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New Characterizations of the Jeribi Essential Spectrum

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Abstract. In this paper, we give several characterizations of the Jeribi essential spectrum of a bounded linear operator defined on a Banach space by using the notion of almost weakly compact operators. As a consequence, we prove the stability of the Jeribi essential spectrum under compact perturbations. Furthermore, some characterizations of the Jeribi essential spectra of 3×3 upper triangular block operator matrix are also given.

1. Introduction

Spectral theory is an important part of functional analysis, it has numerous applications in many branch of mathematics and physics. The study of essential spectra of block operator matrices has been arround for many years and by many authors, see for example, [3], [1] and [4].

The purpose of this paper is to present diverse results on essential spectra. More precisely, we investigate the Jeribi essential spectrum of the sum of two bounded linear operators defined on a Banach space by means of the Jeribi essential spectra of the two operators. We give new characterizations of the Jeribi essential spectrum by means of almost weakly compact operators, and as a consequence of these results, we prove the stability of the Jeribi essential spectrum under compact perturbations. Also, we give some new characterizations of Wolf and Jeribi essential spectra of 3×3 upper triangular block operator matrix by investigating a new decomposition of the upper triangular block operator once using the notion of Fredholm perturbation and another using Tauberian and co Tauberian operators.

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We organize the paper in the following way : In Section 2 we recall some basic definitions, notations and results of different classes of operators (Fredholm, semi-Fredholm, Tauberian, co Tauberian, almost weakly compact and others). Our main results are presented in section 3.

2. Preliminary Results

The goal of this section consists in establishing some preliminary results which will be required in proving our main results. Let X be a Banach space, denote by $\mathcal{L}(X)$ (respectively $\mathcal{W}(X)$) the space of all bounded (respectively weakly compact) linear operators on X. For $T \in \mathcal{L}(X)$, the dimension of the null space $\ker(T)$, denoted $\alpha(T)$, is called the nullity of T. The codimension of the range of T, denoted $\beta(T)$, is called the defect of T. An operator T is said to be Fredholm if its nullity and defect are both finite, denote by $\Phi(X)$ the set of Fredholm operators on X. The sets of upper and lower semi-Fredholm operators are defined respectively by

$$\Phi_+(X) = \{T \in \mathcal{L}(X) : \text{ such that } \alpha(T) < \infty \text{ and } R(T) \text{ is closed}\};$$

$$\Phi_-(X) = \{T \in \mathcal{L}(X) : \text{ such that } \beta(T) < \infty\}.$$

The index of T, $\text{ind}(T)$, is defined as $\text{ind}(T) = \alpha(T) - \beta(T)$. An operator $F \in \mathcal{L}(X)$ is called a Fredholm perturbation if $F + T \in \Phi(X)$ for each $T \in \Phi(X)$, denote by $\mathcal{F}(X)$ the set of Fredholm perturbations.

The class of almost weakly compact operators as those operators which have bounded inverses only on reflexive subspaces. An operator T is said to be almost weakly compact operator if whenever T has a bounded inverse on a closed subspace M, then M is reflexive. The set of all almost weakly compact operators on X will denoted by $\mathcal{AW}(X)$.

Theorem 2.1. *[6] Let T be almost weakly compact operator and S be compact. Then $T + S$ is almost weakly compact operator.*

In order to recall the Tauberian and co Tauberian operators, let us denote X^{**} the second dual of X, T^* the conjugate of T and T^{**} the second conjugate of T. These classes of operators were introduced and investigated respectively by [7] and [12]. An operator $T \in \mathcal{L}(X)$ is said to be Tauberian whenever T^{**} preserves the natural embedding of X into its double dual. An operator $T \in \mathcal{L}(X)$ is said to be co Tauberian if T^* is Tauberian. The sets of Tauberian and co Tauberian operators on X are denoted respectively by $\mathcal{T}(X)$ and $\mathcal{CT}(X)$. For a detailed study of the properties of Tauberian and co Tauberian operators we refer to [5], [12] and [7]. In the sequel, we need the following sets $\sigma_{\mathcal{T}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{T}(X)\}$; and $\sigma_{\mathcal{CT}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{CT}(X)\}$. There are several definitions of essential spectrum of a bounded linear operator on a Banach space. In this paper, we are concerned with the following

$$\sigma_{e,4}(T) = \{\lambda \in \mathbb{C} : (\lambda I - T) \notin \Phi(X)\}$$

$$\sigma_{e,5}(T) = \mathbb{C} \setminus \{\lambda \in \mathbb{C} : (\lambda I - T) \in \Phi(X) \text{ and } \operatorname{ind}(\lambda I - T) = 0\}$$

$$\sigma_{e_j}(T) = \bigcap_{K \in \mathcal{W}(X)} \sigma(T + K)$$

where σ denote the spectrum. We close this section by recalling that an operator T is said to be Riesz operator if $\sigma_{e,4}(T) = \{0\}$.

3. Main Results

First, we investigate the Jeribi essential spectrum of the sum of two bounded linear operators defined on a Banach space by means of the Jeribi essential spectra of the two operators.

Theorem 3.1. *Let T, $L \in \mathcal{L}(X)$. If there exist W, $W' \in \mathcal{W}(X)$ such that $(T - W)(L - W') \in \mathcal{F}(X)$. Then*

$$\sigma_{e_j}(T + L) \setminus \{0\} \subset \left[\sigma_{e_j}(T) \cup \sigma_{e_j}(L)\right] \setminus \{0\}.$$

Furthermore, if $TL = LT$, $TW' = W'T$, $WL = LW$ and $WW' = W'W$ then

$$\sigma_{e_j}(T + L) \setminus \{0\} = \left[\sigma_{e_j}(T) \cup \sigma_{e_j}(L)\right] \setminus \{0\}.$$

Proof. Let us consider $\lambda \notin \left[\sigma_{e_j}(T) \cup \sigma_{e_j}(L)\right] \setminus \{0\}$, then $\lambda \notin \sigma_{e_j}(T)$ and $\lambda \notin \sigma_{e_j}(L)$. So there exist $W, W' \in \mathcal{W}(X)$ such that $\lambda \in \rho(T+W)$ and $\lambda \in \rho(L+W')$. This implies that $(\lambda - T - W) \in \Phi(X)$ with $\operatorname{ind}(\lambda - T - W) = 0$ and $(\lambda - L - W') \in \Phi(X)$ with $\operatorname{ind}(\lambda - L - W') = 0$. By the use of [10, Theorem 5, p 156] we get $(\lambda - T - W)(\lambda - L - W') \in \Phi(X)$ with $\operatorname{ind}(\lambda - T - W)(\lambda - L - W') = 0$. The operator $(\lambda - T - W)(\lambda - L - W')$ can be written as

$$(\lambda - T - W)(\lambda - L - W') = \lambda(\lambda - T - L - W'') + (T - W)(L - W') \tag{3.1}$$

where $W'' = W + W'$ belongs to $\mathcal{W}(X)$. From hypothesis and the last equation (3.1), we see that $(\lambda - T - L - W'') \in \Phi(X)$ with $\operatorname{ind}(\lambda - T - L - W'') = 0$. This means that $\lambda \notin \sigma_{e_5}(T + L + W'')$. So, $\lambda \notin \sigma_{e_j}(T + L)$ since the Jeribi essential spectrum is invariant under weakly compact perturbations. Hence

$$\sigma_{e_j}(T + L) \setminus \{0\} \subset \left[\sigma_{e_j}(T) \cup \sigma_{e_j}(L)\right] \setminus \{0\}.$$

In order to prove the inverse inclusion, suppose that $TL = LT$, $TW' = W'T$, $WL = LW$ and $WW' = W'W$. Let $\lambda \notin \sigma_{e_j}(T + L) \setminus \{0\}$ then there exists $W_1 \in \mathcal{W}(X)$ such that $(\lambda - T - L - W_1) \in \Phi(X)$ and $\operatorname{ind}(\lambda - T - L - W_1) = 0$. This implies that $(\lambda - T - L - W_1) \in \mathcal{T}(X)$, since the class of Tauberian operator is invariant under weakly compact operators, we can write the operator W_1 as $W_1 = W + W'$ where $W, W' \in \mathcal{W}(X)$. Similarly as equation (3.1), we have

$$(\lambda - T - W)(\lambda - L - W') = \lambda(\lambda - T - L - W_1) + (T - W)(L - W') \tag{3.2}$$

$$(\lambda - L - W')(\lambda - T - W) = \lambda(\lambda - T - L - W_1) + (L - W')(T - W) \tag{3.3}$$

Using hypothesis, we get $(\lambda - T - W)(\lambda - L - W') \in \Phi(X)$ with $\text{ind}((\lambda - T - W)(\lambda - L - W')) = 0$ and $(\lambda - L - W')(\lambda - T - W) \in \Phi(X)$ with $\text{ind}((\lambda - L - W')(\lambda - T - W)) = 0$. By the use of [10, Theorem 6, p 157], we see that $(\lambda - L - W') \in \Phi(X)$ with $\text{ind}(\lambda - L - W') = 0$ and $(\lambda - T - W) \in \Phi(X)$ with $\text{ind}(\lambda - T - W) = 0$. Consequently, we get $\lambda \notin \sigma_{e,5}(T + W) \setminus \{0\}$ and $\lambda \notin \sigma_{e,5}(L + W') \setminus \{0\}$. Hence $\lambda \notin \left[\sigma_{e_j}(T) \cup \sigma_{e_j}(L)\right] \setminus \{0\}$. □

In the next result, we give a characterization of the Jeribi essential spectrum by means of almost weakly compact operators.

Theorem 3.2. *Let $T \in \mathcal{L}(X)$. Then*

$$\sigma_{e_j}(T) = \bigcap_{K \in \mathcal{AW}(X)} \sigma(T + K).$$

Proof. The inclusion $\bigcap_{K \in \mathcal{AW}(X)} \sigma(T + K) \subset \bigcap_{K \in \mathcal{W}(X)} \sigma(T + K) = \sigma_{e_j}(T)$ is always satisfies since $\mathcal{W}(X) \subset \mathcal{AW}(X)$. In order to prove the inverse inclusion, suppose that $\lambda \notin \bigcap_{K \in \mathcal{AW}(X)} \sigma(T + K)$. Then there exists an almost weakly operator K on X such that $\lambda \in \rho(T + K)$. This means that whenever the operator K has a bounded inverse on a closed subspace M, M is reflexive. On the subspace M, by the use of [5, Proposition 2.1.3], the operator K is Tauberian and weakly compact. So, K is weakly compact on M and $\lambda \in \rho(T + K)$. This implies that $\lambda \notin \sigma_{e_j}(T)$. Hence

$$\sigma_{e_j}(T) \subset \bigcap_{K \in \mathcal{AW}(X)} \sigma(T + K).$$

□

It is obviously that the Jeribi essential spectrum is invariant under Fredholm perturbations. i.e $\sigma_j(T) = \sigma_j(T + F)$ for all $F \in \mathcal{F}(X)$. Now we can easily obtain another characterization of the Jeribi essential spectrum. We have the following Theorem.

Theorem 3.3. *Let $T \in \mathcal{L}(X)$. Then the Jeribi essential spectrum is given by*

$$\sigma_j(T) = \bigcap_{K \in \mathcal{AW}(X)} \sigma_{e_5}(T + K).$$

Proof. From the stability of the Jeribi essential spectrum under Fredholm perturbation and Theorem (3.2), we have

$$\sigma_j(T) = \bigcap_{F\in\mathcal{F}(X)} \sigma_j(T+F) = \bigcap_{F\in\mathcal{F}(X)} \left(\bigcap_{K\in\mathcal{AW}(X)} \sigma(T+F+K) \right) = \bigcap_{K\in\mathcal{AW}(X)} \left(\bigcap_{F\in\mathcal{F}(X)} \sigma(T+F+K) \right).$$

Using [9, Theorem 3.4], we deduce that the $\sigma_j(T) = \bigcap_{K\in\mathcal{AW}(X)} \sigma_{e_5}(T+K)$. □

Corollary 3.1. *Let $T \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$. Then $\sigma_{e_j}(T+K) = \sigma_{e_j}(T)$.*

Proof. The proof of the corollary is immediately deduced from Theorem (3.2) and Theorem (2.1) □

Now, we consider the following 3×3 block operator matrices defined on X^3 by $T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix}$, $W = \begin{pmatrix} W_{11} & 0 & 0 \\ 0 & W_{22} & 0 \\ 0 & 0 & W_{33} \end{pmatrix}$ where $T_{ij} \in \mathcal{L}(X)$ and $W_{ii} \in \mathcal{W}(X)$ for $i = 1, \ldots, 3$.

In the following main theorem, we characterize the Wolf essential spectrum of 3×3 block operator matrix T.

Theorem 3.4. *If T_{11} is a Riesz operator and $T_{23} \in \mathcal{F}(X)$, then*

$$\sigma_{e,4}(T)\backslash\{0\} = [\sigma_{e,4}(T_{22}) \cup \sigma_{e,4}(T_{33})]\backslash\{0\}.$$

Proof. Let $\lambda \notin [\sigma_{e,4}(T_{22}) \cup \sigma_{e,4}(T_{33})]\backslash\{0\}$, then $(\lambda - T_{22}) \in \Phi(X)$ and $(\lambda - T_{33}) \in \Phi(X)$. We have

$$\lambda - T = \begin{pmatrix} \lambda - T_{11} & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} & -T_{23} \\ 0 & 0 & \lambda - T_{33} \end{pmatrix} = A_1 \times A_2 \times A_3 \times A_4 + B, \text{where}$$

$A_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & \lambda - T_{22} & 0 \\ 0 & 0 & \lambda - T_{33} \end{pmatrix}$, $A_2 = \begin{pmatrix} I & -T_{12} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$, $A_3 = \begin{pmatrix} I & 0 & -T_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$,
$A_4 = \begin{pmatrix} \lambda - T_{11} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_{23} \\ 0 & 0 & 0 \end{pmatrix}$. Since block operator matrices $A_1, \ldots, A_4$ are Fredholm, then $A_1 \times A_2 \times A_3 \times A_4$ is Fredholm. So, $(\lambda - T) \in \Phi(X)$ by using stability of Fredholm operator by Fredholm perturbation B. Hence $\lambda \notin \sigma_{e,4}(T)\backslash\{0\}$. For the inverse inclusion, let $\lambda \notin \sigma_{e,4}(T)\backslash\{0\}$. Then

$$\lambda - T = \begin{pmatrix} \lambda - T_{11} & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} & -T_{23} \\ 0 & 0 & \lambda - T_{33} \end{pmatrix} \in \Phi(X).$$

By using stability of Fredholm operator by Fredholm perturbation $-B$, we get

$$\begin{pmatrix} \lambda - T_{11} & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} & 0 \\ 0 & 0 & \lambda - T_{33} \end{pmatrix} \in \Phi(X).$$

Since $A_2 \times A_3 \times A_4 \in \Phi(X)$, then using [11, Theorem 5.13] we get $A_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & \lambda - T_{22} & 0 \\ 0 & 0 & \lambda - T_{33} \end{pmatrix} \in \Phi(X)$. Consequently, $(\lambda - T_{22}) \in \Phi(X)$ and $(\lambda - T_{33}) \in \Phi(X)$. Hence $\lambda \notin [\sigma_{e,4}(T_{22}) \cup \sigma_{e,4}(T_{33})] \backslash \{0\}$. □

In our next theorem, we state a characterization of the Jeribi essential spectrum of the upper block operator matrix T.

Theorem 3.5. *If $\sigma_j(T_{11}) = \{0\}$ and $T_{23} \in \mathcal{F}(X)$, then*

$$[\sigma_{e,4}(T_{22}) \cup \sigma_{e,4}(T_{33})] \backslash \{0\} \subset \sigma_j(T) \backslash \{0\} \subset \left[\sigma_j(T_{22}) \cup \sigma_j(T_{33})\right] \backslash \{0\}.$$

Proof. Let $\lambda \notin \sigma_j(T) \backslash \{0\}$, then using [2, Lemma 3.1] there exists $W \in \mathcal{W}(X)$ such that $(\lambda - T - W) \in \Phi(X)$ with $\operatorname{ind}(\lambda - T - W) = 0$. We have

$$\begin{aligned} \lambda - T - W &= \begin{pmatrix} \lambda - T_{11} - W_{11} & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} - W_{22} & -T_{23} \\ 0 & 0 & \lambda - T_{33} - W_{33} \end{pmatrix} \\ &= A_1' \times A_2 \times A_3 \times A_4' + B \end{aligned}$$

where $A_1' = \begin{pmatrix} I & 0 & 0 \\ 0 & \lambda - T_{22} - W_{22} & 0 \\ 0 & 0 & \lambda - T_{33} - W_{33} \end{pmatrix}$, A_2, A_3, B are defined in the proof of Theorem (3.4) and $A_4' = \begin{pmatrix} \lambda - T_{11} - W_{11} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$. Using stability of Fredholm operator $(\lambda - T - W)$ by Fredholm pertubation $-B$ and [11, Theorem 5.13] we get

$$A_1' = \begin{pmatrix} I & 0 & 0 \\ 0 & \lambda - T_{22} - W_{22} & 0 \\ 0 & 0 & \lambda - T_{33} - W_{33} \end{pmatrix} \in \Phi(X)$$

since A_2, A_3 and A_4' are Fredholm. This implies that $(\lambda - T_{22} - W_{22}) \in \Phi(X)$ and $(\lambda - T_{33} - W_{33}) \in \Phi(X)$. Hence

$$\lambda \notin [\sigma_{e,4}(T_{22} + W_{22}) \cup \sigma_{e,4}(T_{33} + W_{33})] \backslash \{0\}.$$

It is well known that Jeribi essential spectrum is invariant under weakly compact operators, so we get

$$[\sigma_{e,4}(T_{22}) \cup \sigma_{e,4}(T_{33})] \setminus \{0\} \subset \sigma_j(T - W) \setminus \{0\} = \sigma_j(T) \setminus \{0\}.$$

For the inverse inclusion, suppose that $\lambda \notin \left[\sigma_j(T_{22}) \cup \sigma_j(T_{33})\right] \setminus \{0\}$. Then there exist $W_{11}, W_{22} \in \mathcal{W}(X)$ such that $(\lambda - T_{22} - W_{22}) \in \Phi(X)$ with $\mathrm{ind}(\lambda - T_{22} - W_{22}) = 0$ and $(\lambda - T_{33} - W_{33}) \in \Phi(X)$ with $\mathrm{ind}(\lambda - T_{33} - W_{33}) = 0$. We see that

$$A_1' \times A_2 \times A_3 \times A_4' = \begin{pmatrix} \lambda - T_{11} - W_{11} & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} - W_{22} & 0 \\ 0 & 0 & \lambda - T_{33} - W_{33} \end{pmatrix}$$

is a Fredholm with zero index. So, $(\lambda - T - W) \in \Phi(X)$ with $\mathrm{ind}(\lambda - T - W) = 0$ by using stability of Fredholm operator by Fredholm pertubation B. Hence $\lambda \notin \sigma_j(T) \setminus \{0\}$. □

Finally, we conclude this paper by giving another characterization of the Jeribi essential spectrum of the block operator matrix T by using the notion of Tauberian and co Tauberian operators.

Theorem 3.6. *Let T the block operator matrix defined above. Then we have*

$$\sigma_\tau(T_{11}) \cup \sigma_{c\tau}(T_{33}) \subset \sigma_j(T) \subset \bigcup_{i=1}^{3} \sigma_j(T_{ii}).$$

Proof. Suppose that $\lambda \notin \sigma_j(T)$. Then using [2, Lemma 3.1] with $S = I$ the identity operator, there exists $W \in \mathcal{W}(X)$ such that $(\lambda - T - W) \in \Phi(X)$ and $\mathrm{ind}(\lambda - T - W) = 0$. So, the block matrix operator

$$(\lambda - T - W) = \begin{pmatrix} \lambda - T_{11} - W_{11} & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} - W_{22} & -T_{23} \\ 0 & 0 & \lambda - T_{33} - W_{33} \end{pmatrix} \in \Phi(X).$$

By the fact that $\Phi_+(X) \subset \tau(X)$, $\Phi_-(X) \subset c\tau(X)$ and using [1, Proposition 2.1], we get $(\lambda - T_{11} - W_{11}) \in \tau(X)$ and $(\lambda - T_{33} - W_{33}) \in c\tau(X)$. Since Tauberian and co Tauberian operators are invariant under weakly compact operators, we deduce that $\lambda \notin \sigma_\tau(T_{11})$ and $\lambda \notin \sigma_{c\tau}(T_{33})$. For the second inclusion, Suppose that $\lambda \notin \bigcup_{i=1}^{3} \sigma_j(T_{ii})$, then there exist $W_{11}, \dots, W_{33} \in \mathcal{W}(X)$ such that $(\lambda - T_{ii} - W_{ii}) \in \Phi(X)$ with ind $(\lambda - T_{ii} - W_{ii}) = 0$ for $i = 1, \dots, 3$. By using [8, Lemma 1.33], the triangular block operator matrix

$$\begin{pmatrix} \lambda - T_{11} - W_{11} & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} - W_{22} & -T_{23} \\ 0 & 0 & \lambda - T_{33} - W_{33} \end{pmatrix}$$

is a Fredholm with zero index. This means that $(\lambda - T - W) \in \Phi(X)$ with $\mathrm{ind}(\lambda - T - W) = 0$ which implies that $\lambda \notin \sigma_j(T)$. □

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