12) $\Rightarrow$ (1): Let $x \in X$ and V be any $\star$-open set of Y containing $F(x)$. By (12), we have

$$x \in F^+(V) \subseteq \text{Int}^\star(\text{Cl}(\text{Int}^\star(F^+(s\text{Cl}_{\mathscr{J}}(V)))))$$

and hence F is upper almost α-$\star$-continuous at x by Theorem 3.1. This shows that F is upper almost α-$\star$-continuous. □

Definition 3.3. [5] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

(1) *upper* α-$\star$-*continuous at a point* $x \in X$ *if for each* $\star$-*open set* V *of* Y *such that* $F(x) \subseteq V$, *there exists an* α-$\star$-*open set* U *of* X *containing* x *such that* $F(U) \subseteq V$;

(2) *lower* α-$\star$-*continuous at a point* $x \in X$ *if for each* $\star$-*open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$, *there exists an* α-$\star$-*open set* U *of* X *containing* x *such that* $F(z) \cap V \neq \emptyset$ *for every* $z \in U$;

(3) *upper (resp. lower)* α-$\star$-*continuous if* F *is upper (resp. lower)* α-$\star$-*continuous at each point of* X.

Remark 3.1. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, *the following implication holds:*

upper α-$\star$-*continuity* $\Rightarrow$ *upper almost* α-$\star$-*continuity.*

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 3.1. *Let* $X = \{1, 2, 3\}$ *with a topology* $\tau = \{\emptyset, \{1, 2\}, X\}$ *and an ideal* $\mathscr{I} = \{\emptyset, \{3\}\}$. *Let* $Y = \{a, b, c\}$ *with a topology* $\sigma = \{\emptyset, \{a, b\}, Y\}$ *and an ideal* $\mathscr{J} = \{\emptyset\}$. *A multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ *is defined as follows:* $F(1) = \{c\}$ *and* $F(2) = \{a\}$ *and* $F(3) = \{a, b\}$. *Then,* F *is upper almost* α-$\star$-*continuous but* F *is not upper* α-$\star$-*continuous.*

Theorem 3.4. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, *the following properties are equivalent:*

(1) F *is lower almost* α-$\star$-*continuous;*

(2) *for each* $x \in X$ *and each* $\star$-*open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$, *there exists an* α-$\star$-*open set* U *of* X *containing* x *such that* $U \subseteq F^-(sCl_{\mathscr{J}}(V))$;

(3) *for each* $x \in X$ *and each* $R^\star$-$\mathscr{J}$-*open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$, *there exists an* α-$\star$-*open set* U *of* X *containing* x *such that* $U \subseteq F^-(V)$;

(4) $F^-(V)$ *is* α-$\star$-*open in* X *for every* $R^\star$-$\mathscr{J}$-*open set* V *of* Y;

(5) $F^+(K)$ *is* α-$\star$-*closed in* X *for every* $R^\star$-$\mathscr{J}$-*closed set* K *of* Y;

(6) $F^-(V) \subseteq \star\alpha Int(F^-(sCl_{\mathscr{J}}(V)))$ *for every* $\star$-*open set* V *of* Y;

(7) $\star\alpha Cl(F^+(sInt_{\mathscr{J}}(K))) \subseteq F^+(K)$ *for every* $\star$-*closed set* K *of* Y;

(8) $\star\alpha Cl(F^+(Cl^\star(Int(K)))) \subseteq F^+(K)$ *for every* $\star$-*closed set* K *of* Y;

(9) $\star\alpha Cl(F^+(Cl^\star(Int(Cl^\star(B))))) \subseteq F^+(Cl^\star(B))$ *for every subset* B *of* Y;

(10) $Cl^\star(Int(Cl^\star(F^+(Cl^\star(Int(K)))))) \subseteq F^+(K)$ *for every* $\star$-*closed set* K *of* Y;

(11) $Cl^\star(Int(Cl^\star(F^+(sInt_{\mathscr{J}}(K))))) \subseteq F^+(K)$ *for every* $\star$-*closed set* K *of* Y;

(12) $F^-(V) \subseteq Int^\star(Cl(Int^\star(F^-(sCl_{\mathscr{J}}(V)))))$ *for every* $\star$-*open set* V *of* Y.

Proof. The proof is similar to that of Theorem 3.3. □

Definition 3.4. *A subset* A *of an ideal topological space* $(X, \tau, \mathscr{I})$ *is said to be:*

(1) $pre^\star_{\mathscr{I}}$-*open* [11] *if* $A \subseteq Int^\star(Cl(A))$;

(2) $pre^\star_{\mathscr{I}}$-*closed* [11] *if its complement is* $pre^\star_{\mathscr{I}}$-*open;*

(3) *strong* β-$\mathscr{I}$-*open* [15] *if* $A \subseteq Cl^\star(Int(Cl^\star(A)))$;

(4) *strong β-$\mathscr{I}$-closed* [13] *if its complement is strong β-$\mathscr{I}$-open.*

Theorem 3.5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is upper almost α-$\star$-continuous;*
(2) *$\star\alpha Cl(F^-(V)) \subseteq F^-(Cl^\star(V))$ for every strong β-$\mathscr{J}$-open set V of Y;*
(3) *$\star\alpha Cl(F^-(V)) \subseteq F^-(Cl^\star(V))$ for every semi-$\mathscr{J}$-open set V of Y;*
(4) *$F^+(V) \subseteq \star\alpha Int(F^+(Int^\star(Cl(V))))$ for every pre$^\star_\mathscr{J}$-open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any strong β-$\mathscr{J}$-open set of Y. Then, $\mathrm{Cl}^\star(V)$ is $R^\star$-$\mathscr{J}$-closed in Y. By (1) and Theorem 3.3, we have $F^-(\mathrm{Cl}^\star(V))$ is α-$\star$-closed in X and hence

$$\star\alpha\mathrm{Cl}(F^-(V)) \subseteq \star\alpha\mathrm{Cl}(F^-(\mathrm{Cl}^\star(V))) = F^-(\mathrm{Cl}^\star(V)).$$

(2) $\Rightarrow$ (3): This is obvious since every semi-$\mathscr{J}$-open set is strong β-$\mathscr{J}$-open.

(3) $\Rightarrow$ (1): Let K be any $R^\star$-$\mathscr{J}$-closed set of Y. Then, K is semi-$\mathscr{J}$-open in Y and by (3),

$$\star\alpha\mathrm{Cl}(F^-(K)) \subseteq F^-(\mathrm{Cl}^\star(K)) = F^-(K).$$

Thus, $F^-(K)$ is α-$\star$-closed in X and hence F is upper almost α-$\star$-continuous by Theorem 3.3.

(1) $\Rightarrow$ (4): Let V be any pre$^\star_\mathscr{J}$-open set of Y. Then, we have $\mathrm{Int}^\star(\mathrm{Cl}(V))$ is $R^\star$-$\mathscr{J}$-open in Y. By (1) and Theorem 3.3, $F^+(\mathrm{Int}^\star(\mathrm{Cl}(V)))$ is α-$\star$-open in X. Thus,

$$F^+(V) \subseteq F^+(\mathrm{Int}^\star(\mathrm{Cl}(V))) = \star\alpha\mathrm{Int}(F^+(\mathrm{Int}^\star(\mathrm{Cl}(V)))).$$

(4) $\Rightarrow$ (1): Let V be any $R^\star$-$\mathscr{J}$-open set of Y. Then, V is pre$^\star_\mathscr{J}$-open in Y and by (4),

$$F^+(V) \subseteq \star\alpha\mathrm{Int}(F^+(\mathrm{Int}^\star(\mathrm{Cl}(V)))) = \star\alpha\mathrm{Int}(F^+(V)).$$

This shows that $F^+(V)$ is α-$\star$-open in X. It follows from Theorem 3.3 that F is upper almost α-$\star$-continuous. □

Theorem 3.6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower almost α-$\star$-continuous;*
(2) *$\star\alpha Cl(F^+(V)) \subseteq F^+(Cl^\star(V))$ for every strong β-$\mathscr{J}$-open set V of Y;*
(3) *$\star\alpha Cl(F^+(V)) \subseteq F^+(Cl^\star(V))$ for every semi-$\mathscr{J}$-open set V of Y;*
(4) *$F^-(V) \subseteq \star\alpha Int(F^-(Int^\star(Cl(V))))$ for every pre$^\star_\mathscr{J}$-open set V of Y.*

Proof. The proof is similar to that of Theorem 3.5. □

Definition 3.5. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be almost α-$\star$-continuous if $f^{-1}(V)$ is α-$\star$-open in X for every $R^\star$-$\mathscr{J}$-open set V of Y.*

Corollary 3.1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is almost α-$\star$-continuous;*
(2) *for each $x \in X$ and each $\star$-open set V of Y containing $f(x)$, there exists an α-$\star$-open set U of X containing x such that $f(U) \subseteq sCl_{\mathscr{J}}(V)$;*
(3) *for each $x \in X$ and each $R^\star$-$\mathscr{J}$-open set V of Y containing $f(x)$, there exists an α-$\star$-open set U of X containing x such that $f(U) \subseteq V$;*
(4) *for each $x \in X$ and each $\star$-open set V of Y containing $f(x)$, there exists an α-$\star$-open set U of X containing x such that $f(U) \subseteq Int^\star(Cl(V))$;*
(5) *$f^{-1}(K)$ is α-$\star$-closed in X for every $R^\star$-$\mathscr{J}$-closed set K of Y;*
(6) *$f^{-1}(V) \subseteq \star\alpha Int(f^{-1}(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set V of Y;*
(7) *$\star\alpha Cl(f^{-1}(sInt_{\mathscr{J}}(K))) \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(8) *$\star\alpha Cl(f^{-1}(Cl^\star(Int(K)))) \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(9) *$\star\alpha Cl(f^{-1}(Cl^\star(Int(Cl^\star(B))))) \subseteq f^{-1}(Cl^\star(B))$ for every subset B of Y;*
(10) *$Cl^\star(Int(Cl^\star(f^{-1}(Cl^\star(Int(K)))))) \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(11) *$Cl^\star(Int(Cl^\star(f^{-1}(sInt_{\mathscr{J}}(K))))) \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(12) *$f^{-1}(V) \subseteq Int^\star(Cl(Int^\star(f^{-1}(sCl_{\mathscr{J}}(V)))))$ for every $\star$-open set V of Y.*

Corollary 3.2. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is almost α-$\star$-continuous;*
(2) *$\star\alpha Cl(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ for every strong β-$\mathscr{J}$-open set V of Y;*
(3) *$\star\alpha Cl(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ for every semi-$\mathscr{J}$-open set V of Y;*
(4) *$f^{-1}(V) \subseteq \star\alpha Int(f^{-1}(Int^\star(Cl(V))))$ for every $pre^\star_{\mathscr{J}}$-open set V of Y.*

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