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On Crossed Product Rings Over $p.q.$-Baer and Quasi-Baer Rings

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Abstract. In this paper, we consider a ring R and a monoid M equipped with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to \mathrm{Aut}(R)$. The main objective of our study is to investigate the conditions under which the crossed product structure $R \rtimes M$ is $p.q.$-Baer and quasi-Baer rings, and how this property relates to the $p.q.$-Baer property of R and the existence of a generalized join in $I(R)$ for M-indexed subsets, where $I(R)$ denotes the set of ideals of R. Additionally, we prove a connection between R being a left $p.q.$-Baer ring and the CM-quasi-Armendariz property. Moreover, we prove that for any element $\phi^2 = \phi$, there exist an idempotent element $e^2 = e$ such that $\phi = c_e$. We then prove that R is quasi-Baer if and only if the crossed product structure $R \rtimes M$ is quasi-Baer. Finally, we present novel results regarding various constructions for crossed products.

1. Introduction

Throughout this paper, R denotes a ring with unity. Recall that a ring R is considered (quasi-) Baer if, for every nonempty subset (or every right ideal) of R, the right annihilator of that subset (or right ideal) can be generated by an idempotent element from R. In the publication [1], Kaplansky introduced the concept of Baer rings as a means to abstract properties displayed by AW^*-algebras and von Neumann algebras.

This paper focuses on discussing several results related to the crossed product structure $R \rtimes M$ under specific additional conditions. To prove these results, we consider a ring R and a monoid M with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to \mathrm{Aut}(R)$. Our main objective is to investigate the conditions under which the crossed product structure $R \rtimes M$ becomes a right $p.q.$-Baer ring. We also explore the relationship between this property and the right $p.q.$-Baer property of R,

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as well as the existence of a generalized join in $I(R)$ for M-indexed subsets (where $I(R)$ represents the set of ideals of R). Additionally, we prove a connection between R being a left $p.q.$-Baer ring and the CM-quasi-Armendariz property. We prove that for any element $\phi^2 = \phi$ in R, there exist an idempotent element $e^2 = e$ such that $\phi = c_e$. Furthermore, we prove that R is quasi-Baer if and only if the crossed product structure $R \rtimes M$ also possesses the quasi-Baer property. Moreover, if R satisfies both the M-compatibility condition and the CM-quasi-Armendariz property, we show that $(R \rtimes M)$ is a left $p.q.$-Baer ring if and only if, for any M-indexed subset H of R, there exist an idempotent $e \in l_R(\Sigma_{s\in M} R\omega_s(H))$ such that $l_R(\omega_u(\Sigma_{s\in M} R\omega_s(H))) = R\omega_u(e)$ for any $u \in M$. Finally, we present novel results regarding various constructions for crossed products.

In [2], Clark provided a definition for quasi-Baer rings and employed these rings to prove a criterion for determining the isomorphism between a finite-dimensional algebra with unity over an algebraically closed field and a semigroup algebra of twisted matrix units. In [3], the notion of principally quasi-Baer rings was introduced as a generalization of quasi-Baer rings. A ring R is said to be left principally quasi-Baer, or simply left $p.q.$-Baer, if the left annihilator of any principal left ideal of R can be generated by an idempotent element. Similarly, we can define right $p.q.$-Baer rings. A ring is considered to be a $p.q.$-Baer ring if it satisfies the property of being both a right $p.q.$-Baer ring and a left $p.q.$-Baer ring. In other words, a ring is labeled as $p.q.$-Baer if it possesses the property that the left annihilator of any principal left ideal can be generated by an idempotent, and at the same time, the right annihilator of any principal right ideal can also be generated by an idempotent. Observe that biregular rings and quasi-Baer rings are $p.q.$-Baer. For more comprehensive information and specific examples of left $p.q.$-Baer rings, I recommend referring to the following sources [3], [4], [5], [6], [7] and [8]. A ring R is classified as a left APP-ring if, for any element b in R, the left annihilator $l_R(Rb)$ is a right s-unital ideal of R. This concept serves as a generalization that encompasses both left $p.q.$-Baer rings and right PP-rings. It is evident that every left $p.q.$-Baer ring falls within the category of left APP-rings. Consequently, the class of left APP-rings encompasses all biregular rings and all quasi-Baer rings. A ring R is referred to as a right PP-ring (or left PP-ring) if the right (or left) annihilator of any element in R can be generated by an idempotent element. If a ring satisfies both the right and left PP conditions, it is termed a PP-ring. It is evident that every Baer ring falls under the category of PP-ring.

Let Y be a nonempty subset of the ring R. The left annihilator of Y in R is denoted as $l_R(Y)$, while the right annihilator of Y in R is denoted as $r_R(Y)$.

2. Crossed Products Type Construction

According to [9], a ring R is said to be M-Armendariz of crossed product type (M-quasi Armendariz, respectively), or simply CM-Armendariz (CM-quasi Armendariz, respectively), if satisfies the following condition, for any elements $\phi = \sum_{i=1}^{n} a_i g_i$ and $\psi = \sum_{j=1}^{m} b_j h_j \in R \rtimes M$, where a_i, b_j are elements of R and g_i, h_j are elements of M, if the product $\phi\psi = 0$ ($\phi(R \rtimes M)\psi = 0$, respectively), then $a_i\omega_{g_i}(b_j) = 0$

$(a_i R\omega_{g_i s}(b_j) = 0)$ for all i and j. Ali's work [10], a ring R is said to be strongly M-reflexive of crossed product type with respect to the given twisting map f and action map ω (or simply, strongly CM-reflexive) if for any $\phi = c_1 l_1 + c_2 l_2 + \cdots + c_n l_n$ and $\psi = a_1 h_1 + a_2 h_2 + \cdots + a_m h_m \in R \rtimes M$ satisfying that $\phi(R \rtimes M)\psi = 0$ implies that $c_i \omega_{l_i}(\omega_g(Ra_j)) = 0$, then $\psi(R \rtimes M)\phi = 0$ for each i, j and for all $g, l_i, h_j \in M$. Consider a monoid M and a monoid homomorphism $\omega : M \to \mathrm{Aut}(R)$, where $\mathrm{Aut}(R)$ represents the group of automorphisms of the ring R. For any $g \in M$, we denote the automorphism $\omega(g)$ as ω_g. The crossed product $R \rtimes M$ over the ring R is defined as the set of all finite sums of the form $R \rtimes M = \{x_g g | x_g \in R, g \in M\}$. The addition in this crossed product is defined component-wise, and the multiplication is determined using two rules: the action rule and the twisting rule. Specifically, for any $h, g \in M$ and $x \in R$, we have the following definitions, $gx = \omega_g(x)h$, which means that multiplying an element x of R by g and in the crossed product yields the result $hg = f(h, g)hg$, where $f : M \times M \to U(R)$ is a twisted function and $U(R)$ denotes the set of units (invertible elements) of R. This rule implies that multiplying two elements h and g in the crossed product results in $f(h, g)hg$. The twisted function f and the action ω of M on R satisfy the following conditions:

$\omega_h(\omega_g(x)) = f(h, g)\omega_h(\omega_g(x)f(h, g)^{-1})$ and $\omega_h(f(g, k))f(h, gk) = f(h, g)f(h, gk)$, $f(1, h) = f(h, 1) = 1$. It is important to note that the crossed product construction is a general way of constructing a ring using a monoid and a ring. If the twisting function f in the crossed product $R \rtimes M$ is trivial, meaning that $f(a, b) = 1$ for all $a, b \in M$, then the resulting structure is known as the skew monoid ring. In this case, $R \rtimes M$ coincides with the skew monoid ring construction.

On the other hand, if both the twisting function f and the action ω are trivial, then the resulting structure is a monoid ring denoted by $R[M]$. This monoid ring construction is obtained when both the twisting function and the action are trivial, and it is referred to as $R[M]$ in literature references such as [11] and [12]. An ordered monoid is a monoid M in which its elements are linearly ordered with respect to the relation $<$, satisfying the following conditions, for all $a, b, z \in M$, if $a < b$, then $za < zb$ and $az < bz$. In other words, the order on M is compatible with the monoid multiplication. It is a well-known fact, as stated in [12], that torsion-free nilpotent groups and free groups are examples of ordered groups. Therefore, any submonoid of a torsion-free nilpotent group or a free group is also an ordered monoid. This means that if we consider a subset of elements from a torsion-free nilpotent group or a free group and restrict the operation to form a monoid, the resulting structure will still be an ordered monoid. A monoid M is referred to as a unique product monoid, or $u.p.$-monoid, if it satisfies the following property: for any two nonempty finite subsets X and Y of M, there exists a unique element $h \in M$ that can be expressed in the form $h = uv$ with $u \in X$ and $v \in Y$. According to Hirano's work [13], it has been shown that if a ring R is quasi-Baer (or left principally quasi-Baer), and M is an ordered monoid, then the monoid ring RM also possesses the quasi-Baer property (or left principally quasi-Baer property). In their work [14], Nasr-Isfahani and Moussavi introduced a ring R with an endomorphism ω and defined it as ω-weakly rigid if the condition $cRt = 0$ holds if and only if

$c\,\omega(Rt) = 0$ for any $c, t \in R$. It is worth noting that the category of ω-rigid rings and ω-compatible rings is a limited one, and it is evident that every ω-compatible ring falls under the category of ω-weakly rigid rings. However, there exist several classes of ω-weakly rigid rings that do not belong to the category of ω-compatible rings. By [15], R is α-rigid if and only if R is α-compatible and reduced.

According to [14], any prime ring that has an automorphism ω is considered to be ω-weakly rigid. If a monoid homomorphism $\omega : M \to Aut(R)$ is weakly-rigid (compatible), it means that the ring R is also weakly rigid (compatible) with respect to each $g \in M$ under the automorphism ω_g.

3. *p.q.*-Baer and Quasi-Baer Rings of Crossed Product Type

In this section, we discuss various constructions and extensions under which the class of quasi-Baer and $p.q.$-Baer is closed over crossed product $R \rtimes M$, where R is a ring and M is a monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$. The following theorem characterizes left $p.q.$-Baer is closed over crossed product $R \rtimes M$ in terms of M-indexed subsets of the coefficient ring R. An ideal I of a ring R is considered to be right s-unital if there exists an element $e \in I$ for every $t \in I$ such that $te = t$. A ring is referred to as a left APP-ring if the left annihilator $l_R(Rt)$ is right s-unital as an ideal of R for any element $t \in R$.

Lemma 3.1. *[16, Lemma 1.1]. If M is a u.p.-monoid, then M is cancellative ($i.e.$, for $\lambda, k, \mu \in M$, if $\mu\lambda = k\lambda$ or $\lambda\mu = \lambda k$, then $\mu = k$).*

Lemma 3.2. *Let R be a ring, M be a strictly totally ordered monoid, with twisting $f : M \times M \to U(R)$ and with action $\omega : M \to Aut(R)$ and R is M-compatible, then for any $b \in R$,*

$$r_R(bR \rtimes M) = r_{(R\rtimes M)}(c_b(R \rtimes M)).$$

Proof. Let $\phi = \Sigma_{i=1}^{n} a_i k_i$ and $\psi = \Sigma_{j=1}^{m} b_j h_j \in R \rtimes M$ for $a_i, b_j \in R$, $k_i, h_j \in M$ satisfying $\phi(R \rtimes M)\psi = 0$ which implies that $a_i\omega_{k_i}(\omega_{g_s}(Rb_j))f(k_i, h_j)(k_ih_j) = 0$. In the following,we freely use the fact that $\omega_{k_i}(R)f(k_i, h_j) = Rf(k_i, h_j) = R$ for any $k_i, h_j \in M$ and any $a_i, b_j \in R$. Let $\psi \in r_{(R\rtimes M)}(c_b(R \rtimes M))$. Then for every $r \in R$,

$$(c_b\phi c_r\psi)(s) = \sum_{(a_i,b_j)\in X_s(k_i,h_j)} ba_i\omega_{k_i}(\omega_{g_s}(Rb_j))f(k_i, h_j)(k_ih_j) = ba_i\omega_{g_i}(\omega_{g_s}(Rb_j)) = 0.$$

By Lemma 3.1, M is cancellative. Thus $a_iRb_j = 0$ for every $s \in M$ and ω is an automorphism. Hence $\psi \in r_R(bR \rtimes M)$.

Conversely, let $\phi \in r_R(bR \rtimes M)$. Since ω is a map from M to $Aut(R)$, there exist $c_1, c_2, \ldots, c_n \in R$ such that $b_j = \omega_{h_j}(c_j)$ for $j = 1, 2, \ldots, n$. So $c_j \in r_R(\omega_c(a_iR))$ for every $c \in M$. Thus by M-compatible of R for every $\phi \in (R \rtimes M)$,

$$a_i\omega_{k_i}(\omega_{g_s}(Rb_j))f(k_i, h_j)(k_ih_j) = a_i\omega_{k_i}(\omega_{g_s}(Rb_j)) = 0.$$

For any $s \in M, t \in R$, $(c_b\phi\psi)(s) = c_b a_s \omega_{k_s}(R) f(k_s, c)\omega_{g_s}(b_t) f(k_s c, h_t)(k_s h_s) = 0$ and hence $c_b a_s \omega_{k_s}(R) f(k_s, c)\omega_{g_{sc}}(b_t) f(k_s c, h_t) = 0$. Thus, $c_b a_s \omega_{k_s}(R)\omega_{g_{sc}}(b_t) = 0$ since $\omega_{g_s}(R) f(g_s, c) = R$. This shows that $c_b a_s \omega_{k_s}(R\omega_{gc}(b_t)) = 0$ and so $\omega_{k_s}(c_s R\omega_c(b_t)) = 0$, which implies that $c_b(c_s R\omega_c(b_t)) = 0$ since ω_{k_s} is a ring automorphism. Therefore, $\phi \in r_{(R \rtimes M)}(c_b(R \rtimes M))$ and the result follows. □

The following result appeared in Lemma 2 [17].

Lemma 3.3. *Let R be a ring and $(S, \leq)$ a strictly totally ordered monoid satisfying that $0 \leq s$ for all $s \in S$. If $\phi \in [[R^{S,\leq}]]$ is a left semicemtral idempotent, then $\phi(0) \in R$ is a left semicentral idempotent and $\phi[[R^{S,\leq}]] = c_{\phi(0)}[[R^{S,\leq}]]$.*

Let $I(R)$ be the set of all idempotents of R. G be a subset of $I(R)$. We say that G is S-indexed if there exists an artinian and narrow subset I of S such that G is indexed by I (see [18]).

Definition 3.1. *[19] Let G be an S-indexed subset of $I(R)$. We say that G has a generalized join in $I(R)$ if there exists an idempotent $e \in I(R)$ such that*
(1) $gR(1-e) = 0$ for any $g \in G$, and
(2) If $f \in I(R)$ is such that $gR(1-f) = 0$ for any $g \in G$, then $eR(1-f) = 0$.

Theorem 3.1. *Let R be a ring, M be a strictly totally ordered monoid, with twisting $f : M \times M \to U(R)$ and with action $\omega : M \to Aut(R)$ and R is M-compatible satisfying the condition that $0 \leq s$ for all $s \in M$. If $R \rtimes M$ is right p.q.Baer, then R is right p.q.Baer and any M-indexed subset of $I(R)$ has a generalized join in $I(R)$.*

Proof. Suppose $\phi = \Sigma_{i=1}^{n} c_i k_i$ and $\psi = \Sigma_{j=1}^{m} a_j h_j \in R \rtimes M$ for $c_i, a_j \in R, k_i, h_j \in M$. We remember that, since M is a strictly totally ordered monoid, we have $k_1 h_1 \preceq k_i h_j$ for $i \neq 1$ or $j \neq 1$ and M is cancellative by Lemma 3.1. Let b be an element of R. Then, by Lemma 3.2, $r_{R \rtimes M}(c_b R \rtimes M) = r_R(bR) \rtimes M$. Alternatively, due to the right $p.q$. Baer property of $R \rtimes M$, a left semicentral idempotent $\phi \in R \rtimes M$ exists, satisfying the equation $r_{R \rtimes M}(c_b R \rtimes M) = \phi R \rtimes M$. Our aim is to demonstrate that $r_R(bR) = \phi(0)R$, where $\phi(0)^2 = \phi(0)$. This result will prove that R satisfies the $p.q$.Baer property. By Lemma 3.3, $\phi(0)$ is an idempotent of R and $\phi R \rtimes M = c_{\phi(0)} R \rtimes M$. Thus, by compatibility for any $r \in R, s, k \in M$, $c_b c_r c_{\phi(0)} = b\omega_k(\omega_s(R\phi(0))) f(k, e)(kh) = 0$ and $b\omega_k(\omega_s(R\phi(0))) = 0$ which implies that $bR\phi(0) = 0$. Hence $\phi(0) \in r_R(bR)$.
Conversely, assume that $d \in r_R(bR)$. Then for any $g \in R \rtimes M$ and any $v \in M$, $(c_b g c_d)(v) = b\omega_k(\omega_{g_v}(d)) f(k, e)(k\ell) = 0$ and $b\omega_k(\omega_{g_v}(d)) = 0$ implies that $bg(v)d = 0$. Thus, $c_b g c_d = 0$. This means that $c_d \in r_{R \rtimes M}(c_b R \rtimes M)$. So $c_d = c_{\phi(0)} h$ for some $h \in R \rtimes M$, which implies that $d \in \phi(0)R$. Thus, $r_R(bR) = \phi(0)R$. This means that R is right $p.q$.Baer.

Suppose that G is an M-indexed subset of $I(R)$. Then there exist an artinian and narrow subset I of M such that $G = \{e_s \in I(R) \mid s \in I\}$. Define $\phi \in R \rtimes M$ via

$$\phi(s) = \begin{cases} e_s, & s \in I; \\ 0, & s \notin I. \end{cases}$$

Since $R \rtimes M$ is right $p.q.$Baer, there exist a left semicentral idempotent $\lambda \in R \rtimes M$ such that $r_{R\rtimes M}(\phi R \rtimes M) = \lambda R \rtimes M$. By Lemma 3.3, $\lambda(0)$ is an idempotent of R and $c_{\lambda(0)} R \rtimes M = \lambda R \rtimes M$. Thus, $r_{R\rtimes M}(\phi R \rtimes M) = c_{\lambda(0)} R \rtimes M$. Now for any $r \in R$, $(\phi c_r c_{\lambda(0)})(s) = a_i \omega_{k_i}(\omega_{g_s}(r\lambda(0))) f(k_i, e)(k_i \lambda(0)) = 0$ and $a_i \omega_{k_i}(\omega_{g_s}(r\lambda(0))) = 0$. Therefore, $\phi(s) r\lambda(0) = 0$. Thus, $e_s r\lambda(0) = 0$, for all $s \in I$ since ω is an automorphism. Let $g = 1 - \lambda(0)$. Then $e_s r(1-\phi) = 0$, for all $r \in R$. Thus, $e_s R(1-\phi) = 0$. Suppose that e is an idempotent of R such that $e_s R(1-e) = 0$. Then $e_s re = e_s r$, for all $r \in R$. Thus, for any $b \in R$ and for any $\psi \in R \rtimes M$, any $t \in M$,

$$(\phi\psi c_b c_{1-e})(t) = \sum_{(a_i,b_j)\in X_t(\phi,\psi)} a_i \omega_{k_i}(\omega_{g_t}(b(1-e))) f(k_i, h_j)(k_i h_j) = a_i \omega_{k_i}(\omega_{g_t}(b(1-e))) = 0.$$

This means that $c_b c_{1-e} \in r_{R\rtimes M}(\phi R \rtimes M)$ for all $b \in R$. Thus, $c_b c_{1-e} = c_{\phi(0)} c_b c_{1-e}$, which implies that $gb(1-e) = 0$ for all $b \in R$. Hence g is a generalized join of the M-indexed subset G. □

Corollary 3.1. *[8, Theorem 3] Let R be a ring such that $S_\ell(R) = B(R)$. Then $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and any countable family of idempotents in R has a generalized join in $I(R)$.*

Theorem 3.2. *Assuming a ring R, a strictly ordered monoid M, a twisting map $f : M \times M \to U(R)$, and an action map $\omega : M \to Aut(R)$. If R is simultaneously M-compatible and CM-quasi-Armendariz, then the following conditions are equivalent:*
(1) $(R \rtimes M)$ is left p.q.-Baer.
(2) For any M-indexed subset H of R there exist an idempotent $e \in l_R(\Sigma_{s\in M} R\omega_s(H))$ such that $l_R(\omega_u(\Sigma_{s\in M} R\omega_s(H))) = R\omega_u(e)$ for any $u \in M$.
(3) For any M-indexed subset H of R and for any $u \in M$ there exist an idempotent $e \in l_R(\Sigma_{s\in M} R\omega_s(H))$ such that $l_R(\omega_u(\Sigma_{s\in M} R\omega_s(H))) = R\omega_u(e)$.

Proof. Set $B = R \rtimes M$ and $\phi = \Sigma_{i=1}^n c_i k_i, \psi = \Sigma_{j=1}^m a_j h_j \in B$ for $c_i, a_j \in R, k_i, h_j \in M$.

(1) ⇒ (2) Let H be an M-indexed subset of R. By Lemma 2.5 [20] for some $\varphi = \Sigma_{q=1}^r v_q z_q \in H$ we have $H \subseteq \varphi(M) \subseteq H \cup \{0\}$. Since H is left $p.q.$-Baer, $l_B(B\varphi) = B\theta$ for some $\theta = \theta^2 \in B$ by Lemma 3.2. Since $B\theta = l_B(B\varphi)$ is an ideal of $B, \theta B \subseteq B\theta$ and thus for any $\theta = \Sigma_{p=1}^q b_p \ell_p$ for $b_p \in R$

and $\ell_p \in M$,

$$\begin{aligned}\theta B(1-\theta) = \;& b_p\omega_{\ell_p}[c_i\omega_{k_i}(\omega_{g_s}(ra_j))f(k_i,h_j)(k_ih_j)[1-(\omega_{\ell_p}(b_p))]]\\ =\;& b_p\omega_{\ell_p}(c_i\omega_{k_i}(\omega_{g_s}(ra_j))f(k_i,h_j)(k_ih_j))\\ -\;& b_p\omega_{\ell_p}(c_i\omega_{k_i}(\omega_{g_s}(ra_j))f(k_i,h_j)(k_ih_j))(\omega_{\ell_p}(b_p))\\ =\;& b_p\omega_{\ell_p}(c_i\omega_{k_i}(\omega_{g_s}(ra_j))) - b_p\omega_{\ell_p}(c_i\omega_{k_i}(\omega_{g_s}(ra_j))(\omega_{\ell_p}(b_p))\\ =\;& c_i\omega_{k_i}(\omega_{g_s}(ra_j)) - c_i\omega_{k_i}(\omega_{g_s}(ra_j)) = 0.\end{aligned}$$

Since ω is an automorphism and R is CM-quasi-Armendariz, $\theta(1).(1-\theta)(1) = 0$. Hence, it can be deduced that $e = \theta(1)$ is an idempotent element in the ring R. Furthermore, again since $\theta B(1-\theta) = 0$ and R is CM-quasi-Armendariz for any $t \in M$ we have $0 = \theta(t)\omega_t((1-\theta)(1)) = \theta(t)(1-\omega_t(e))$, and thus $\theta(t) = \theta(t)\omega_t(e) = (\theta c_e)(t)$, which shows that $\theta = \theta c_e$. Hence $l_B(B\varphi) = B\theta \subseteq Bc_e$. On the other hand, since $\theta B\varphi = 0$ and R is CM- quasi-Armendariz, for any $s, z_q \in M, v_q \in R, \varphi = \Sigma_{q=1}^{r} v_q z_q$ we have $0 = b_p\omega_{\ell_p}(1)(\omega_s(R(v_q))f(\ell_p, z_q)(\ell_p z_q) = b_p\omega_{\ell_p}(1)\omega_s(R(v_q)) = eR\omega_s(v_q)$, which implies $c_e \in l_B(B\varphi)$, and $l_B(B\varphi) = Bc_e$ follows. Note that since e is an idempotent of R, c_e is an idempotent of B.

We can demonstrate two things: Firstly, it is shown that e belongs to the left ideal $l_R(\Sigma_{s\in M} R\omega_s(H))$, and secondly, we prove that $l_R(\omega_u(\Sigma_{s\in M} R\omega_s(H))) = R\omega_u(e)$ for any $u \in M$.

Consider any arbitrary element $d \in H$. We can express $d = \varphi(x)$ for some $x \in M$. Since $c_e B\varphi = 0$ and R satisfies the CM-quasi-Armendariz property, we have $0 = c_e(1)R\omega_s(\varphi(x)) = eR\omega_s(d)$ for all $s \in M$. This implies that e belongs to $l_R(\Sigma_{s\in M} R\omega_s(H))$. Furthermore, it also implies that $R\omega_u(e) \subseteq l_R(\omega_u(\Sigma_{s\in M} R\omega_s(H)))$.

To prove the reverse inclusion, consider an element $y \in l_R(\omega_u(\Sigma_{s\in M} R\omega_s(H)))$. Since $\varphi(M) \subseteq H \cup \{0\}$, we can deduce that $\zeta_y^u l_B(B\varphi) = Bc_e$. Consequently, we have $\zeta_y^u = \zeta_y^u c_e$. This implies that $y = \zeta_y^u(u) = (\zeta_y^u c_e)(u) = \zeta_y^u(u)\omega_u(c_e(1)) = y\omega_u(e)$. Therefore, we conclude that $y \in R\omega_u(e)$, hence $l_R(\omega_u(\Sigma_{s\in M} R\omega_s(H))) \subseteq R\omega_u(e)$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Suppose $\varphi \in B$ and $H = \varphi(M)$. According to Lemma 2.5 [20], it can be concluded that the set H is indexed by M.

By substituting $u = 1$ into (3), it becomes evident that there exists an idempotent element $e \in R$ such that $l_R(\Sigma_{s\in M} R\omega_s(H)) = Re$. It is evident that c_e is an idempotent element of B. To complete the proof, it is enough to show that $l_B(B\varphi) = Bc_e$. Since $e \in l_R(\Sigma_{s\in M} R\omega_s(H))$, it implies that Bc_e is a subset of $l_B(B\varphi)$.

On the contrary, if $\phi \in l_B(B\varphi)$, then considering $\phi = \Sigma_{i=1}^{n} c_i k_i$ and $\varphi = \Sigma_{q=1}^{r} v_q z_q$, where c_i, k_i, v_q, z_q are elements of B, and R satisfies the CM-quasi-Armendariz property, we can observe that $\phi B\varphi = 0$. Hence, for any k_i, s, $z_q \in M$, we obtain $c_i\omega_{k_i}(R\omega_s(v_q))f(k_i, z_q)(k_i z_q) = 0$. As a result, we conclude that $c_i\omega_{k_i}(R\omega_s(v_q)) = 0$.

Consequently, it follows that $c_i \in l_R(\omega_u(\Sigma_{s\in M} R\omega_s(H)))$ for any $u \in M$. Thus, according to equation (3), for each $u \in M$, there exists an idempotent element $e_u \in l_R(\Sigma_{s\in M} R\omega_s(H))$ such that $\phi(u) \in R\omega_u(e_u)$. Since $l_R(\Sigma_{s\in M} R\omega_s(H)) = Re$, for any $u \in M$, we have $e_u = e_u e$, and consequently, $\phi(u) = \phi(u)\omega_u(e_u) = \phi(u)\omega_u(e_u e) = \phi(u)\omega_u(e_u)\omega_u(e) = \phi(u)\omega_u(e) = (\phi c_e)(u)$ since ω is an automorphism. Therefore, we can conclude that $\phi = \phi c_e \in Bc_e$, which implies that $l_B(B\varphi) = Bc_e$, as required. □

Theorem 3.3. *Assuming a ring R, a strictly totally ordered monoid M with a twisting map $f : M \times M \to U(R)$, and an action map $\omega : M \to Aut(R)$ that is compatible with the multiplication in M. If R is a left p.q.-Baer ring, then R is CM-quasi-Armendariz.*

Proof. The proof presented here is a modified version of the proof provided in Proposition 2.9 [21]. Consider the elements $\phi = c_1k_1 + c_2k_2 + \cdots + c_nk_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ belonging to $R \rtimes M$. Assume that $\phi(R \rtimes M)\psi = 0$. Given that M is a strictly totally ordered monoid, we can make the assumption that $k_i \preceq k_j$ and $h_i \preceq h_j$ whenever $i < j$. Based on this assumption, we make the claim that $c_i\omega_{k_i}(\omega_g(Ra_j)) = 0$ for all i and j.

To prove this claim, let r be an element of R. Then, we observe that $\phi(re)\psi = 0$ since $\phi(R \rtimes M)\psi = 0$. Thus, we have

$$\begin{aligned} 0 = \;& \phi(re)\psi = c_1rf(k_1, e)a_1f(k_1, h_1)k_1h_1 + \cdots + [c_nrf(k_n, e)a_{m-2}f(k_n, h_{m-2})k_nh_{m-2} \\ +\;& c_{n-1}rf(k_{n-1}, e)a_{m-1}f(k_{n-1}, h_{m-1})k_{n-1}h_{m-1} + c_{n-2}rf(k_{n-2}, e)k_mf(k_{n-2}, h_m)k_{n-2}h_m] \\ +\;& [c_nrf(k_n, e)a_{m-1}f(k_n, h_{m-1})k_nh_{m-1} + a_{n-1}rf(k_{n-1}, e)a_mf(k_{n-1}, h_m)k_{n-1}h_m] \\ +\;& c_nrf(k_n, e)a_mf(k_n, h_m)k_nh_m. \end{aligned} \tag{2.2}$$

Consequently, we can deduce that $c_nrf(k_n, e)a_mf(k_n, h_m) = 0$ because k_nh_m has the highest order among all the $k_ih_j's$ terms. As a result, we obtain $c_nrf(k_n, e)a_m = 0$. This indicates that c_n belongs to $\ell_R(Rf(k_n, e)a_m) = \ell_R(Ra_m)$. Therefore, we can conclude that $\ell_R(Ra_m) = Re_m$ for some idempotent e_m as assumed

By substituting r with re_m in Eq. (2.2), we arrive at the following expression.

$$\begin{aligned} 0 = \;& c_1re_mf(k_1, e)a_1f(k_1, h_1)k_1h_1 + \cdots + [c_nre_mf(k_n, e)a_{m-2}f(k_n, h_{m-2})k_nh_{m-2} \\ +\;& c_{n-1}re_mf(k_{n-1}, e)a_{m-1}f(k_{n-1}, h_{m-1})k_{n-1}h_{m-1}] \\ +\;& c_nre_mf(k_n, e)a_{m-1}f(k_n, h_{m-1})k_nh_{m-1}. \end{aligned} \tag{2.3}$$

So $c_nre_mf(k_n, e)a_{m-1}f(k_n, h_{m-1}) = 0$, because k_nh_{m-1} is of highest order in $\{k_ih_j|1 \le i \le n, 1 \le j \le m\}$ $\{k_{n-1}h_m, k_nh_m\}$. Hence $c_nre_mf(k_n, e)a_{m-1} = 0$. Since Re_m is an ideal of R and $e_m \in Re_m$, we have $e_mr \in Re_m$ and thus $e_mr = e_mre_m$ for all $r \in R$. Furthermore, we can also observe that $c_n = c_ne_m$ since $c_n \in \ell_R(Ra_m) = Re_m$. So $c_nrf(k_n, e)a_{m-1} = c_ne_mrf(k_n, e)a_{m-1} = c_ne_mre_mf(k_n, e)a_{m-1} = c_nre_mf(k_n, e)a_{m-1}$
$= 0$. This implies that $c_n \in \ell_R(Ra_m + Ra_{m-1})$. Consequently, we can conclude that $\ell_R(Ra_m + Ra_{m-1}) = Re_{m-1}$ for a certain idempotent $e_{m-1} \in R$, as R is a left $p.q.$-Baer. By substituting r with

re_{m-1} in Eq. (2.3), we obtain the equation $c_n re_{m-1} f(k_n, e) a_{m-2} f(k_n, h_{m-2}) = 0$ using the same reasoning as before. This demonstrates that $c_n \in \ell_R(Ra_m + Ra_{m-1} + Ra_{m-2})$. By continuing this process, we can deduce that $c_n R a_t = 0$ for all $t = 1, 2, \dots, m$. Therefore, we have $(c_1k_1 + c_2k_2 + \cdots + c_{n-1}k_{n-1})(R \rtimes M)(a_1h_1 + a_2h_2 + \cdots + a_mh_m) = 0$. By utilizing induction on $m + n$, we can prove that $c_i \omega_{k_i}(\omega_g(Ra_j)) = 0$. As a result, we can conclude that R is CM-quasi-Armendariz. □

Corollary 3.2. *[22, Corollary 2.4] Let M be a u.p.-monoid and R a right PP-ring or a left p.q.-Baer ring. Then R is M-quasi-Armendariz.*

In [23], it was proven that a ring R is quasi-Baer if and only if $R[[S, \omega]]$ is quasi-Baer. Based on this result, we can conclude the following.

Theorem 3.4. *Assuming a ring R and a strictly ordered monoid M with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$, we consider the case where R is both M-compatible and CM-quasi-Armendariz. Additionally, given any element $\phi^2 = \phi$ in R, there exist an idempotent element $e^2 = e$ such that $\phi = c_e$. Then R is quasi-Baer if and only if $R \rtimes M$ is a quasi-Baer.*

Proof. ($\Rightarrow$) Let $U \in R \rtimes M$ is a subset, since R is quasi-Baer, there exist $e^2 = e \in R$ such that $r_R(C_U) = eR$, where $C_U R$ denotes generated by C_U subset of R. We want to show that $r_{R\rtimes M}(U) = C_e R \rtimes M$. For any $\phi = \sum_{i=1}^{n} c_i k_i, \psi = \sum_{j=1}^{m} a_j h_j \in R \rtimes M$ for $c_i, a_j \in R, s, k_i, h_j \in M$ and $\varphi = \sum_{r=1}^{p} b_r g_r \in U$ for $\rho \in M$

$$(\varphi\psi C_e\phi)(s) \sum_{(b_r, a_j, c_i) \in X_s(\varphi, \psi, \phi)} b_r \omega_{g_r}(a_j \omega_{\rho_s}(ec_i)) f(g_r, ek_i)(eg_r k_i).$$

Because $b_r a_j \in C_U R$ and $ec_i \in eR$, so $(\varphi\psi C_e\phi)(s) = b_r \omega_{g_r}(a_j \omega_{\rho_s}(ec_i)) = 0$ since ω is an automorphism and M is cancellative by Lemma 3.1, therefore $\varphi\psi C_e \phi = 0$. This means that $r_{R\rtimes M}(U) \supseteq C_e R \rtimes M$. Conversely, let $\phi \in r_{R\rtimes M}(U), \varphi \in U$, then $\varphi(R \rtimes M)\phi = 0$. Because R is S-compatible CM-quasi-Armendariz, we have $b_r \omega_{g_r}(R\omega_{\rho_s}(c_i)) f(g_r, k_i)(g_r k_i) = 0$ so, $b_r \omega_{g_r}(R\omega_{\rho_s}(c_i)) = 0$. This means $c_i \in r_R(C_U R)$. Therefore, there exist $r_s \in R$ such that $\phi(s) = er_s$. We have map $\theta : M \to R$ as follows

$$\theta(s) = \begin{cases} r_s, & s \in supp(\phi); \\ 0, & s \in M - supp(\phi), \end{cases}$$

so because $supp(\theta) = supp(\phi)$ we have $\theta \in R \rtimes M$. Easy to show $\phi = C_e\theta \in C_e R \rtimes M$, So $r_{R\rtimes M}(U) \subseteq C_e R \rtimes M$. Thus, $r_{R\rtimes M}(U) = C_e R \rtimes M$. Therefore, $R \rtimes M$ is quasi-Baer.

($\Leftarrow$) For any subset $Q \in R$, let $V = \{\phi \in R \rtimes M \mid c_i \in Q\}$ and let $VR \rtimes M$ denotes the subsets of $R \rtimes M$, which is generated by V. Therefore, there exist $e^2 = e \in R$ such that

$$r_{R\rtimes M}(VR \rtimes M) = C_e R \rtimes M.$$

We can show that $r_R(Q) = eR$. For any $c_i \in Q, b_r \in R$, we have $(C_{c_i} C_e C_{b_r})(0) = c_i \omega_{k_i}(e\omega_{\rho_s}(b_r)) f(k_i, g_r)(k_i g_r) = c_i \omega_{k_i}(e\omega_{\rho_s}(b_r)) = c_i e b_r = 0$. So $eR \subseteq r_R(Q)$, and let $a \in r_R(Q)$, because $C_a \in r_{R\rtimes M}(VR \rtimes M)$, there exist $\psi \in R \rtimes M$ such that $C_a = C_e\psi$. So $a = C_e\psi(0) = e\psi(0) \in eR$, this means that $r_R(Q) \subseteq eR$, so $r_R(Q) = eR$. Therefore, R is quasi-Baer. □

Corollary 3.3. *Assuming a ring R and a strictly ordered monoid M with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$. If R is M-compatible quasi-Baer, then R is CM-quasi-Armendariz.*

Proof. If R is a quasi-Baer ring, then it is a left $p.q.$-Baer ring, as shown in [24]. Therefore, the result can be deduced from Theorem 3.3. □

Corollary 3.4. *Assuming a ring R and a strictly ordered monoid M with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$. If R is M-compatible quasi-Baer, then R is strongly CM-reflexive.*

Proof. If R is a quasi-Baer ring, it is a left $p.q.$-Baer ring, as stated in [24]. Consequently, R is strongly CM-reflexive by [10]. □

Corollary 3.5. *[24, Proposition 2.3] For any ring, we have the following implications:*
(1) *right $PP \Rightarrow$ left APP.*
(2) *quasi-Baer $\Rightarrow$ $p.q.$-Baer $\Rightarrow$ left APP $\Rightarrow$ quasi-Armendariz.*

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