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A Solution of a General Functional Equation Involved in Psychological Theory of Learning and Stability Results

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Abstract. The psychological learning theory (PLT) in the formation of moral verdict is represented by the choice-practice paradigm. It involves weighing the effects of various options and choosing one to put into practice. This manuscript is devoted to presenting a general functional equation (FE) for observing animal behavior in such situations. The proposed equation can be used to explain a number of well-known learning and psychological theories. The existence and uniqueness of the solution to a given equation are demonstrated using fixed point (FP) techniques. Furthermore, the stability of the solution to the provided FE is explored in the sense of Hyers-Ulam-Rassias (HUR) and Hyers-Ulam (HU). Ultimately, to emphasize the importance of our results, two examples are presented.

1. Introduction and Preliminaries Work

The study of intellectual, perceptual, and cognitive processes as models in mathematics is known as mathematical psychology. As an alternative, learning in both humans and animals can be seen as a series of choices among many input options. Preference sequences are frequently unexpected, demonstrating that response choices are made at random even in simple recurrent studies in well-controlled environments. Consequently, it is useful to represent variations in response likelihood over trials by including systematic changes in a series of options. According to this viewpoint, the majority

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of learning research is focused on clarifying the likelihood of the occurrence of the events that make up a random technique over particular actions.

The behavior of even the most superficial learning experiment may be represented using stochastic processes, according to recent mathematical learning research. As a result, it is only partially original (for further information, check [1,2]). However, after 1950, two important qualities emerged, according to research mostly done by Bush, Estes, and Mosteller. First, it is essential that all models that are suggested have an inclusive learning approach. Second, the analysis cannot be concealed from the statistical characteristics of such models [3,4].

Using the equation shown below, Istrațéscu [5] proposed the idea that predator animals could be involved in 1976 by eating two different kinds of food.

$$\Re(\varkappa) = \varkappa\Re\left(\xi_1 + (1-\xi_1)\varkappa\right) + (1-\varkappa)\Re\left((1-\xi_2)\varkappa\right), \text{ for each } \varkappa \in J = [0,1], \tag{1.1}$$

where $0 < \xi_1 \leq \xi_2 < 1$ and $\Re : J \to \mathbb{R}$ such that $\Re(0) = 0$ and $\Re(1) = 1$. Theoretically, a Markov process with state space J and probabilities of transition from state $\varkappa$ to state $\xi_1 + (1-\xi_1)\varkappa$ and from state $\varkappa$ to state $(1-\xi_2)\varkappa$ defines this behavior as follows:

$$\begin{cases} Q\left(\varkappa \to \xi_1 + (1-\xi_1)\varkappa\right) = \varkappa, \\ Q\left(\varkappa \to (1-\xi_1)\varkappa\right) = 1-\varkappa, \end{cases}$$

respectively. A predator will target a particular class of prey if the initial chance of this class being chosen is equal to $\varkappa$, the solution $\Re$ in the FE (1.1) indicates the ultimate probability of an event.

Bush and Mosteller's experimental investigations [6] were discussed by Turab and Sintunavarat [7] in 2019. They used their related probabilities to analyze the behavior of a paradise fish and then presented the resulting model

$$\Re(\varkappa) = \varkappa\Re\left(\xi_1\varkappa + (1-\xi_1)\right) + (1-\varkappa)\Re\left(\xi_2\varkappa\right), \text{ for each } \varkappa \in J, \tag{1.2}$$

where $0 < \xi_1 \leq \xi_2 < 1$ and $\Re : J \to \mathbb{R}$ such that $\Re(0) = 0$ and $\Re(1) = 1$.

In 2020, the following FE was proposed by the authors in [8] using the aforementioned methodology:

$$\Re(\varkappa) = \varkappa\Re\left(\xi_1\varkappa + (1-\xi_1)\beta_1\right) + (1-\varkappa)\Re\left(\xi_2\varkappa + (1-\xi_2)\beta_2\right), \text{ for each } \varkappa \in J, \tag{1.3}$$

where $0 < \xi_1 \leq \xi_2 < 1$, $\beta_1, \beta_2 \in J$ and $\Re : J \to \mathbb{R}$. This study examined a specific psychological barrier displayed by dogs when kept indoors in a small closet using the FE described above.

Berinde and Khan [9] developed the ideas mentioned above and explored whether the equation shown below has a unique solution:

$$\Re(\varkappa) = \varkappa\Re\left(\Im_1\varkappa\right) + (1-\varkappa)\Re\left(\Im_2\varkappa\right), \text{ for each } \varkappa \in J, \tag{1.4}$$

where $\Im_1, \Im_2 : J \to J$ are given contraction mappings (CMs) and $\Re : J \to \mathbb{R}$. These kinds of functional equations are also used to explain the relationship between predatory animals and their two choices of prey. In contrast, a few scholars talked about the stability [10], and convergence findings [11] of the FEs.

In 2020, authors [12] considered a FE, which is a generalization of the concept described in [9] for an arbitrary interval $[\varrho_1, \varrho_2]$.

$$\Re(\varkappa) = \left(\frac{\varkappa - \varrho_1}{\varrho_2 - \varrho_1}\right) \Re(\Im_1 \varkappa) + \left(1 - \frac{\varkappa - \varrho_1}{\varrho_2 - \varrho_1}\right) \Re(\Im_2 \varkappa), \text{ for each } \varkappa \in [\varrho_1, \varrho_2], \tag{1.5}$$

where $\Im_1, \Im_2 : [\varrho_1, \varrho_2] \to [\varrho_1, \varrho_2]$ are CMs and $\Re : [\varrho_1, \varrho_2] \to \mathbb{R}$.

Considering the transitional operators in Table 1, numerous results have been reached from diverse investigations on the conduct of both people and animals in situations teaching likelihood; see [13–17] for more information.

Table1: Operators for habit-formation and reward-extinction

Interactions	**Lift side results**	**Hand side results**
Operators in models of habit formation		
Reinforcement	$\xi_1 \varkappa$	$\xi_1 \varkappa + 1 - \xi_1$
Non-Reinforcement	$\xi_2 \varkappa$	$\xi_2 \varkappa + 1 - \xi_2$
Operators of the rewards-extinction model		
Reinforcement	$\xi_1 \varkappa$	$\xi_1 \varkappa + 1 - \xi_1$
Non-Reinforcement	$\xi_2 \varkappa + 1 - \xi_2$	$\xi_2 \varkappa$

Based on the reward and chosen side framework presented in Table 1 by Bush and Wilson [6]. Writers [18] expanded the mentioned study in 2022 by assuming the FE below:

$$\Re(\varkappa) = \tau\vartheta(\varkappa)\Re(\Im_1 \varkappa) + (1-\tau)\vartheta(\varkappa)\Re(\Im_2 \varkappa) + \tau(1-\vartheta(\varkappa))\Re(\Im_3 \varkappa) + (1-\tau)(1-\vartheta(\varkappa))\Re(\Im_4 \varkappa), \tag{1.6}$$

for each $\varkappa \in J$, where $\Im_i : J \to J$, $(i = 1, 2, 3, 4)$ are given maps, $\vartheta : J \to J$ denotes the proportional event probability, with a fixed percentage of occurrences falling inside a value of $\tau \in J$, and $\Re : [\varrho_1, \varrho_2] \to \mathbb{R}$ is an unknown.

Turab recently proposed the following FE [19] to extend the aforementioned model (1.6) to a random interval $[\varrho_1, \varrho_2]$ in order to continue the study in [6]:

$$\begin{aligned} \Re(\varkappa) &= \left(\frac{\varkappa - \varrho_1}{\varrho_2 - \varrho_1}\right)\left(\frac{\tau - \varrho_1}{\varrho_2 - \varrho_1}\right)\Re(\Im_1 \varkappa) + \left(\frac{\varkappa - \varrho_1}{\varrho_2 - \varrho_1}\right)\left(1 - \frac{\tau - \varrho_1}{\varrho_2 - \varrho_1}\right)\Re(\Im_2 \varkappa) \\ &+ \left(1 - \frac{\varkappa - \varrho_1}{\varrho_2 - \varrho_1}\right)\left(\frac{\tau - \varrho_1}{\varrho_2 - \varrho_1}\right)\Re(\Im_3 \varkappa) + \left(1 - \frac{\varkappa - \varrho_1}{\varrho_2 - \varrho_1}\right)\left(1 - \frac{\tau - \varrho_1}{\varrho_2 - \varrho_1}\right)\Re(\Im_4 \varkappa), \end{aligned} \tag{1.7}$$

for each $\varkappa \in [\varrho_1, \varrho_2]$, where $\Im_i : [\varrho_1, \varrho_2] \to [\varrho_1, \varrho_2]$, $(i = 1, 2, 3, 4)$ are given maps, $\tau \in [0, 1]$ and $\Re : [\varrho_1, \varrho_2] \to \mathbb{R}$ is an unknown.

The results of numerous studies that have looked at how various animals response to discovering novel probabilistic patterns has substantially altered; see [20] for more details. An issue that frequently arises in these investigations is: What would happen to the building if an animal refused to leave the middle and stayed there?

Neimark's ground-breaking research on behavior in two-choice perspective situations must be taken into account in order to provide a proper response to this question; see [21]. During tests, individuals were given the option to select which of two lights would be turned on, while others could choose to do nothing. Hence, the third type of event are the "blank trials," as she referred to them.

Motivated by the study mentioned above, we present the following general FE:

$$\Re(\varkappa)=\theta_1(\varkappa)\Re\left(\Im_1\varkappa\right)+\theta_2(\varkappa)\Re\left(\Im_2\varkappa\right)+\theta_3(\varkappa)\Re\left(\Im_3\varkappa\right)+\cdots+\theta_n(\varkappa)\Re\left(\Im_n\varkappa\right), \tag{1.8}$$

for each $\varkappa\in J$, where $\Re:J\to\mathbb{R}$, $\Im_i:J\to J$, are given maps and $\theta_i:J\to J$, $(i=1,2,...,n)$ are the associated probabilities of the events with

$$\theta_n(\varkappa)=\left(1-\theta_1(\varkappa)-\theta_2(\varkappa)-\cdots-\theta_{n-1}(\varkappa)\right),\ n\in\mathbb{N}.$$

In this instance, the Banach FP theorem will be used to show that the suggested equation (1.8) has a unique solution (US). In addition, the stability of the suggested solution to the FE (1.8) will be evaluated using the conclusions put out by HU and HUR. Finally, two examples are provided to illustrate the importance of our study in this area..

The requirements that ensure there are solutions to the fitted system are also a focus of the philosophy of FP theory. It contains techniques that can be used to handle problems in a number of fields of mathematics. For the most recent research in this field, we suggest [22–28] and the references therein.

The subsequent result is necessary for the development to proceed.

Theorem 1.1. *[29] A self-mapping $\Re$, which is defined on a complete metric space (J,d), owns a unique FP if it is a contraction, i.e., there is $\nu<1$ so that $d\left(\Re\ell,\Re\widetilde{\ell}\right)\leq\nu d\left(\ell,\widetilde{\ell}\right)$, for all $\ell,\widetilde{\ell}\in J$. In addition, the Picard iteration (PI) $\ell_n=\Re^n\ell_0$, for all $\{\ell_n\}\subset J$ and $\ell_0\in J$ converges to a unique FP of $\Re$.*

2. Assumptions and Main Results

Let $J\in[0,1]$ and Θ be a family of all real-valued continuous functions $\Re:J\to\mathbb{R}$ with $\sup_{\ell\neq\widetilde{\ell}}\frac{\left|\Re\ell-\Re\widetilde{\ell}\right|}{\left|\ell-\widetilde{\ell}\right|}<\infty$ and $\Re(0)=0$. Clearly, $(\Theta,\|.\|)$ is a BS under the norm

$$\|\Re\|=\sup_{\ell\neq\widetilde{\ell}}\frac{\left|\Re\ell-\Re\widetilde{\ell}\right|}{\left|\ell-\widetilde{\ell}\right|},\ \Re\in\Theta. \tag{2.1}$$

To reach our purpose in this study, we impose the following hypotheses:

(H_1) There is $\psi_{\neq\emptyset}\subset\Upsilon=\{\Re\in\Theta:\Re(1)\leq 1\}$ so that $(\psi,\|.\|)$ is a BS, where $\|.\|$ is described as (2.1).

(H_2) The maps $\theta_i:J\to J$ with $\theta_1(0)=0=\theta_2(0)$, for $i=1,2,...,n$, fulfill the inequalities below:

$$d\left(\theta_i\ell,\theta_i\widetilde{\ell}\right)\leq d\left(\ell,\widetilde{\ell}\right),\ \text{for all}\ \ell,\widetilde{\ell}\in J,$$

and

$$|\theta_i(\varkappa)| \leq \nu_i, \text{ for all } \varkappa \in J \text{ and } \nu_i \geq 0.$$

(H$_3$) The maps $\Im_i : J \to J$ with $\Im_3(0) = \Im_4(0) = ... = \Im_n(0) = 0$, for $i = 1, 2, ..., n$, fulfill the inequalities shown below:

$$d\left(\Im_i \ell, \Im_i \widetilde{\ell}\right) \leq \sigma_i d\left(\ell, \widetilde{\ell}\right), \text{ for all } \ell, \widetilde{\ell} \in J, \text{ and } \sigma_i < 1,$$

and

$$|\Im_i(\varkappa)| \leq \eta_i, \text{ for all } \varkappa \in J \text{ and } \eta_i \geq 0.$$

(H$_4$) (HUR-stability [30]) For $\Re \in \psi$ and for a function $\varpi : \psi \to [0, \infty)$ with $d(Z\Re, \Re) \leq \varpi(\Re)$, there is a unique $\Re^* \in \psi$ so that $Z\Re^* = \Re^*$ and $d(\Re, \Re^*) \leq \Delta\varpi(\Re)$, where $\Delta > 0$.

(H$_5$) (HU-stability [31]) For $\Re \in \psi$ and for $\rho > 0$ with $d(Z\Re, \Re) \leq \rho$, there is a unique $\Re^* \in \psi$ so that $Z\Re^* = \Re^*$ and $d(\Re, \Re^*) \leq \Delta\rho$, where $\mu > 0$.

Now, we present the first main result in this part.

Theorem 2.1. *Assume that the hypotheses $(H_1) - (H_3)$ hold, then the general FE (1.8) possesses a US provided that $\mho < 1$, where*

$$\mho = \sum_{i=1}^{n} \sigma_i \nu_i + \sum_{i=1}^{n} \eta_i, \ i \in \mathbb{N}. \tag{2.2}$$

Moreover, for all $k \in \mathbb{N}$, the sequence $\{\Re_k\}$ in ψ converges to the US of (1.8) with

$$\Re_k(\varkappa) = \theta_1(\varkappa)\Re_{k-1}(\Im_1\varkappa) + \theta_2(\varkappa)\Re_{k-1}(\Im_2\varkappa) + \theta_3(\varkappa)\Re_{k-1}(\Im_3\varkappa) + \cdots + \theta_n(\varkappa)\Re_{k-1}(\Im_n\varkappa). \tag{2.3}$$

Proof. Define the operator $Z : \psi \to \psi$ as

$$(Z\Re)(\varkappa) = \theta_1(\varkappa)\Re(\Im_1\varkappa) + \theta_2(\varkappa)\Re(\Im_2\varkappa) + \theta_3(\varkappa)\Re(\Im_3\varkappa) + \cdots + \theta_n(\varkappa)\Re(\Im_n\varkappa), \tag{2.4}$$

for all $\varkappa \in J$ and $\Re \in \psi$. For all $\Re \in J$, we get $(Z\Re)(0) = 0$. As $Z : \psi \to \psi$ is continuous, then $\|Z\Re\| \leq \infty$, for all $\Re \in \psi$. There is no doubt that the unique FP of the operator Z is equivalent to the US to Equation (1.8).

Since Z is linear, one has

$$\|Z\Re_1 - Z\Re_2\| = \|Z(\Re_1 - \Re_2)\|, \text{ for all } \Re_1, \Re_2 \in \psi.$$

Finally, we develop the following framework for examining the expression $\|Z\Re_1 - Z\Re_2\|$: for all $\Re_1, \Re_2 \in \psi$,

$$\wp_{\lambda,\chi} = \frac{Z(\Re_1 - \Re_2)(\lambda) - Z(\Re_1 - \Re_2)(\chi)}{\lambda - \chi}, \ \lambda, \chi \in J, \ \lambda \neq \chi.$$

For all $\lambda, \chi \in J$ with$\lambda \neq \chi$, one has

$$\begin{aligned}
\wp_{\lambda,\chi} &= \frac{1}{\lambda-\chi}\left(\theta_1(\lambda)\Re\left(\Im_1\lambda\right)+\theta_2(\lambda)\Re\left(\Im_2\lambda\right)+\theta_3(\lambda)\Re\left(\Im_3\lambda\right)+\cdots+\theta_n(\lambda)\Re\left(\Im_n\lambda\right)\right.\\
&\left.-\theta_1(\chi)\Re\left(\Im_1\chi\right)-\theta_2(\chi)\Re\left(\Im_2\chi\right)-\theta_3(\chi)\Re\left(\Im_3\chi\right)-\cdots-\theta_n(\chi)\Re\left(\Im_n\chi\right)\right)\\
&= \frac{1}{\lambda-\chi}\left(\left[\theta_1(\lambda)\Re\left(\Im_1\lambda\right)-\theta_1(\lambda)\Re\left(\Im_1\chi\right)\right]+\left[\theta_2(\lambda)\Re\left(\Im_2\lambda\right)-\theta_2(\lambda)\Re\left(\Im_2\chi\right)\right]\right.\\
&\left[\theta_3(\lambda)\Re\left(\Im_3\lambda\right)-\theta_3(\lambda)\Re\left(\Im_3\chi\right)\right]+\cdots+\left[\theta_n(\lambda)\Re\left(\Im_n\lambda\right)-\theta_n(\lambda)\Re\left(\Im_n\chi\right)\right]\\
&+\left[\theta_1(\lambda)\Re\left(\Im_1\chi\right)-\theta_1(\chi)\Re\left(\Im_1\chi\right)\right]+\left[\theta_2(\lambda)\Re\left(\Im_2\chi\right)-\theta_2(\chi)\Re\left(\Im_2\chi\right)\right]\\
&\left.+\left[\theta_3(\lambda)\Re\left(\Im_3\chi\right)-\theta_3(\chi)\Re\left(\Im_3\chi\right)\right]+\cdots+\left[\theta_n(\lambda)\Re\left(\Im_n\chi\right)-\theta_n(\chi)\Re\left(\Im_n\chi\right)\right]\right),
\end{aligned}$$

which yields,

$$\begin{aligned}
\left|\wp_{\lambda,\chi}\right| &\le \left|\frac{1}{\lambda-\chi}\right|\left(\left|\theta_1(\lambda)\Re\left(\Im_1\lambda\right)-\theta_1(\lambda)\Re\left(\Im_1\chi\right)\right|+\left|\theta_2(\lambda)\Re\left(\Im_2\lambda\right)-\theta_2(\lambda)\Re\left(\Im_2\chi\right)\right|\right.\\
&+\left|\theta_3(\lambda)\Re\left(\Im_3\lambda\right)-\theta_3(\lambda)\Re\left(\Im_3\chi\right)\right|+\cdots+\left|\theta_n(\lambda)\Re\left(\Im_n\lambda\right)-\theta_n(\lambda)\Re\left(\Im_n\chi\right)\right|\\
&+\left|\theta_1(\lambda)\Re\left(\Im_1\chi\right)-\theta_1(\chi)\Re\left(\Im_1\chi\right)\right|+\left|\theta_2(\lambda)\Re\left(\Im_2\chi\right)-\theta_2(\chi)\Re\left(\Im_2\chi\right)\right|\\
&\left.+\left|\theta_3(\lambda)\Re\left(\Im_3\chi\right)-\theta_3(\chi)\Re\left(\Im_3\chi\right)\right|+\cdots+\left|\theta_n(\lambda)\Re\left(\Im_n\chi\right)-\theta_n(\chi)\Re\left(\Im_n\chi\right)\right|\right).
\end{aligned}$$

Our goal in this situation is to benefit from the norm specification that is given in Equation (2.1). As a result, we can write

$$\begin{aligned}
\left|\wp_{\lambda,\chi}\right| &\le \frac{\left|\theta_1(\lambda)\Re\left(\Im_1\lambda\right)-\theta_1(\lambda)\Re\left(\Im_1\chi\right)\right|}{\left|\Im_1\lambda-\Im_1\chi\right|}\times\frac{\left|\Im_1\lambda-\Im_1\chi\right|}{|\lambda-\chi|}+\frac{\left|\theta_2(\lambda)\Re\left(\Im_2\lambda\right)-\theta_2(\lambda)\Re\left(\Im_2\chi\right)\right|}{\left|\Im_2\lambda-\Im_2\chi\right|}\times\frac{\left|\Im_2\lambda-\Im_2\chi\right|}{|\lambda-\chi|}\\
&+\frac{\left|\theta_3(\lambda)\Re\left(\Im_3\lambda\right)-\theta_3(\lambda)\Re\left(\Im_3\chi\right)\right|}{\left|\Im_3\lambda-\Im_3\chi\right|}\times\frac{\left|\Im_3\lambda-\Im_3\chi\right|}{|\lambda-\chi|}+\cdots\\
&+\frac{\left|\theta_n(\lambda)\Re\left(\Im_n\lambda\right)-\theta_n(\lambda)\Re\left(\Im_n\chi\right)\right|}{\left|\Im_n\lambda-\Im_n\chi\right|}\times\frac{\left|\Im_n\lambda-\Im_n\chi\right|}{|\lambda-\chi|}\\
&+\frac{\left|\theta_1(\lambda)-\theta_1(\chi)\right|}{\lambda-\chi}\times\frac{\left|\Re\left(\Im_1\chi\right)-\Re(0)\right|}{\left|\Im_1\chi-0\right|}\times\left|\Im_1\chi\right|+\frac{\left|\theta_2(\lambda)-\theta_2(\chi)\right|}{\lambda-\chi}\times\frac{\left|\Re\left(\Im_2\chi\right)-\Re(0)\right|}{\left|\Im_2\chi-0\right|}\times\left|\Im_2\chi\right|\\
&+\frac{\left|\theta_3(\lambda)-\theta_3(\chi)\right|}{\lambda-\chi}\times\frac{\left|\Re\left(\Im_3\chi\right)-\Re(0)\right|}{\left|\Im_3\chi-0\right|}\times\left|\Im_3\chi\right|+\cdots\\
&+\frac{\left|\theta_n(\lambda)-\theta_n(\chi)\right|}{\lambda-\chi}\times\frac{\left|\Re\left(\Im_n\chi\right)-\Re(0)\right|}{\left|\Im_n\chi-0\right|}\times\left|\Im_n\chi\right|.
\end{aligned}$$

It follows from (H_2), (H_3) and (2.1) that

$$\begin{aligned}
\left|\wp_{\lambda,\chi}\right| &\le \sigma_1\nu_1\left\|\Re_1-\Re_2\right\|+\sigma_2\nu_2\left\|\Re_1-\Re_2\right\|+\sigma_3\nu_3\left\|\Re_1-\Re_2\right\|+\cdots+\sigma_n\nu_n\left\|\Re_1-\Re_2\right\|\\
&+\eta_1\left\|\Re_1-\Re_2\right\|+\eta_2\left\|\Re_1-\Re_2\right\|+\eta_3\left\|\Re_1-\Re_2\right\|+\cdots+\eta_n\left\|\Re_1-\Re_2\right\|\\
&= \left(\sum_{i=1}^{n}\sigma_i\nu_i+\sum_{i=1}^{n}\eta_i\right)\left\|\Re_1-\Re_2\right\|\\
&= \mho\left\|\Re_1-\Re_2\right\|,
\end{aligned}$$

where $\mho$ is defined by (2.2). Hence

$$d\left(Z\Re_1, Z\Re_2\right)=\left\|Z\Re_1-Z\Re_2\right\|\le\mho\left\|\Re_1-\Re_2\right\|=\mho d\left(\Re_1,\Re_2\right),$$

where $d : \psi \times \psi \to \mathbb{R}$ is induced metric on ψ. As $\mho < 1$, then Z is a contraction. We infer that the PI defined in (2.3) converges to the US of Equation (1.8) on J because of the important FP finding assumed by Theorem 1.1. □

Here, we present the following more relaxed conditions as opposed to the original circumstances (H_2) and (H_3):

$(\widetilde{H}_2)$ The maps $\theta_i : J \to J$ with $\theta_1(0) = 0 = \theta_2(0)$, for $i = 1, 2, ..., n$, satisfy the following:

$$d\left(\theta_i \ell, \theta_i \widetilde{\ell}\right) \le \mu_i d\left(\ell, \widetilde{\ell}\right), \text{ for all } \ell, \widetilde{\ell} \in J,$$

and

$$|\theta_i(\varkappa)| \le \widehat{\nu}, \text{ for all } \varkappa \in J \text{ and } \widehat{\nu} \ge 0.$$

$(\widetilde{H}_3)$ The maps $\Im_i : J \to J$ with $\Im_3(0) = \Im_4(0) = ... = \Im_n(0) = 0$, for $i = 1, 2, ..., n$, fulfill the inequalities shown below:

$$d\left(\Im_i \ell, \Im_i \widetilde{\ell}\right) \le \widehat{\sigma}_i d\left(\ell, \widetilde{\ell}\right), \text{ for all } \ell, \widetilde{\ell} \in J, \text{ and } \widehat{\sigma}_i < 1,$$

with $\widehat{\sigma}_1 \le \widehat{\sigma}_2 \le \widehat{\sigma}_3 \le ... \le \widehat{\sigma}_n$ and

$$|\Im_i(\varkappa)| \le \widehat{\eta}, \text{ for all } \varkappa \in J \text{ and } \eta \ge 0.$$

We can draw the following conclusion from the aforementioned findings:

Corollary 2.1. *In the light of assumptions* (H_1), $(\widetilde{H}_2)$, (H_3), *the general FE (1.8) owns a US, if* $\widetilde{\mho} < 1$, *where*

$$\widetilde{\mho} = \widehat{\nu} \sum_{i=1}^{n} \sigma_i + \sum_{i=1}^{n} \mu_i \eta_i, \ i \in \mathbb{N}.$$

Moreover, for all $k \in \mathbb{N}$, *the sequence* $\{\Re_k\}$ *in* ψ *converges to the US of (1.8) with*

$$\Re_k(\varkappa) = \theta_1(\varkappa)\Re_{k-1}(\Im_1\varkappa) + \theta_2(\varkappa)\Re_{k-1}(\Im_2\varkappa) + \theta_3(\varkappa)\Re_{k-1}(\Im_3\varkappa) + \cdots + \theta_n(\varkappa)\Re_{k-1}(\Im_n\varkappa).$$

Corollary 2.2. *Under the hypotheses* (H_1), (H_2) *and* $\left(\widetilde{H}_3\right)$, *the general FE (1.8) has a US, if* $\widehat{\mho} < 1$, *where*

$$\widehat{\mho} = \widehat{\sigma}_n \sum_{i=1}^{n} \nu_i + \widehat{\eta}, \ i \in \mathbb{N}.$$

Moreover, for all $k \in \mathbb{N}$, *the sequence* $\{\Re_k\}$ *in* ψ *converges to the US of (1.8) with*

$$\Re_k(\varkappa) = \theta_1(\varkappa)\Re_{k-1}(\Im_1\varkappa) + \theta_2(\varkappa)\Re_{k-1}(\Im_2\varkappa) + \theta_3(\varkappa)\Re_{k-1}(\Im_3\varkappa) + \cdots + \theta_n(\varkappa)\Re_{k-1}(\Im_n\varkappa).$$

After that, we'll talk about a less stringent criterion than $(\widetilde{H}_2)$ that can be used to prove that there is only one possible solution to equation (1.8).

$\left(\widehat{H}_2\right)$ The maps $\theta_i : J \to J$ with $\theta_1(0) = 0 = \theta_2(0)$, for $i = 1, 2, ..., n$, satisfy the following:

$$d\left(\theta_i \ell, \theta_i \widetilde{\ell}\right) \leq \mu_i d\left(\ell, \widetilde{\ell}\right), \text{ for all } \ell, \widetilde{\ell} \in J,$$

with $\mu_1 \leq \mu_1 \leq ... \leq \mu_2$, and

$$|\theta_i(\varkappa)| \leq \widehat{\nu}, \text{ for all } \varkappa \in J \text{ and } \widehat{\nu} \geq 0.$$

The previous conclusion directly leads to the next corollary.

Corollary 2.3. *Assume that* (H_1), $(\widehat{H}_2)$ *and* $\left(\widetilde{H}_3\right)$ *are true, then the general FE (1.8) has a US, if* $\overline{\mho} < 1$, *where*

$$\overline{\mho} = \widehat{\sigma}_n \widehat{\nu} + \mu_n \widehat{\eta} \ n \in \mathbb{N}.$$

Moreover, for all $k \in \mathbb{N}$, *the sequence* $\{\Re_k\}$ *in* ψ *converges to the US of (1.8) with*

$$\Re_k(\varkappa) = \theta_1(\varkappa)\Re_{k-1}(\Im_1\varkappa) + \theta_2(\varkappa)\Re_{k-1}(\Im_2\varkappa) + \theta_3(\varkappa)\Re_{k-1}(\Im_3\varkappa) + \cdots + \theta_n(\varkappa)\Re_{k-1}(\Im_n\varkappa).$$

Remark 2.1. *Using the suggested iteration strategy, we cannot anticipate rapid convergence because the PI converges linearly. We might overcome this barrier with the appropriate accelerative method, for additional study in this area, we refer to [32–34].*

Remark 2.2. *The proposed equation (1.8) generalizes a wide range of FEs that are already present in the literature as follows:*

(i) *If we take* $\theta_1(\varkappa) = \varkappa$, $\Im_1\varkappa = \xi_1 + (1-\xi_1)\varkappa$, $\theta_2(\varkappa) = (1-\varkappa)$, $\Im_2\varkappa = (1-\xi_2)\varkappa$, *and* $\theta_3(\varkappa) = \theta_4(\varkappa) = ... = \theta_n(\varkappa) = 0$, *for* $0 < \xi_1 \leq \xi_2 < 1$ *and for* $n \in \mathbb{N}$ *in (1.8), we have the FE (1.1), where* $\Re : J \to \mathbb{R}$, $\Re(0) = 0$ *and* $\Re(1) = 1$.

(ii) *Putting* $\theta_1(\varkappa) = \varkappa$, $\Im_1\varkappa = \xi_1\varkappa + (1-\xi_1)$, $\theta_2(\varkappa) = (1-\varkappa)$, $\Im_2\varkappa = \xi_2\varkappa$ *and* $\theta_3(\varkappa) = \theta_4(\varkappa) = ... = \theta_n(\varkappa) = 0$, *for* $0 < \xi_1 \leq \xi_2 < 1$ *and for* $n \in \mathbb{N}$ *in (1.8), we get the FE (1.2), where* $\Re : J \to \mathbb{R}$, $\Re(0) = 0$ *and* $\Re(1) = 1$.

(iii) *Setting* $\theta_1(\varkappa) = \varkappa$, $\Im_1\varkappa = \xi_1\varkappa + (1-\xi_1)\beta_1$, $\theta_2(\varkappa) = (1-\varkappa)$, $\Im_2\varkappa = \xi_2\varkappa + (1-\xi_2)\beta_2$ *and* $\theta_3(\varkappa) = \theta_4(\varkappa) = ... = \theta_n(\varkappa) = 0$, *for* $0 < \xi_1 \leq \xi_2 < 1$, $\beta_1, \beta_2 \in J$ *and for* $n \in \mathbb{N}$ *in (1.8), we obtain the FE (1.3), where* $\Re : J \to \mathbb{R}$.

(iv) *Letting* $\theta_1(\varkappa) = \varkappa$, $\theta_2(\varkappa) = (1-\varkappa)$ *and* $\theta_3(\varkappa) = \theta_4(\varkappa) = ... = \theta_n(\varkappa) = 0$, *for* $n \in \mathbb{N}$ *in (1.8), one has the FE (1.4), where* $\Re : J \to \mathbb{R}$ *and* $\Im_1, \Im_2 : J \to J$ *are given CMs.*

(v) *If we set* $\theta_1(\varkappa) = \frac{\varkappa - \varrho_1}{\varrho_2 - \varrho_1}$, $\theta_2(\varkappa) = (1 - \frac{\varkappa - \varrho_1}{\varrho_2 - \varrho_1})$ *and* $\theta_3(\varkappa) = \theta_4(\varkappa) = ... = \theta_n(\varkappa) = 0$, *for* $n \in \mathbb{N}$ *in (1.8), we have the FE (1.5), where* $\Re : [\varrho_1, \varrho_2] \to \mathbb{R}$ *and* $\Im : [\varrho_1, \varrho_2] \to [\varrho_1, \varrho_2]$ *are CMs, for arbitrary interval* $[\varrho_1, \varrho_2]$.

(vi) *If we let* $\theta_1(\varkappa) = \tau\vartheta(\varkappa)$, $\theta_2(\varkappa) = (1-\tau)\vartheta(\varkappa)$, $\theta_3(\varkappa) = \tau(1-\vartheta(\varkappa))$, $\theta_4(\varkappa) = (1-\tau)(1-\vartheta(\varkappa))$ *and* $\theta_5(\varkappa) = \theta_6(\varkappa) = ... = \theta_n(\varkappa) = 0$, *for* $n \in \mathbb{N}$ *in (1.8), we obtain the FE (1.6), where* $\Im_i : J \to J$, $(i = 1, 2, 3, 4)$ *are given maps and* $\vartheta : J \to J$ *is the proportional event probability for each* $\tau, \varkappa \in J$.

(vi) *If we put* $\theta_1(\varkappa) = \left(\frac{\varkappa-\varrho_1}{\varrho_2-\varrho_1}\right)\left(\frac{\tau-\varrho_1}{\varrho_2-\varrho_1}\right)$, $\theta_2(\varkappa) = \left(\frac{\varkappa-\varrho_1}{\varrho_2-\varrho_1}\right)\left(1-\frac{\tau-\varrho_1}{\varrho_2-\varrho_1}\right)$, $\theta_3(\varkappa) = \left(1-\frac{\varkappa-\varrho_1}{\varrho_2-\varrho_1}\right)\left(\frac{\tau-\varrho_1}{\varrho_2-\varrho_1}\right)$, $\theta_4(\varkappa) = \left(1-\frac{\varkappa-\varrho_1}{\varrho_2-\varrho_1}\right)\left(1-\frac{\tau-\varrho_1}{\varrho_2-\varrho_1}\right)$ *and* $\theta_5(\varkappa) = \theta_6(\varkappa) = \ldots = \theta_n(\varkappa) = 0$, *for* $n \in \mathbb{N}$ *in (1.8), we obtain the FE (1.7), where* $\Re : [\varrho_1, \varrho_2] \to \mathbb{R}$, $\Im_i : [\varrho_1, \varrho_2] \to [\varrho_1, \varrho_2]$, $(i = 1, 2, 3, 4)$ *are given mappings and* $\varkappa \in [\varrho_1, \varrho_2]$.

3. Stability Analysis

Almost all disciplines of mathematical computing should be concerned with the following main issue: Is there ever an equivalency between a mathematical entity that approximates a criteria and one that clearly meets that attribute? When we turn our attention to FEs, we could run into the problem of determining whether an equation's solution that deviates only slightly from an equation is close to the supplied equation's solution. Therefore, it is necessary to discuss the stability of the suggested equation (1.8) below; for more information, see [35–44].

Theorem 3.1. *Assume that all requirements of Theorem 2.1 are verified. Then* $Z\Re = \Re$ *fulfills* (H_4), *where* Z *is described as (2.4).*

Proof. Consider $\Re \in \psi$ with $d(Z\Re, \Re) \leq \varpi(\Re)$. Thank to Theorem 2.1, there exists a US $\Re^* \in \psi$ with $Z\Re^* = \Re^*$. Therefore, we can write

$$d(\Re, \Re^*) \leq d(\Re, Z\Re) + d(Z\Re, \Re^*) \leq \varpi(\Re) + d(Z\Re, Z\Re^*) \leq \varpi(\Re) + \mho d(\Re, \Re^*),$$

where $\varpi : \psi \to [0, \infty)$ and $\mho$ is given by (2.2). As a result,

$$d(\Re, \Re^*) \leq \Delta\varpi(\Re),$$

where $\Delta = \frac{1}{1-\mho}$. □

Also, we can draw this conclusion about the stability of the HU type from this analysis.

Corollary 3.1. *Assume that all hypotheses of Theorem 2.1 are satisfied. Then* $Z\Re = \Re$ *fulfills* (H_5), *where* Z *is described as (2.4).*

4. Supportive Applications

To support and strengthen the theoretical results, we present the two examples below:

Example 4.1. *Consider the following FE:*

$$\begin{aligned} \Re(\varkappa) &= \left(\frac{\delta_1(\varkappa)}{3}\right)\Re\left(\frac{\phi_1}{8} + \frac{(1-\phi_1)\varkappa}{8}\right) + \left(\frac{\delta_2(\varkappa)}{3}\right)\Re\left(\frac{\phi_2}{6} + \frac{(1-\phi_2)\varkappa}{6}\right) \\ &+ \left(\frac{\delta_3(\varkappa)}{3}\right)\Re\left(\frac{\phi_3}{4} + \frac{(1-\phi_3)\varkappa}{4}\right) + \left(\frac{(1-\delta_1)\varkappa}{3}\right)\Re\left(\frac{\phi_4\varkappa}{7}\right) \\ &+ \left(\frac{(1-\delta_2)\varkappa}{3}\right)\Re\left(\frac{\phi_5\varkappa}{10}\right) + \left(\frac{(1-\delta_3)\varkappa}{3}\right)\Re\left(\frac{\phi_6\varkappa}{13}\right) + (1-\varkappa)\Re\left(\frac{\phi_7}{15}\right), \quad (4.1) \end{aligned}$$

where $\Re : J \to \mathbb{R}$*. Equation (4.1) is a special case of Equation (1.8) with* $\theta_i, \Im_i : J \to J$*,* $(i = 1, 2, ..., 7)$*, and*

$$\begin{cases} \theta_1(\varkappa) = \frac{\delta_1(\varkappa)}{3}, \\ \theta_2(\varkappa) = \frac{\delta_2(\varkappa)}{3}, \\ \theta_3(\varkappa) = \frac{\delta_3(\varkappa)}{3}, \\ \theta_4(\varkappa) = \frac{(1-\delta_1)\varkappa}{3}, \\ \theta_5(\varkappa) = \frac{(1-\delta_2)\varkappa}{3}, \\ \theta_6(\varkappa) = \frac{(1-\delta_3)\varkappa}{3}, \\ \theta_7(\varkappa) = 1-\varkappa, \end{cases} \quad and \quad \begin{cases} \Im_1(\varkappa) = \frac{\phi_1}{8} + \frac{(1-\phi_1)\varkappa}{8}, \\ \Im_2(\varkappa) = \frac{\phi_2}{6} + \frac{(1-\phi_2)\varkappa}{6}, \\ \Im_3(\varkappa) = \frac{\phi_3}{4} + \frac{(1-\phi_3)\varkappa}{4}, \\ \Im_4(\varkappa) = \frac{\phi_4\varkappa}{7}, \\ \Im_5(\varkappa) = \frac{\phi_5\varkappa}{10}, \\ \Im_6(\varkappa) = \frac{\phi_6\varkappa}{13}, \\ \Im_7(\varkappa) = \frac{\phi_7\varkappa}{15}, \end{cases} \tag{4.2}$$

$\theta_8(\varkappa) = \theta_9(\varkappa) = ... = \theta_n(\varkappa) = 0$*, for each* $\varkappa \in J$*, where* $\phi_i \in (0, 1)$*,* $(i = 1, 2, ..., 7)$*.*

The use of the FE (4.1) to analyze a paradise fish's behavior (Figure 1) in a three-choice scenario (A, C, or B) has many implications for PLT. In this context, $\Im_i$ *represent the seven occurrences, depending on the chosen side and reward, whereas* θ_i *denote their respective probability (Table 2), where* $i = 1, 2, ..., 7$*.*

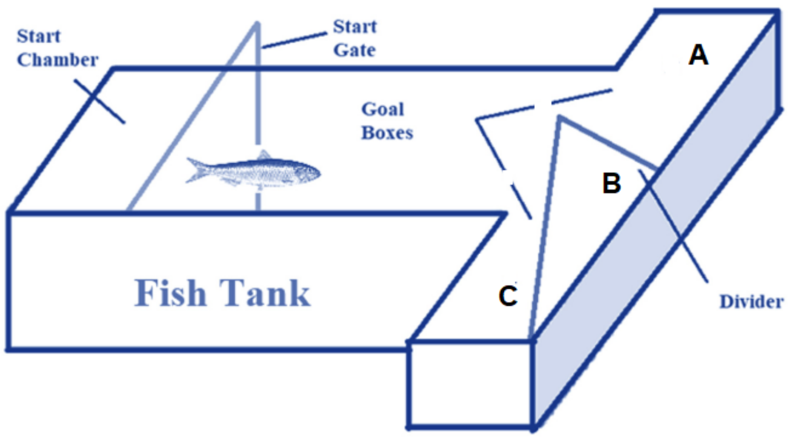

Figure 1: The activity of the paradise fesh under three-option circumstances in a fish tank.

Table2: Seven potential outcomes and their associated probabilities

Reactions (left or right side)	Results (reward or no reward)	Associated probabilities
A (left)	**Food side (reward)**	$\theta_1(\varkappa) = \frac{\delta_1\varkappa}{3}$
B (right)	**Non-food side (no reward)**	$\theta_2(\varkappa) = \frac{\delta_2\varkappa}{3}$
C (left)	**Food side (reward)**	$\theta_3(\varkappa) = \frac{\delta_3\varkappa}{3}$
A (left)	**Non-food side (no reward)**	$\theta_4(\varkappa) = \frac{(1-\delta_1)\varkappa}{3}$
B (right)	**Food side (reward)**	$\theta_5(\varkappa) = \frac{(1-\delta_2)\varkappa}{3}$
C (left)	**Non-food side (no reward)**	$\theta_6(\varkappa) = \frac{(1-\delta_3)\varkappa}{3}$
No side selection (A or B or C)	**Blank trial**	$\theta_4(\varkappa) = 1-\varkappa$

To demonstrate that the suggested equation (4.1) possesses a US, we will use Theorem 2.1. Obviously, the maps θ_i and $\Im_i$, $i = 1, 2, \dots, 7$, defined in (4.2) verify hypotheses (H_2) and (H_3) respectively, with coefficients

$$\begin{cases} |\theta_1(\varkappa)| \le \nu_1 = \frac{\delta_1}{3}, \\ |\theta_2(\varkappa)| \le \nu_2 = \frac{\delta_2}{3}, \\ |\theta_3(\varkappa)| \le \nu_3 = \frac{\delta_3}{3}, \\ |\theta_4(\varkappa)| \le \nu_4 = \frac{1-\delta_1}{3}, \\ |\theta_5(\varkappa)| \le \nu_5 = \frac{1-\delta_2}{3}, \\ |\theta_6(\varkappa)| \le \nu_6 = \frac{1-\delta_3}{3}, \\ |\theta_7(\varkappa)| \le \nu_7 = 1, \end{cases} \quad \begin{cases} \sigma_1 = \frac{(1-\phi_1)}{8}, \\ \sigma_2 = \frac{(1-\phi_2)}{6}, \\ \sigma_3 = \frac{(1-\phi_3)}{4}, \\ \sigma_4 = \frac{\phi_4}{7}, \\ \sigma_5 = \frac{\phi_5}{10}, \\ \sigma_6 = \frac{\phi_6}{13}, \\ \sigma_7 = \frac{\phi_7}{15}, \end{cases} \quad \text{and} \quad \begin{cases} |\Im_1(\varkappa)| \le \eta_1 = \frac{1}{8}, \\ |\Im_2(\varkappa)| \le \eta_2 = \frac{1}{6}, \\ |\Im_3(\varkappa)| \le \eta_3 = \frac{1}{4}, \\ |\Im_4(\varkappa)| \le \eta_4 = \frac{1}{7}, \\ |\Im_5(\varkappa)| \le \eta_5 = \frac{1}{10}, \\ |\Im_6(\varkappa)| \le \eta_6 = \frac{1}{13}, \\ |\Im_7(\varkappa)| \le \eta_7 = \frac{1}{15}, \end{cases}$$

for all $\varkappa, \delta_1, \delta_2, \delta_3 \in J$. If

$$\begin{aligned} \mho &= \sum_{i=1}^{7} \sigma_i \nu_i + \sum_{i=1}^{n} \eta_i = \frac{(1-\phi_1)\delta_1}{24} + \frac{(1-\phi_2)\delta_2}{18} + \frac{(1-\phi_3)\delta_3}{12} + \frac{(1-\delta_1)\,\phi_4}{21} \\ &\quad + \frac{(1-\delta_2)\,\phi_5}{30} + \frac{(1-\delta_3)\,\phi_5}{39} + \frac{\phi_7}{15} + \frac{2027}{2184} \\ &< 1, \end{aligned}$$

then all requirements of Theorem 2.1 are fulfilled. Hence, the FE (4.1) has at least one solution.

Furthermore, imagine that we make an estimate computation, say $\Re_0(\varkappa) = \varkappa$, for all $\varkappa \in J$, the solution to equation (4.1) will be reached in the following iteration, for each $n \in \mathbb{N}$,

$$\begin{aligned} \Re_1(\varkappa) &= \left(\frac{\delta_1(\varkappa)}{3}\right)\Re_0\left(\frac{\phi_1}{8} + \frac{(1-\phi_1)\varkappa}{8}\right) + \left(\frac{\delta_2(\varkappa)}{3}\right)\Re_0\left(\frac{\phi_2}{6} + \frac{(1-\phi_2)\varkappa}{6}\right) \\ &\quad + \left(\frac{\delta_3(\varkappa)}{3}\right)\Re_0\left(\frac{\phi_3}{4} + \frac{(1-\phi_3)\varkappa}{4}\right) + \left(\frac{(1-\delta_1)\varkappa}{3}\right)\Re_0\left(\frac{\phi_4\varkappa}{7}\right) \\ &\quad + \left(\frac{(1-\delta_2)\,\varkappa}{3}\right)\Re_0\left(\frac{\phi_5\varkappa}{10}\right) + \left(\frac{(1-\delta_3)\,\varkappa}{3}\right)\Re_0\left(\frac{\phi_6\varkappa}{13}\right) + (1-\varkappa)\Re_0\left(\frac{\phi_7}{15}\right), \\ \Re_2(\varkappa) &= \left(\frac{\delta_1(\varkappa)}{3}\right)\Re_1\left(\frac{\phi_1}{8} + \frac{(1-\phi_1)\varkappa}{8}\right) + \left(\frac{\delta_2(\varkappa)}{3}\right)\Re_1\left(\frac{\phi_2}{6} + \frac{(1-\phi_2)\varkappa}{6}\right) \\ &\quad + \left(\frac{\delta_3(\varkappa)}{3}\right)\Re_1\left(\frac{\phi_3}{4} + \frac{(1-\phi_3)\varkappa}{4}\right) + \left(\frac{(1-\delta_1)\varkappa}{3}\right)\Re_1\left(\frac{\phi_4\varkappa}{7}\right) \\ &\quad + \left(\frac{(1-\delta_2)\,\varkappa}{3}\right)\Re_1\left(\frac{\phi_5\varkappa}{10}\right) + \left(\frac{(1-\delta_3)\,\varkappa}{3}\right)\Re_1\left(\frac{\phi_6\varkappa}{13}\right) + (1-\varkappa)\Re_1\left(\frac{\phi_7}{15}\right), \\ &\;\;\vdots \\ \Re_n(\varkappa) &= \left(\frac{\delta_1(\varkappa)}{3}\right)\Re_{n-1}\left(\frac{\phi_1}{8} + \frac{(1-\phi_1)\varkappa}{8}\right) + \left(\frac{\delta_2(\varkappa)}{3}\right)\Re_{n-1}\left(\frac{\phi_2}{6} + \frac{(1-\phi_2)\varkappa}{6}\right) \\ &\quad + \left(\frac{\delta_3(\varkappa)}{3}\right)\Re_{n-1}\left(\frac{\phi_3}{4} + \frac{(1-\phi_3)\varkappa}{4}\right) + \left(\frac{(1-\delta_1)\varkappa}{3}\right)\Re_{n-1}\left(\frac{\phi_4\varkappa}{7}\right) \\ &\quad + \left(\frac{(1-\delta_2)\,\varkappa}{3}\right)\Re_{n-1}\left(\frac{\phi_5\varkappa}{10}\right) + \left(\frac{(1-\delta_3)\,\varkappa}{3}\right)\Re_{n-1}\left(\frac{\phi_6\varkappa}{13}\right) + (1-\varkappa)\Re_{n-1}\left(\frac{\phi_7}{15}\right). \end{aligned}$$

Further, we get

$$\begin{aligned}\Delta &= \frac{1}{1-\mho} \\ &= \frac{1}{1-\left[\frac{(1-\phi_1)\delta_1}{24}+\frac{(1-\phi_2)\delta_2}{18}+\frac{(1-\phi_3)\delta_3}{12}+\frac{(1-\delta_1)\phi_4}{21}+\frac{(1-\delta_2)\phi_5}{30}+\frac{(1-\delta_3)\phi_5}{39}+\frac{\phi_7}{15}+\frac{2027}{2184}\right]} > 0.\end{aligned}$$

Now, assume that the function $\Re \in J$ *fulfills*

$$d\left(Z\Re, \Re\right) \leq \varpi\left(\Re\right).$$

Hence, using Theorem 3.1, we deduce that the US $\Re^* \in J$ *of the FE (4.1) is HUR with*

$$Z\Re^* = \Re^* \text{ and } d\left(\Re, \Re^*\right) \leq \Delta\varpi\left(\Re\right).$$

Example 4.2. *Consider the following FE:*

$$\Re(\varkappa) = \varkappa\Re\left(\frac{6\varkappa+7}{41}\right) + (1-\varkappa)\,\Re\left(\frac{\varkappa}{23}\right) + \varkappa\Re\left(\frac{5+\varkappa}{35}\right). \tag{4.3}$$

Equation (4.3) is a special form of Equation (1.8) with $\theta_j, \Im_i : J \to J$, $(i = 1, 2, 3)$, *and*

$$\begin{cases} \theta_1(\varkappa) = \varkappa, \\ \theta_2(\varkappa) = (1-\varkappa), \\ \theta_3(\varkappa) = \varkappa, \end{cases} \quad \text{and} \quad \begin{cases} \Im_1(\varkappa) = \frac{6\varkappa+7}{41}, \\ \Im_2(\varkappa) = \frac{\varkappa}{23}, \\ \Im_3(\varkappa) = \frac{5+\varkappa}{35}, \end{cases} \tag{4.4}$$

$\theta_4(\varkappa) = \theta_5(\varkappa) = \dots = \theta_n(\varkappa) = 0$, *for each* $\varkappa \in J$. *In order to prove that the FE (4.4) has a US, we apply Theorem 2.1. Clearly, the maps* θ_i *and* $\Im_i$, $i = 1, 2, 3$ *defined in (4.4) satisfy the assumptions* (H_2) *and* (H_3) *respectively, with coefficients*

$$\begin{cases} |\theta_1(\varkappa)| \leq \nu_1 = 1, \\ |\theta_2(\varkappa)| \leq \nu_2 = 1, \\ |\theta_3(\varkappa)| \leq \nu_3 = 1, \end{cases} \quad \begin{cases} \sigma_1 = \frac{6}{41}, \\ \sigma_2 = \frac{1}{23}, \\ \sigma_3 = \frac{1}{35}, \end{cases} \quad \text{and} \quad \begin{cases} |\Im_1(\varkappa)| \leq \eta_1 = \frac{13}{41}, \\ |\Im_2(\varkappa)| \leq \eta_2 = \frac{1}{23}, \\ |\Im_3(\varkappa)| \leq \eta_3 = \frac{6}{35}, \end{cases}$$

for all $\varkappa \in J$. *Further,* $\mho = \frac{3538}{4715} < 1$. *Hence, all axioms of Theorem 2.1 are fulfilled. Thus, the FE (4.3) has at least one solution.*

In addition, imagine that we make an estimate computation, say $\Re_0(\varkappa) = \varkappa$, *for all* $\varkappa \in J$, *the solution to equation (4.3) will converge in the following iteration for each* $n \in \mathbb{N}$,

$$\begin{aligned} \Re_1(\varkappa) &= \varkappa\Re_0\left(\frac{6\varkappa+7}{41}\right) + (1-\varkappa)\,\Re_0\left(\frac{\varkappa}{23}\right) + \varkappa\Re_0\left(\frac{5+\varkappa}{35}\right), \\ \Re_2(\varkappa) &= \varkappa\Re_1\left(\frac{6\varkappa+7}{41}\right) + (1-\varkappa)\,\Re_1\left(\frac{\varkappa}{23}\right) + \varkappa\Re_1\left(\frac{5+\varkappa}{35}\right), \\ &\vdots \\ \Re_n(\varkappa) &= \varkappa\Re_{n-1}\left(\frac{6\varkappa+7}{41}\right) + (1-\varkappa)\,\Re_{n-1}\left(\frac{\varkappa}{23}\right) + \varkappa\Re_{n-1}\left(\frac{5+\varkappa}{35}\right). \end{aligned}$$

Also, we get

$$\Delta = \frac{1}{1-\mho} = \frac{1}{1-\frac{3538}{4715}} = \frac{4715}{1177} > 0.$$

Now, assume that the function $\Re \in J$ fulfills

$$d\left(Z\Re, \Re\right) \le \varpi\left(\Re\right).$$

Hence, using Theorem 3.1, we deduce that the US $\Re^* \in J$ of the FE (4.3) is HUR with

$$Z\Re^* = \Re^* \text{ and } d\left(\Re, \Re^*\right) \le \Delta\varpi\left(\Re\right).$$

5. Conclusion

The goal of mathematical psychology, a branch of psychology, is to use mathematics to illustrate difficulties in PLT. From this point, the bulk of learning research tries to compare outcomes across multiple trials in order to determine the random chance that contributes to a stochastic process. In the context of PLT, our work created a large class of FEs that may be used to analyze a variety of studies. The Banach FP theorem has been used to determine whether the given equation (1.8) has a US. We gave a quick stability study of the solution to the suggested FE. Ultimately, the key points were explained with the help of two examples.

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