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Instabilities and Stabilities of Additive Functional Equation in Paranormed Spaces

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Abstract. In this paper, we solve the general solution in vector space and prove the Hyers-Ulam stability of the following additive functional equation

$$f\left(\left(\frac{x_1+x_2}{2}\right)-\sum_{i=3}^{n}x_i\right)+f\left(\left(\frac{\sum_{i=2}^{n}x_i}{2}\right)-x_1\right)+f\left(\left(\frac{x_1+\sum_{i=3}^{n}x_i}{2}\right)-x_2\right)$$
$$=f\left(x_1-x_2\right)+f\left(x_2-\sum_{i=3}^{n}x_i\right)+f\left(\sum_{i=3}^{n}x_i-x_1\right)$$

in paranormed spaces by using the direct and fixed point methods. Also we present its pertinent counter examples for instabilities.

1. Introduction

A classical question in the theory of functional equations is the following "Whether it is true that a function which approximately satisfies a functional equation ϵ must be close to an exact solution ϵ? If the problem accepts a solution, can we say that the equation ϵ is stable". The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. The stability question is: How do the solutions of the inequality differ from those of the given functional equation?

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In the fall of 1940, Stanislaw M. Ulam [36] gave a wide-ranging talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exists $\delta(\epsilon) > 0$ such that if $h : G_1 \to G_2$ satisfies the inequality

$$d\left(h(x * y), h(x) \diamond h(y)\right) < \delta \quad \forall\ x, y \in G_1,$$

then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$.

If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In the next year, D.H. Hyers [12] gave a affirmative answer to this question for additive groups under the assumption that groups are Banach spaces. He brilliantly answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces. The result of Hyers is stated as follows:

Theorem 1.1. *Let $f : E_1 \to E_2$ be a function between Banach spaces such that*

$$\left\|f(x+y) - f(x) - f(y)\right\| \le \epsilon \tag{1.1}$$

for all $x, y \in E_1$ and $\epsilon > 0$ is a constant. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{1.2}$$

exists for each $x \in E_1$ and $A : E_1 \to E_2$ is unique additive mapping satisfying

$$\left\|f(x) - A(x)\right\| \le \epsilon \tag{1.3}$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Taking this famous result into consideration, the additive Cauchy functional equation $f(x+y) = f(x) + f(y)$ is said to have the Hyers-Ulam stability on (E_1, E_2) if for every function $f : E_1 \to E_2$ satisfying the inequality (1.1) for some $\epsilon \ge 0$ and for all $x, y \in E_1$, there exists an additive function $A : E_1 \to E_2$ such that $f - A$ is bounded on E_1.The method in (1.2) provided by Hyers which produces the additive function A will be called a direct method. This method is the most important and powerful tool to study the stability of various functional equations.

It is possible to prove a stability result similar to Hyers functions that do not have bounded Cauchy difference. T. Aoki (1950) [2] first generalized the Hyers theorem for unbounded Cauchy difference having sum of norms $\left(\|x\|^p + \left\|y\right\|^p\right)$.

The same result was rediscovered by Th. M. Rassias [30] in 1978 and proved a generalization of Hyers theorem for additive mappings. This stability result is named Hyers-Ulam-Rassias stability or Hyers-Ulam-Aoki-Rassias stability for the functional equation.In 1982 J.M. Rassias [28], followed the innovative approach of Rassias theorem in which he replaced the factor $\|x\|^p + \left\|y\right\|^p$ by $\|x\|^p \left\|y\right\|^q$ with $p + q \ne 1$. Later this stability result was called Ulam-Gavruta-Rassias stability of functional equation.

In 1990, Th.M. Rassias during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem in [31] can also be proved for value of p greater or equal to 1. In 1991, Gajda [10] provided an affirmative solution to Th.M. Rassias' question for p strictly greater than one. In 1994, P. Găvruţa [11] provided a further generalization of Th.M. Rassias [30] theorem in which he replaced the bound $\epsilon\left(\|x\|^p + \|y\|^p\right)$ by a general control function $\phi(x,y)$. This stability result is called Generalized Hyers-Ulam-Rassias stability of functional equation. In 2008, a special case of Găvruţa's theorem for the unbounded Cauchy difference was obtained by K. Ravi, M. Arunkumar and J.M. Rassias [32] by considering the summation of both the sum and the product of two p-norms in the sprit of Rassias approach and is named J. M. Rassias Stability of functional equation. Later authors used many spaces to stabilize the equations of additive functions, giving flexible results [1,3,6,13,19,25].

Functional equations can be used to study a wide range of problems, such as describing the behavior of physical systems, solving mathematical puzzles, and understanding data.

The equation of the additive function is

$$f(x+y) = f(x) + f(y). \tag{1.4}$$

Since $f(x) = kx$ is a solution of the functional equation (1.4), each solution of the additive functional equation is called an additive map.

In this article, we propose the stability of n variable additive functional equation

$$\begin{aligned} f\left(\left(\frac{x_1+x_2}{2}\right) - \sum_{i=3}^{n} x_i\right) + f\left(\left(\frac{\sum_{i=2}^{n} x_i}{2}\right) - x_1\right) + f\left(\left(\frac{x_1 + \sum_{i=3}^{n} x_i}{2}\right) - x_2\right) \\ = f(x_1 - x_2) + f\left(x_2 - \sum_{i=3}^{n} x_i\right) + f\left(\sum_{i=3}^{n} x_i - x_1\right) \end{aligned} \tag{1.5}$$

using direct and fixed-point methods in paranormed spaces.

2. Solution of the Functional Equation (1.5)

In this section, we give the general solution of the functional equation (1.5) in vector spaces.

Theorem 2.1. *Let X and Y be real vector spaces. A mapping $f : X \to Y$ satisfies the functional equation (1.4) if and only if $f : X \to Y$ satisfies the functional equation (1.5) for all $x_1, x_2, \cdots, x_n \in X$.*

Proof. Assume that $f : X \to Y$ satisfies the functional equation (1.4).

Setting $x = y = 0$ in (1.4), we get $f(0) = 0$. Letting $y = -x$ in (1.4), we obtain $f(-x) = -f(x)$ for all $x \in X$. Therefore f is an odd mapping. Replacing y by x in (1.4), we get $f(2x) = 2f(x)$ for all $x \in X$.

Replacing (x,y) by $\left(\frac{x_1+x_3}{2}, x_2\right)$ in (1.4) and using oddness, we get

$$f\left(\frac{x_1+x_3}{2} - x_2\right) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_3) - f(x_2) \tag{2.1}$$

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) by $\left(\frac{x_2+x_3}{2}, x_1\right)$ in (1.4), we get

$$f\left(\frac{x_2+x_3}{2} - x_1\right) = \frac{1}{2}f(x_2) + \frac{1}{2}f(x_3) - f(x_1) \tag{2.2}$$

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) by $\left(\frac{x_1+x_2}{2}, x_3\right)$ in (1.4), we get

$$f\left(\frac{x_1+x_2}{2} - x_3\right) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) - f(x_3) \tag{2.3}$$

for all $x_1, x_2, x_3 \in X$. Adding (2.1), (2.2), (2.3) and using oddness, we obtain

$$\begin{aligned} f\left(\left(\frac{x_1+x_2}{2}\right) - x_3\right) + f\left(\left(\frac{x_2+x_3}{2}\right) - x_1\right) + f\left(\left(\frac{x_3+x_1}{2}\right) - x_2\right) \\ = f\left(x_1 - x_2\right) + f\left(x_2 - x_3\right) + f\left(x_3 - x_1\right) \end{aligned} \tag{2.4}$$

for all $x_1, x_2, x_3 \in X$. Finally, replacing x_3 by $x_3 + x_4 + \cdots + x_n$, we have (1.5) for all $x_1, x_2, \cdots, x_n \in X$.

Conversely, assume that $f : X \to Y$ satisfies the functional equation (1.5). Setting $x_4 = x_5 = \cdots = x_n = 0$ in (1.5), we get

$$\begin{aligned} f\left(\frac{x_1+x_2}{2} - x_3\right) + f\left(\frac{x_2+x_3}{2} - x_1\right) + f\left(\frac{x_3+x_1}{2} - x_2\right) \\ = f(x_1 - x_2) + f(x_2 - x_3) + f(x_3 - x_1) \end{aligned} \tag{2.5}$$

for all $x_1, x_2, x_3 \in X$. Setting $x_2 = x_1, x_3 = -x_1$ in (2.5) and finally replacing x_1 by $-x$, we obtain

$$2f(x) = f(2x) \tag{2.6}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (2.6) , we get

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.7}$$

for all $x \in X$. Replacing (x_1, x_2, x_3) by $(x, -x, 0)$ in (2.5) and using (2.6) and (2.7), we get

$$f\left(-x\right) = -f(x) \tag{2.8}$$

for all $x \in X$. Hence f is an odd mapping. Letting $x_1 = x, x_2 = y, x_3 = 0$ in (2.5) and using the oddness, we obtain

$$f(x+y) + f(y-2x) + f(x-2y) = 2f(x-y) + 2f(y) - 2f(x) \tag{2.9}$$

for all $x, y \in X$. Interchanging x and y in (2.9) and using the oddness, we obtain

$$f(x+y) + f(x-2y) + f(-2x+y) = -2f(x-y) + 2f(x) - 2f(y) \tag{2.10}$$

for all $x, y \in X$. Subtracting (2.10) from (2.9), we get

$$f(x-y) = f(x) - f(y) \tag{2.11}$$

for all $x, y \in X$. Replacing y by $-y$ and using the oddness, we have (1.4) for all $x, y \in X$. $\square$

3. Basic Concepts on Paranormed Spaces

We recall some basic facts concerning Fréchet spaces.

Definition 3.1. *[37] Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that*

(P1) $P(0) = 0$;
(P2) $P(-x) = P(x)$;
(P3) $P(x + y) \leq P(x) + P(y)$ *(triangle inequality);*
(P4) *If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).*

The pair (X, P) is called a **paranormed space** *if P is a* **paranorm** *on X.*

Definition 3.2. *[37] The paranorm is called total if, in addition, we have*

(P5) $P(x) = 0$ *implies* $x = 0$.

Definition 3.3. *[37] A Fréchet space is a total and complete paranormed space.*

4. Stability Results: Hyers' Direct Method

In this section, we investigate the Hyers-Ulam stability of the functional equation (1.5) in paranormed spaces using direct method.

Throughout this section, let (U, P) be a Fréchet space and $(V, \|\cdot\|)$ be a Banach space.

For the convenience, we define a mapping $F : U^n \to V$ by

$$\begin{aligned} F(x_1, x_2, \cdots, x_n) &= f\left(\left(\frac{x_1 + x_2}{2}\right) - \sum_{i=3}^{n} x_i\right) \\ &\quad + f\left(\left(\frac{\sum_{i=2}^{n} x_i}{2}\right) - x_1\right) + f\left(\left(\frac{x_1 + \sum_{i=3}^{n} x_i}{2}\right) - x_2\right) \\ &\quad - f(x_1 - x_2) - f\left(x_2 - \sum_{i=3}^{n} x_i\right) - f\left(\sum_{i=3}^{n} x_i - x_1\right) \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in U$.

Theorem 4.1. *Let $j \in \{-1, 1\}$ be fixed and $\alpha : U^n \to [0, \infty)$ be a function with the condition*

$$\sum_{n=0}^{\infty} 2^{nj} \xi\left(\frac{x_1}{2^{nj}}, \frac{x_2}{2^{nj}}, \cdots, \frac{x_n}{2^{nj}}\right) < +\infty \tag{4.1}$$

for all $x_1, x_2, \cdots, x_n \in U$. Suppose that a mapping $f : U^n \to V$ satisfies the following inequality

$$P\left(F(x_1, x_2, \cdots, x_n)\right) \leq \xi(x_1, x_2, \cdots, x_n) \tag{4.2}$$

for all $x_1, x_2, \cdots, x_n \in U$. Then there exists a unique additive mapping $\mathcal{A} : U^n \to V$ such that

$$P\left(f(x) - \mathcal{A}(x)\right) \leq \sum_{k=\frac{1-j}{2}}^{\infty} 2^{kj} \xi\left(\frac{x}{2^{kj}}, \frac{x}{2^{kj}}, 0, \cdots, 0\right) \tag{4.3}$$

for all $x \in U$. The mapping $\mathcal{A}(x)$ is defined by

$$P\left(\lim_{n\to\infty} 2^{nj} f\left(\frac{x}{2^{nj}}\right) - A(x)\right) \to 0 \tag{4.4}$$

for all $x \in U$.

Proof. Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, x, 0, \cdots, 0)$ in (4.2), we get

$$P\left(2f\left(\frac{x}{2}\right) - f(x)\right) \leq \xi(x, x, 0, \cdots, 0) \tag{4.5}$$

for all $x \in U$. For any $m, n > 0$, we simplify

$$P\left(2^m f\left(\frac{x}{2^m}\right) - 2^n f\left(\frac{x}{2^n}\right)\right) \leq \sum_{k=m}^{n-1} 2^k \xi\left(\frac{x}{2^k}, \frac{x}{2^k}, 0, \cdots, 0\right) \tag{4.6}$$

for all $x \in U$ and all $m, n \geq 0$. It follows from (4.6) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is Cauchy sequence. Since V is complete, there exists a mapping $\mathcal{A} : U^n \to V$ by

$$P\left(\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) - \mathcal{A}(x)\right) \to 0$$

for all $x \in U$. By continuity of multiplication, we have

$$P\left(\lim_{n\to\infty} t_n \, 2^n f\left(\frac{x}{2^n}\right) - t\mathcal{A}(x)\right) \to 0$$

Letting $m = 0$ and $n \to \infty$ in (4.6), we see that (4.3) holds for all $x \in U$. To show that $\mathcal{A}$ satisfies (1.5), replacing $(x_1, x_2, \cdots, x_n)$ by $\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \cdots, \frac{x_n}{2^n}\right)$ in (4.2), we get

$$P\left(2^n \, F\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \cdots, \frac{x_n}{2^n}\right)\right) \leq 2^n \xi\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \cdots, \frac{x_n}{2^n}\right)$$

for all $x_1, x_2, \cdots, x_n \in U$. Letting $n \to \infty$ in the above inequality and using the definition of $\mathcal{A}(x)$, we see that

$$P\left(\mathcal{A}\left(x_1, x_2, \cdots, x_n\right)\right) = 0 \tag{4.7}$$

for all $x_1, x_2, \cdots, x_n \in U$. Using condition $(P5)$ in (4.7), we obtain

$$\begin{aligned}\mathcal{A}\left(\left(\frac{x_1 + x_2}{2}\right) - \sum_{i=3}^{n} x_i\right) &+ \mathcal{A}\left(\left(\frac{\sum_{i=2}^{n} x_i}{2}\right) - x_1\right) + \mathcal{A}\left(\left(\frac{x_1 + \sum_{i=3}^{n} x_i}{2}\right) - x_2\right) \\ &= \mathcal{A}\left(x_1 - x_2\right) + \mathcal{A}\left(x_2 - \sum_{i=3}^{n} x_i\right) + \mathcal{A}\left(\sum_{i=3}^{n} x_i - x_1\right)\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in U$. Hence $\mathcal{A}$ satisfies (1.5) for all $x_1, x_2, \cdots, x_n \in U$. In order to prove that $\mathcal{A}(x)$ is unique, let $\mathcal{A}'(x)$ be another additive mapping satisfying (1.5) and (4.3). Then

$$\begin{aligned}P(\mathcal{A}(x) - \mathcal{A}'(x)) &= 2^m \left\{P\left(\mathcal{A}\left(\frac{x}{2^m}\right) - \mathcal{A}'\left(\frac{x}{2^m}\right)\right)\right\} \\ &\leq 2^m \left\{P\left(\mathcal{A}\left(\frac{x}{2^m}\right) - f\left(\frac{x}{2^m}\right)\right) + P\left(f\left(\frac{x}{2^m}\right) - \mathcal{A}'\left(\frac{x}{2^m}\right)\right)\right\} \\ &\leq \sum_{k=0}^{\infty} 2^{k+m} \xi\left(\frac{x}{2^{k+m}}, \frac{x}{2^{k+m}}, 0, \cdots, 0\right) \to 0 \ as \ m \to \infty\end{aligned}$$

for all $x \in U$. Thus $P(\mathcal{A}(x) - \mathcal{A}'(x)) = 0$ for all $x \in U$. Hence, we have $\mathcal{A}(x) = \mathcal{A}'(x)$. Therefore $\mathcal{A}(x)$ is unique. Thus the mapping $\mathcal{A} : U^n \to V$ is a unique additive mapping. Thus the theorem holds for $j = 1$

For $j = -1$, we can prove the result by a similar method. This completes the proof. □

From Theorem 4.1, we obtain the following corollary concerning the Hyers-Ulam stability for the functional equation (1.5).

Corollary 4.1. *Let $F : U^n \to V$ be a mapping and assume that there exist real numbers σ and s such that*

$$P\left(F(x_1, x_2, \cdots, x_n)\right) \leq \begin{cases} \sigma, & \\ \sigma\left\{P(x_1)^s + P(x_2)^s + \cdots + P(x_n)^s\right\}, & s \neq 1 \\ \sigma\left\{P(x_1)^s P(x_2)^s \cdots P(x_n)^s + \left\{P(x_1)^{ns} + \cdots + P(x_n)^{ns}\right\}\right\}, & s \neq \frac{1}{n} \end{cases} \tag{4.8}$$

for all $x_1, x_2, \cdots, x_n \in U$. Then there exists a unique additive mapping $\mathcal{A} : U^n \to V$ such that

$$P\left(f(x) - \mathcal{A}(x)\right) \leq \begin{cases} \dfrac{\sigma}{|1|} \\ \dfrac{2\sigma P(x)^s}{|2 - 2^s|} \\ \dfrac{2\sigma P(x)^{ns}}{|2 - 2^{ns}|} \end{cases} \tag{4.9}$$

for all $x \in U$.

5. Alternative Stability Results: Fixed Point Method

In this section, we prove the Hyers-Ulam stability of the functional equation (1.5) in paranormed spaces by using the fixed point method.

Now, we will recall the fundamental results in fixed point theory.

Theorem 5.1. *[20] Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

$(B1)$ $$d(T^n x, T^{n+1} x) = \infty \quad \forall\, n \geq 0,$$

or

$(B2)$ there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;

(ii)The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0} x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L}\, d(y, Ty)$ for all $y \in Y$.

Many researchers have applied the fixed point alternative method to prove the Hyers-Ulam stability of functional equations (see [4,5,7,26]).

Theorem 5.2. *Let $F : U^n \to V$ be a mapping for which there exists a function $\xi : U^n \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \mu_i^n \xi\left(\frac{x_1}{\mu_i^n}, \frac{x_2}{\mu_i^n}, \cdots, \frac{x_n}{\mu_i^n}\right) = 0 \tag{5.1}$$

where $\mu_0 = \frac{1}{2}$ and $\mu_1 = 2$ such that the functional inequality

$$P\left(F(x_1, x_2, \cdots, x_n)\right) \le \xi(x_1, x_2, \cdots, x_n) \tag{5.2}$$

for all $x_1, x_2, \cdots, x_n \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \xi\left(x, x, 0, \cdots, 0\right)$$

has the property

$$\gamma(x) = L\, \mu_i\, \gamma\left(\frac{x}{\mu_i}\right), \tag{5.3}$$

then there exists a unique additive mapping $\mathcal{A} : U^n \to V$ satisfying the functional equation (1.5) and

$$P\left(f(x) - \mathcal{A}(x)\right) \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{L^{1-i}}{1-L}\xi\left(2x, 2x, 0, \cdots, 0\right) \tag{5.4}$$

for all $x \in U$.

Proof. Consider the set $\Omega = \{p/p : U^n \to V,\ p(0,0) = 0\}$ and introduce the generalized metric on Ω, $d(p, q) = d_\gamma(p, q) = \inf\{K \in (0, \infty) : P\left(p(x) - q(x)\right) \le K\gamma(x), x \in U\}$. It is easy to see that (Ω, d) is complete.

Define $T : \Omega^n \to \Omega$ by $Tp(x, x) = \mu_i p\left(\frac{x}{\mu_i}\right)$ for all $x \in U$. Now $p, q \in \Omega$ imply that $d(Tp, Tq) \le Ld(p, q)$ for all $p, q \in \Omega$, i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, x, 0, \cdots, 0)$ in (5.2), we get

$$P\left(2f\left(\frac{x}{2}\right) - f(x)\right) \le \xi(x, x, 0, \cdots, 0) \tag{5.5}$$

for all $x \in U$. By using (5.3) for the case $i = 1$, it reduces to

$$P\left(2f\left(\frac{x}{2}\right) - f(x)\right) \le \ \xi(x, x, 0, \cdots, 0) \le \gamma(x)$$

for all $x \in U$,

$$\text{i.e.,} \quad d_\gamma(Tf, f) \le 1 \Rightarrow d(Tf, f) \le L^0 < \infty.$$

Again replacing $x = 2x$ in (5.5), we get

$$P\left(2f\left(x\right) - f\left(2x\right)\right) \le \xi\left(2x, 2x, 0, \cdots, 0\right). \tag{5.6}$$

By using (5.3) for the case $i = 0$, it reduces to

$$P\left(2f\left(x\right) - f\left(2x\right)\right) \le \ \xi\left(2x, 2x, 0, \cdots, 0\right) \le L\gamma(x)$$

for all $x \in U$,

$$\text{i.e.,} \quad d_\gamma(f, Tf) \le L \Rightarrow d(f, Tf) \le L^1 < \infty.$$

In both cases, we obtain $d(g,Tg) \leq L^{1-i}$. Therefore $(A1)$ holds. By $(A2)$, it follows that there exists a fixed point $\mathcal{A}$ of T in Ω such that

$$P\left(\lim_{n\to\infty}\mu_i^n\left(f\left(\frac{x}{\mu_i^{n+1}}\right)-f\left(\frac{x}{\mu_i^n}\right)\right)-\mathcal{A}(x)\right)\to 0 \tag{5.7}$$

for all $x \in U$. To prove $\mathcal{A}: U^n \to V$ is additive. Replacing $(x_1, x_2, \cdots, x_n)$ by $\left(\frac{x_1}{\mu_i^n}, \frac{x_2}{\mu_i^n}, \cdots, \frac{x_n}{\mu_i^n}\right)$ in (5.2) and multiply by μ_i^n, it follows from (5.1) that

$$\begin{aligned} P\left(\mathcal{A}(x_1,x_2,\cdots,x_n)\right) &= \lim_{n\to\infty}P\left(\mu_i^nF\left(\frac{x_1}{\mu_i^n},\frac{x_2}{\mu_i^n},\cdots,\frac{x_n}{\mu_i^n}\right)\right)\\ &\leq \lim_{n\to\infty}\mu_i^n\xi\left(\frac{x_1}{\mu_i^n},\frac{x_2}{\mu_i^n},\cdots,\frac{x_n}{\mu_i^n}\right)=0 \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in U$, i.e., $\mathcal{A}$ satisfies the functional equation (1.5).

By $(A3)$, $\mathcal{A}$ is the unique fixed point of T in the set $\Delta = \{\mathcal{A} \in \Omega : d(f,\mathcal{A}) < \infty\}$, i.e., $\mathcal{A}$ is the unique mapping such that $P\left(f(x)-\mathcal{A}(x)\right) \leq K\gamma(x)$ for all $x \in U$ and $K > 0$. Finally by $(A4)$, we obtain $d(f,\mathcal{A}) \leq \frac{1}{1-L}d(f,Tf)$ which implies $d(f,\mathcal{A}) \leq \frac{L^{1-i}}{1-L}$ which yields $P\left(f(x)-\mathcal{A}(x)\right) \leq \frac{L^{1-i}}{1-L}\gamma(x)$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 5.2 concerning the stability of (1.5).

Corollary 5.1. *Let $F : U^n \to V$ be a mapping and assume that there exist real numbers σ and s such that the inequality (4.8). Then there exists a unique additive mapping $\mathcal{A} : U^n \to V$ such that the inequality (4.9) holds for all $x \in U$.*

Proof. Set

$$\xi(x_1,x_2,\cdots,x_n)=\begin{cases}\sigma\\ \sigma\left\{P(x_1)^s+P(x_2)^s+\cdots+P(x_n)^s\right\}\\ \sigma\left\{P(x_1)^sP(x_2)^s\cdots P(x_n)^s+\left\{P(x_1)^{ns}+P(x_2)^{ns}+\cdots+P(x_n)^{ns}\right\}\right\}\end{cases}$$

for all $x_1, x_2, \cdots, x_n \in U$. Now

$$\begin{aligned}\mu_i^n\xi\left(\frac{x_1}{\mu_i^n},\frac{x_2}{\mu_i^n},\cdots,\frac{x_n}{\mu_i^n}\right)&=\begin{cases}\mu_i^n\sigma,\\ \mu_i^n\sigma\left\{P\left(\frac{x_1}{\mu_i^n}\right)^s+P\left(\frac{x_2}{\mu_i^n}\right)^s+\cdots+P\left(\frac{x_n}{\mu_i^n}\right)^s\right\}\\ \mu_i^n\sigma\left\{\prod_{i=1}^n P\left(\frac{x_i}{\mu_i^n}\right)^s+\sum_{i=1}^n P\left(\frac{x_i}{\mu_i^n}\right)^{ns}\right\}\end{cases}\\ &=\begin{cases}\to 0 \text{ as } n\to\infty\\ \to 0 \text{ as } n\to\infty\\ \to 0 \text{ as } n\to\infty.\end{cases}\end{aligned}$$

Then (5.1) holds.

But we have $\gamma(x) = \xi\left(2x, 2x, 0, \cdots, 0\right)$ has the property $L\,\gamma(x) = \frac{1}{\mu_i}\,\gamma\left(\mu_i x\right)$ for all $x \in U$. Hence

$$\gamma(x) = \xi\left(2x, 2x, 0, \cdots, 0\right) = \begin{cases} \sigma, \\ 2\sigma 2^{s} P(x)^{s}, \\ 2\sigma 2^{ns} P(x)^{ns}. \end{cases}$$

Now,

$$\mu_i \gamma\left(\frac{x}{\mu_i}\right) = \begin{cases} \mu_i \sigma \\ 2\mu_i^{1-s} \sigma P(x)^{s} \\ 2\mu_i^{1-ns} \sigma P(x)^{ns} \end{cases} = \begin{cases} \mu_i \gamma(x) \\ \frac{\mu_i^{1-s}}{2^{s}} \gamma(x) \\ \frac{\mu_i^{1-ns}}{2^{ns}} \gamma(x) \end{cases}$$

for all $x \in U$. Hence the inequality (5.3). Now from (5.4), we prove the following cases.

Case 1: $L = 2^{-1}$ for $s = 0$ if $i = 0$

$$P\left(f(x) - \mathcal{A}(x)\right) \leq P\left(\frac{2^{-1}}{1 - 2^{-1}} \gamma(x)\right) \leq P\left(\frac{\sigma}{1}\right).$$

Case 2: $L = 2$ for $s = 0$ if $i = 1$

$$P\left(f(x) - \mathcal{A}(x)\right) \leq P\left(\frac{1}{1 - 2} \gamma(x)\right) \leq P\left(\frac{\sigma}{-1}\right).$$

Case:3 $L = 2^{s-1}$ for $s < 4$ if $i = 0$

$$P\left(f(x) - \mathcal{A}(x)\right) \leq P\left(\frac{\left(2^{s-1}\right)^{1-0}}{1 - 2^{s-1}}\right) \gamma(x) \leq \frac{2\sigma P(x)^{s}}{2 - 2^{s}}.$$

Case:4 $L = \frac{1}{2^{s-1}}$ for $s > 4$ if $i = 1$

$$P\left(f(x) - \mathcal{A}(x)\right) \leq P\left(\frac{\left(2^{1-s}\right)^{0}}{1 - 2^{1-s}}\right) \gamma(x) \leq \frac{2\sigma P(x)^{s}}{2^{s} - 2}.$$

Case:5 $L = 2^{ns-1}$ for $s < \frac{1}{n}$ if $i = 0$

$$P\left(f(x) - \mathcal{A}(x)\right) \leq P\left(\frac{\left(2^{ns-1}\right)^{1-0}}{1 - 2^{ns-1}}\right) \gamma(x) \leq \frac{2\sigma P(x)^{ns}}{2 - 2^{ns}}.$$

Case:6 $L = \frac{1}{2^{ns-1}}$ for $s > \frac{1}{n}$ if $i = 1$

$$P\left(f(x) - \mathcal{A}(x)\right) \leq P\left(\frac{\left(2^{1-ns}\right)^{0}}{1 - 2^{1-ns}}\right) \gamma(x) \leq \frac{2\sigma P(x)^{ns}}{2^{ns} - 2}.$$

□

6. Counter Examples for Non Stability Cases

In this section, authors discussed the counter examples for non stable cases for the Corollaries 4.1, and 5.1.

Now we will provide an example to illustrate that the functional equation (1.5) is not stable for $s = 1$ in Condition (ii) of Corollary 4.1.

Example 6.1. *Let $\rho : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\rho(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\rho(2^k x)}{2^k} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\Big|F(x_1, x_2, \cdots, x_n)\Big| \le 12\mu \left(|x_1| + |x_2| + \cdots + |x_n|\right) \tag{6.1}$$

for all $x_1, x_2, \cdots, x_n \in \mathbb{R}$. Then there do not exists a mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - \mathcal{A}(x)| \le \beta|x| \qquad \text{for all} \quad x \in \mathbb{R}. \tag{6.2}$$

Proof. *Now*

$$|f(x)| \le \sum_{n=0}^{\infty} \frac{|\rho(2^k x)|}{|2^k|} = \sum_{k=0}^{\infty} \frac{\mu}{2^k} = 2\mu.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (6.1).

If $x_1 = x_2 = \cdots = x_n = 0$ then (6.1) is trivial. If $|x_1| + |x_2| + \cdots + |x_n| \ge \frac{1}{2}$ then the left hand side of (6.1) is less than 12μ. Now suppose that $0 < |x_1| + |x_2| + \cdots + |x_n| < \frac{1}{2}$. Then there exists a positive integer ℓ such that

$$\frac{1}{2^\ell} \le |x_1| + |x_2| + \cdots + |x_n| < \frac{1}{2^{\ell-1}}, \tag{6.3}$$

so that $2^{\ell-1}|x_1| < 1,\ 2^{\ell-1}|x_2| < 1, \cdots, 2^{\ell-1}|x_n| < 1$ and consequently

$$\begin{aligned} &2^{\ell-1}\left(\frac{x_1 + x_2}{2} - x_3 - x_4 + \cdots - x_n\right), 2^{\ell-1}\left(\frac{x_2 + x_3 + x_4 + \cdots + x_n}{2} - x_1\right), \\ &2^{\ell-1}\left(\frac{x_1 + x_3 + x_4 + \cdots + x_n}{2} - x_2\right), -2^{\ell-1}(x_1 - x_2), -2^{\ell-1}(x_2 - x_3 - \cdots - x_n), \\ &-2^{\ell-1}(x_3 + x_4 + \cdots + x_n - x_1) \in (-1, 1). \end{aligned}$$

Therefore for each $k = 0, 1, \ldots, \ell - 1$, we have

$$\begin{aligned} &2^k\left(\frac{x_1 + x_2}{2} - x_3 - x_4 + \cdots - x_n\right), 2^k\left(\frac{x_2 + x_3 + x_4 + \cdots + x_n}{2} - x_1\right), \\ &2^k\left(\frac{x_1 + x_3 + x_4 + \cdots + x_n}{2} - x_2\right), -2^k(x_1 - x_2), -2^k(x_2 - x_3 - \cdots - x_n), \\ &-2^k(x_3 + x_4 + \cdots + x_n - x_1) \in (-1, 1). \end{aligned}$$

and

$$\begin{aligned}&\rho\left(2^k\left(\frac{x_1+x_2}{2}-x_3-x_4+\cdots-x_n\right)\right)+\rho\left(2^k\left(\frac{x_2+x_3+x_4+\cdots+x_n}{2}-x_1\right)\right)\\&+\rho\left(2^k\left(\frac{x_1+x_3+x_4+\cdots+x_n}{2}-x_2\right)\right)-\rho\left(2^k(x_1-x_2)\right)-\rho\left(2^k(x_2-x_3-\cdots-x_n)\right)\\&-\rho\left(2^k(x_3+x_4+\cdots+x_n-x_1)\right)=0\end{aligned}$$

for $k=0,1,\ldots,\ell-1$*. From the definition of* f *and (6.3), we obtain that*

$$\begin{aligned}\Big|F(x_1,x_2,\cdots,x_n)\Big| &\leq \sum_{k=0}^{\infty}\frac{1}{2^k}\Big|\rho\left(2^k\left(\frac{x_1+x_2}{2}-x_3-x_4+\cdots-x_n\right)\right)+\\&\rho\left(2^k\left(\frac{x_2+x_3+x_4+\cdots+x_n}{2}-x_1\right)\right)+\rho\left(2^k\left(\frac{x_1+x_3+x_4+\cdots+x_n}{2}-x_2\right)\right)\\&-\rho\left(2^k(x_1-x_2)\right)-\rho\left(2^k(x_2-x_3-\cdots-x_n)\right)-\rho\left(2^k(x_3+x_4+\cdots+x_n-x_1)\right)\Big|\\&\leq \sum_{k=\ell}^{\infty}\frac{1}{2^k}\Big|\rho\left(2^k\left(\frac{x_1+x_2}{2}-x_3-x_4+\cdots-x_n\right)\right)+\\&\rho\left(2^k\left(\frac{x_2+x_3+x_4+\cdots+x_n}{2}-x_1\right)\right)+\rho\left(2^k\left(\frac{x_1+x_3+x_4+\cdots+x_n}{2}-x_2\right)\right)\\&-\rho\left(2^k(x_1-x_2)\right)-\rho\left(2^k(x_2-x_3-\cdots-x_n)\right)-\rho\left(2^k(x_3+x_4+\cdots+x_n-x_1)\right)\Big|\\&\leq \sum_{k=\ell}^{\infty}\frac{6}{2^k}\mu\leq 12\mu\left(|x_1|+|x_2|+\cdots+|x_n|\right).\end{aligned}$$

Thus f *satisfies (6.1) for all* $x_1,x_2,\cdots,x_n\in\mathbb{R}$ *with* $0<|x_1|+|x_2|+\cdots+|x_n|<1$*.*

We claim that the additive functional equation (1.5) is not stable for $s=1$ *in condition* (ii) *of Corollary 4.1. Suppose on the contrary that there exist a mapping* $\mathcal{A}:\mathbb{R}\to\mathbb{R}$ *and a constant* $\beta>0$ *satisfying (6.2). Since* f *is bounded and continuous for all* $x\in\mathbb{R}$*,* $\mathcal{A}$ *is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 4.1,* $\mathcal{A}$ *must have the form* $\mathcal{A}(x)=cx$ *for any* x *in* $\mathbb{R}$*. Thus we obtain that*

$$|f(x)|\leq(\beta+|c|)\,|x|. \tag{6.4}$$

But we can choose a positive integer m *with* $m\mu>\beta+|c|$*.*

If $x\in\left(0,\frac{1}{2^{m-1}}\right)$*, then* $2^kx\in(0,1)$ *for all* $k=0,1,\ldots,m-1$*. For this* x*, we get*

$$f(x)=\sum_{n=0}^{\infty}\frac{\rho(2^kx)}{2^k}\geq\sum_{n=0}^{m-1}\frac{\mu(2^kx)}{2^k}=m\mu x>(\beta+|c|)\,x$$

which contradicts (6.4). Therefore the additive functional equation (1.5) is not stable in sense of Ulam, Hyers and Rassias if $s=1$*, assumed in the inequality (4.9).*

Now we will provide an example to illustrate that the functional equation (1.5) is not stable for $s=\frac{1}{n}$ in Condition (iii) of Corollary 4.1.

Example 6.2. *Let $\rho : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\rho(x) = \begin{cases} \mu x, & \text{if } |x| < \frac{1}{n} \\ \frac{\mu}{n}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\rho(2^k x)}{2^k} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left|F(x_1, \cdots, x_n)\right| \leq 12\mu \left(|x_1|^{\frac{1}{n}} \cdots |x_n|^{\frac{1}{n}} + |x_1| + \cdots + |x_n|\right) \tag{6.5}$$

for all $x_1, x_2, \cdots, x_n \in \mathbb{R}$. Then there do not exists a mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - \mathcal{A}(x)| \leq \beta|x| \qquad \text{for all} \quad x \in \mathbb{R}. \tag{6.6}$$

Proof. *Now*

$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{|\rho(2^k x)|}{|2^k|} = \sum_{k=0}^{\infty} \frac{\mu}{2^k} = 2\mu.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (6.5).

If $x_1 = x_2 = \cdots = x_n = 0$ then (6.5) is trivial. If $|x_1|^{\frac{1}{n}}|x_2|^{\frac{1}{n}} \cdots |x_n|^{\frac{1}{n}} + |x_1| + |x_2| + \cdots + |x_n| \geq \frac{1}{n}$ then the left hand side of (6.5) is less than 12μ. Now suppose that $0 < |x_1|^{\frac{1}{n}}|x_2|^{\frac{1}{n}} \cdots |x_n|^{\frac{1}{n}} + |x_1| + |x_2| + \cdots + |x_n| < \frac{1}{n}$.

The rest of the proof is similar to the proof of Example 6.1. Therefore the additive functional equation (1.5) is not stable in sense of Ulam, Hyers and Rassias if $s = \frac{1}{n}$, assumed in the inequality (4.9).

7. Conclusion

This article has proved the Hyers-Ulam, Hyers-Ulam-Rassias and Rassias stability results of the n-dimensional additive functional equation in paranormed spaces by using the direct and fixed point methods with suitable counterexamples.

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