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A Graphical Representation of the Truncated Moment of the Solution of a Nonlinear SPDE

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Abstract. We consider a stochastic partial differential equation driven by a Lévy type noise (SPDE). Particular attention is given to the correlation function which measures the moments of the solution. Using the Feynman graph formalism, the solution of the SPDE as well as its truncated moments are given as a sum over specific graphs that are evaluated according to some rules. A remark on some applications will be given at the end of this work.

1. Introduction

The dynamics of many phenomena studied in sciences, engineering and economy are described in many cases by stochastic differential equations (SDEs), see for example [1, 2, 6]. Stochastic partial differential equations (SPDE) is an interesting class of SDEs with numerous applications in different fields, which received a lot of attention during the last 50 years. SPDEs driven by Gaussian noise are intensively studied and they have found many applications in different areas from physics to biology and mathematical finance, see e.g. [4, 17].

The extensions to Lévy type noise (in the sense that random variables with Lévy distribution extended Gaussian random variable) are less studied, particularly when we consider nonlinear SPDEs driven by a Lévy noise, since the complex behaviour of their solutions is behind any further study compared to SPDEs driven by Gaussian noise. Moreover, in many cases there is no explicit distribution of the Lévy process.

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In [4], the authors have studied the asymptotic expansion for SPDEs driven by Lévy noise, and as a particular case, they considered Fitz-Hugh-Nagumo equations due to their impact in neurobiology, see e.g. [3]. Asymptotic expansions of the solution of a class of SDEs or SPDEs are also studied in [4, 6, 7, 25].

Another important class of SPDEs driven by Gaussian noise are the Kardar-Parisi-Zhang equations which describe physical and probabilistic models (interacting particle systems, polymers in random environments, etc.), see e.g. [19]. An extension of the KPZ equation to SPDEs driven by Lévy noise is studied in [17], where the authors have adopted a new approach of the solution using the Feynman graphs and rules, see also [25].

A suitable way to take care of stochastic influence, complexity and randomness is to generalize such class of SPDEs by adding more factors that describe models in financial market, weather forecast, climatic changes, neurobiological process. It is therefore the aim of this paper to study a class of nonlinear SPDEs driven by Lévy noise, which is more general than the previous studied models in [17] and [25, 26]. We shall adopt methods based on Feynman graphs and rules to represent the solution of a given class of a nonlinear SPDE, as well as its truncated moments.

Before describing the contents of the different sections of the present paper, let us mention that to the best of our knowledge the current results seem to be new and not studied before.

The contents of this paper are described as follows:
Section 2 will be reserved to the motivation and description of the problem.In section 3, the analytic solution of the SPDE will be given. Section 4 will be dedicated to the main results of this paper, where we will define new type of Feynman graphs and rules to simplify the analytic solution provided in the previous section. In Section 5, the truncated moments of the solution will be graphically represented. Section 6 is reserved to a remark on some applications.

2. Preliminaries

Consider the stochastic partial differential equation (SPDE):

$$\begin{cases} \dfrac{\partial X}{\partial t}(t,\boldsymbol{x}) = \Delta^p X(t,\boldsymbol{x}) - \mu X(t,\boldsymbol{x}) - \lambda|\nabla X(t,\boldsymbol{x})|^2 + \eta(t,\boldsymbol{x}), \\ X(0,\boldsymbol{x}) = f(\boldsymbol{x})\ ;\ (t,\boldsymbol{x}) \in \Lambda =]0,+\infty[\times L_\delta,\ \ \mu,\ \lambda\ > 0 \end{cases} \tag{2.1}$$

where $L_\delta = \left\{\delta z\ ,\ z \in \mathbb{Z}^d\right\}$ is the lattice on $\mathbb{Z}^d$, η is a general space-time white noise of Lévy type and f is a given initial function. ∇X is the discrete gradiant defined by

$$\nabla X(\boldsymbol{x}) = \delta^{-1}\Big(X(\boldsymbol{x}+\delta\boldsymbol{e}_1) - X(\boldsymbol{x}), \dots, X(\boldsymbol{x}+\delta\boldsymbol{e}_d) - X(\boldsymbol{x})\Big), \tag{2.2}$$

while the p-discrete Laplacian $\Delta^p X$ is defined as $\Delta^p = \Delta(\Delta^{p-1}X)$ and

$$\Delta X(\boldsymbol{x}) = \delta^{-2}\Big(-2dX(\boldsymbol{x}) + \sum_{|\boldsymbol{y}-\boldsymbol{x}|=\delta} X(\boldsymbol{y})\Big), \tag{2.3}$$

where $(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_d)$ is the canonical basis of $\boldsymbol{R}^d$.

Note that ∇X and $\Delta^p X$ can be formulated using a random walk on L_δ. Indeed, let $(Z_n^{\boldsymbol{x}})_n$ be a random walk on L_δ starting at $Z_0^{\boldsymbol{x}} = \boldsymbol{x}$. More precisely, we have

$$\mathbb{P}\Big\{Z_{n+1}^{\boldsymbol{x}} - Z_n^{\boldsymbol{x}} = -\delta \boldsymbol{e}_i\Big\} = \mathbb{P}\Big\{Z_{n+1}^{\boldsymbol{x}} - Z_n^{\boldsymbol{x}} = \delta \boldsymbol{e}_i\Big\} = \frac{1}{2d}, \tag{2.4}$$

for $i = 1, \dots, d$. Then, the discrete Laplacian can be written as

$$\Delta X(\boldsymbol{x}) = 2d\delta^{-2}\mathbb{E}\left[X\left(Z_1^{\boldsymbol{x}}\right) - X\left(Z_0^{\boldsymbol{x}}\right)\right], \tag{2.5}$$

where $\mathbb{E}$ is the expectation with respect to the distribution of $(Z_n^{\boldsymbol{x}})_{n\geq 0}$. By induction on p, we have

$$\Delta^p X(\boldsymbol{x}) = \left(2d\delta^{-2}\right)^p \mathbb{E}\left[\sum_{n=0}^{p}\binom{p}{n}(-1)^n X(Z_n^{\boldsymbol{x}})\right]. \tag{2.6}$$

Let $\boldsymbol{Y}$ be a random variable given by the distribution

$$P\left(\boldsymbol{Y} = \delta \boldsymbol{e}_i\right) = \frac{1}{d}, \tag{2.7}$$

for $i = 1, \dots, d$. Then, $|\nabla X(t, \boldsymbol{x})|^2$ can be written as

$$\begin{aligned} |\nabla X(t,\boldsymbol{x})|^2 &= \delta^{-2}\sum_{i=1}^{d}\left(X(t,\boldsymbol{x}+\delta\boldsymbol{e}_i) - X(t,\boldsymbol{x})\right)^2 \\ &= d\delta^{-2}E\left[\left(X(t,\boldsymbol{x}+\boldsymbol{Y}) - X(t,\boldsymbol{x})\right)^2\right], \end{aligned} \tag{2.8}$$

where E is the expectation with respect to the distribution of $\boldsymbol{Y}$.

The main idea is to solve perturbatively the SDPE (2.1) by letting the solution X having the following form:

$$X(t,\boldsymbol{x}) = \sum_{l=0}^{\infty}(-\lambda)^l X_l(t,\boldsymbol{x}). \tag{2.9}$$

3. Analytic Solution of the SPDEs

In this section, we will first introduce Green's functions associated to the linear operators of equation (2.1). Such class of Green functions will be used to determine the form of the solution.

Denote by $S(\Lambda)$ the Schwartz space of all rapidly decreasing functions on Λ occupied with the Schwartz topology and define the Fourier transform $\mathcal{F} : S(\Lambda) \longrightarrow \mathbb{R}\times\left[0, \frac{2\pi}{\delta}\right]^d$ by

$$\mathcal{F}(f)(t,\boldsymbol{x}) = \int_{\mathbb{R}\times L_\delta} e^{its}e^{iy.x}f(s,\boldsymbol{y})dsd\boldsymbol{y}, \tag{3.1}$$

where

$$\int_{\mathbb{R}\times L_\delta} g(s,\boldsymbol{y})d\boldsymbol{y}ds = \sum_{\boldsymbol{y}\in L_\delta}\delta^d\int_{\mathbb{R}} g(s,\boldsymbol{y})ds. \tag{3.2}$$

The inverse Fourier transform of $\mathcal{F}$ denoted by $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}(f)(t,\boldsymbol{x}) = \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}}\int_{\left[0,\frac{2\pi}{\delta}\right]^d} e^{-its}e^{-iy.x}f(s,\boldsymbol{y})dsd\boldsymbol{y}. \tag{3.3}$$

Similarly, we define the Fourier transform on $S(L_\delta)$ as

$$\hat{\mathcal{F}}(f)(\boldsymbol{x}) = \int_{L_\delta} e^{i\boldsymbol{y}.\boldsymbol{x}} f(\boldsymbol{y}) d\boldsymbol{y}, \tag{3.4}$$

where $\int_{L_\delta} f(\boldsymbol{y}) d\boldsymbol{y} = \sum_{\boldsymbol{y}\in L_\delta} \delta^d f(\boldsymbol{y})$.

Note that its inverse Fourier transform is

$$\hat{\mathcal{F}}^{-1}(f)(\boldsymbol{x}) = \frac{1}{(2\pi)^d} \int_{[0,\frac{2\pi}{\delta}]^d} e^{-i\boldsymbol{y}.\boldsymbol{x}} f(\boldsymbol{y}) d\boldsymbol{y}. \tag{3.5}$$

Finally, we define the convolution products on $S(\Lambda)$ and $S(L_\delta)$ by

$$f \star g(t, \boldsymbol{x}) = \int_\Lambda f(s, \boldsymbol{x}) g(t-s, \boldsymbol{x}-\boldsymbol{y}) ds d\boldsymbol{y}, \text{ for } f, g \in S(\Lambda) \tag{3.6}$$

and

$$f * g(\boldsymbol{x}) = \int_{L_\delta} f(\boldsymbol{x}) g(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{y}, \text{ for } f, g \in S(L_\delta). \tag{3.7}$$

Using the definitions of the p-Laplacian and the Fourier transform given by Equations (2.6) and (3.1) respectively, we have

$$\begin{aligned} \mathcal{F}\left(\Delta^p X\right)(t, \boldsymbol{x}) &= \int_{\mathbb{R}\times L_\delta} e^{its} e^{i\boldsymbol{y}.\boldsymbol{x}} \Delta^p X(s, \boldsymbol{y}) d\boldsymbol{y} ds \\ &= \left(2d\delta^{-2}\right)^p \sum_{n=0}^{p} \binom{p}{n} (-1)^{p-n} \mathbb{E}\left[\int_{\mathbb{R}\times L_\delta} e^{its} e^{i\boldsymbol{y}.\boldsymbol{x}} X(s, Z_n^{\boldsymbol{y}}) d\boldsymbol{y} ds\right] \\ &= \left(2d\delta^{-2}\right)^p \sum_{n=0}^{p} \binom{p}{n} (-1)^{p-n} \mathbb{E}\left[e^{-iZ_n^{\boldsymbol{0}}.\boldsymbol{x}}\right] \mathcal{F}(X)(t, \boldsymbol{x}) \\ &= (2\delta^{-2})^p \left[\sum_{n=1}^{d} \cos\left(\delta \boldsymbol{e}_n.\boldsymbol{x}\right) - d\right]^p \mathcal{F}(X)(t, \boldsymbol{x}). \end{aligned} \tag{3.8}$$

Here, we have used the characteristic function of $Z_n^{\boldsymbol{0}}$ given by

$$\mathbb{E}\left[e^{-iZ_n^{\boldsymbol{0}}.\boldsymbol{x}}\right] = \left(\frac{1}{d}\sum_{k=1}^{d} \cos\left(\delta \boldsymbol{e}_k.\boldsymbol{x}\right)\right)^n. \tag{3.9}$$

Consider the Green's function G, which is nothing else than the kernel of the linear part of the SPDE (2.1),

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta^p + \mu\right) G(t, \boldsymbol{x}) = \delta(t, \boldsymbol{x}), \\ G(t, \boldsymbol{x}) = 0, \text{ for } t < 0, \end{cases} \tag{3.10}$$

Here, $\delta(t, \boldsymbol{x})$ is the Dirac distribution defined by $\delta(t, \boldsymbol{x}) = \delta(t)\delta^{-d}\delta_{\boldsymbol{0},\boldsymbol{x}}$, where $\delta(t)$ is the Dirac distribution on $\mathbb{R}$, $\delta_{\boldsymbol{x},\boldsymbol{y}} = \prod_{l=1}^{d} \delta_{x_l,y_l}$ and $\delta_{a,b}$ is the Kronecker symbol.

Applying the Fourier transform to equation (3.10) and using equation (3.8), we obtain

$$\mathcal{F}(G)(t, \boldsymbol{x}) = \frac{1}{-it + \Gamma_{\delta,\mu}(\boldsymbol{x})}, \tag{3.11}$$

where

$$\Gamma_{\delta,\mu}(\boldsymbol{x}) = \mu - (2\delta^{-2})^p \left[\sum_{n=1}^{d} \cos\left(\delta \boldsymbol{e}_n.\boldsymbol{x}\right) - d\right]^p . \tag{3.12}$$

Lemma 3.1. *For $m > 0$, we have*

$$\int_{-\infty}^{\infty} \frac{e^{its}}{m + is} ds = 2\pi\theta(t)e^{-tm}, \tag{3.13}$$

where $\theta(t) = 1$ if $t > 0$ and $\theta(t) = 0$ if $t < 0$.

Proof. For $t > 0$ and $C_R^- = \{m + is : -R \leq s \leq R\} \cup \{m + Re^{is} : \frac{\pi}{2} \leq s \leq \frac{3\pi}{2}\}$ oriented on the counter-clockwise direction, applying Residue theorem yields

$$\int_{C_R^-} \frac{e^{tz}}{z} dz = 2i\pi. \tag{3.14}$$

This is equivalent to

$$2i\pi = i\int_{-R}^{R} \frac{e^{t(m+is)}}{m + is} ds + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{iRe^{is}e^{t(m+Re^{is})}}{m + Re^{is}} ds. \tag{3.15}$$

Since

$$\left| \frac{iRe^{is}e^{t(m+Re^{is})}}{m + Re^{is}} \right| = \frac{Re^{t(m+R\cos s)}}{\sqrt{m^2 + R^2}} \leq e^{tm},$$

and $\lim\limits_{R\to\infty} \dfrac{Re^{t(m+R\cos s)}}{\sqrt{m^2 + R^2}} = 0$, Lebesgue's dominated convergence theorem gives

$$\lim_{R\to\infty} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{iRe^{is}e^{t(m+Re^{is})}}{m + Re^{is}} ds = 0.$$

Thus, the limit of equation (3.15) is

$$\int_{-\infty}^{\infty} \frac{e^{its}}{m + is} ds = 2\pi e^{-tm}. \tag{3.16}$$

For $t < 0$, following the previous steps with $C_R^+ = \{m + is : -R \leq s \leq R\} \cup \{m + Re^{is} : -\frac{\pi}{2} \leq s \leq \frac{\pi}{2}\}$ oriented on the clockwise direction, we have

$$\int_{-\infty}^{\infty} \frac{e^{its}}{m + is} ds = 0. \tag{3.17}$$

□

By Equation (3.11) and lemma 3.1, we get

$$\begin{aligned} G(t,\boldsymbol{x}) &= \frac{1}{(2\pi)^{d+1}} \int_{[0,\frac{2\pi}{\delta}]^d} \left(\int_{-\infty}^{\infty} \frac{e^{its}}{is + \Gamma_{\delta,\mu}(\boldsymbol{y})} ds \right) e^{i\boldsymbol{x}.\boldsymbol{y}} d\boldsymbol{y} \\ &= \frac{\theta(t)}{(2\pi)^d} \int_{[0,\frac{2\pi}{\delta}]^d} e^{i\boldsymbol{y}.\boldsymbol{x}} e^{-t\Gamma_{\delta,\mu}(\boldsymbol{y})} d\boldsymbol{y}. \end{aligned} \tag{3.18}$$

Lemma 3.2. *For any integer N, there exists a constant C_N such that*

$$|G(t,\boldsymbol{x})| \le C_N \frac{e^{-t\frac{\mu}{2}}}{(1+|\boldsymbol{x}|^2)^N}, \tag{3.19}$$

for any $(t,\boldsymbol{x}) \in \Lambda =]0,+\infty[\times L_\delta$.

Proof. Let Δ_1 be the Laplacian on $\left[0, \frac{2\pi}{\delta}\right]^d$. Then, we have

$$\begin{aligned} G(t,\boldsymbol{x}) &= \frac{\theta(t)}{(2\pi)^d} \int_{\left[0,\frac{2\pi}{\delta}\right]^d} e^{i\boldsymbol{y}.\boldsymbol{x}} e^{-t\Gamma_{\delta,\mu}(\boldsymbol{y})} d\boldsymbol{y}. \\ &= \frac{\theta(t)}{(2\pi)^d(1+|\boldsymbol{x}|^2)^N} \int_{\left[0,\frac{2\pi}{\delta}\right]^d} \left[(1-\Delta_1)^N e^{i\boldsymbol{y}.\boldsymbol{x}}\right] e^{-t\Gamma_{\delta,\mu}(\boldsymbol{y})} d\boldsymbol{y} \\ &= \frac{\theta(t)}{(2\pi)^d(1+|\boldsymbol{x}|^2)^N} \int_{\left[0,\frac{2\pi}{\delta}\right]^d} e^{i\boldsymbol{y}.\boldsymbol{x}} \left[(1-\Delta_1)^N e^{-t\Gamma_{\delta,\mu}(\boldsymbol{y})}\right] d\boldsymbol{y} \\ &= \frac{\theta(t)e^{-\frac{t\mu}{2}}}{(2\pi)^d(1+|\boldsymbol{x}|^2)^N} \int_{\left[0,\frac{2\pi}{\delta}\right]^d} e^{i\boldsymbol{y}.\boldsymbol{x}} \left[(1-\Delta_1)^N e^{-t\Gamma_{\delta,\frac{\mu}{2}}(\boldsymbol{y})}\right] d\boldsymbol{y}. \end{aligned} \tag{3.20}$$

Note that

$$(1-\Delta_1)^N e^{-t\Gamma_{\delta,\frac{\mu}{2}}(\boldsymbol{y})} = P^N(t,\delta,\boldsymbol{y}) e^{-t\Gamma_{\delta,\frac{\mu}{2}}(\boldsymbol{y})} \tag{3.21}$$

where $P^N(t,\delta,\boldsymbol{y})$ is a polynomial function with respect to t. Thus, we obtain

$$C_N := \frac{1}{\delta^d} \sup_{t\in(0,\infty),\boldsymbol{y}\in\left[0,\frac{2\pi}{\delta}\right]^d} \left|(1-\Delta_1)^N e^{-t\Gamma_{\delta,\frac{\mu}{2}}(\boldsymbol{y})}\right| < \infty. \tag{3.22}$$

This means that

$$|G(t,\boldsymbol{x})| \le \frac{e^{-\frac{t\mu}{2}}}{(2\pi)^d(1+|\boldsymbol{x}|^2)^N} \int_{\left[0,\frac{2\pi}{\delta}\right]^d} \delta^d C_N d\boldsymbol{y} = C_N \frac{e^{-t\frac{\mu}{2}}}{(1+|\boldsymbol{x}|^2)^N}. \tag{3.23}$$

□

Lemma 3.3. *Let $\mathcal{C}$ be the set of all measurable functions $f : \Lambda \to \mathbb{R}$ such that for any $k \in \mathbb{N}, \exists N \in \mathbb{N}$ and*

$$\int_\Lambda \frac{|f(x)|^k}{(1+|x|^2)^N} dx < \infty.$$

Then, $\mathcal{C}$ is an algebra under multiplication. In addition, if $f \in \mathcal{C}$, then $G * f \in \mathcal{C}$.

Proof. Cauchy Schwartz inequality

$$\int \frac{|fg|^k(x)}{(1+|x|^2)^N} dx \le \sqrt{\int \frac{|f|^{2k}(x)}{(1+|x|^2)^N} dx} \sqrt{\int \frac{|g|^{2k}(x)}{(1+|x|^2)^N} dx} \tag{3.24}$$

guarantees that $\mathcal{C}$ is an algebra under multiplication.

Moreover, suppose that $f \in \mathcal{C}$. For any $k \in \mathbb{N}$, $\exists N_k \in \mathbb{N}$ such that

$$\int_\Lambda \frac{|f(x)|}{(1+|x|^2)^{N_k}} dx < \infty. \tag{3.25}$$

The triangular inequality leads to

$$1+|y|^2 \le 1+2|y-x|^2+2|x|^2 \le 2(1+|x|^2)(1+|y-x|^2). \tag{3.26}$$

Using the last inequality and lemma 3.2, we have

$$\begin{aligned} |G*f|(x) = \left|\int f(y)G(x-y)dy\right| &\le C_{N_k}\int_\Lambda \frac{|f(y)|}{(1+|x-y|^2)^{N_k}}dy \\ &\le 2^{N_k}C_{N_k}(1+|x|^2)^{N_k}\int_\Lambda \frac{|f(y)|}{(1+|y|^2)^{N_k}}dy \\ &\le A_{N_k}(1+|x|^2)^{N_k}, \end{aligned} \tag{3.27}$$

where $A_{N_k} = 2^{N_k}C_{N_k}\int \frac{|f(y)|}{(1+|y|^2)^{N_k}}dy < \infty$. Then,

$$\int \frac{|G*f|^k(x)}{(1+|x|^2)^{kN_k+2}}dx \le A_{N_k}^k \int \frac{1}{(1+|x|^2)^2}dx < \infty, \tag{3.28}$$

which shows that $G*f \in \mathcal{C}$. □

Similarly, let $\hat{G}_t$ be the Green's function satisfying the equation

$$\begin{cases} \frac{\partial \hat{G}_t(\boldsymbol{x})}{\partial t} = \Delta^p \hat{G}_t(\boldsymbol{x}) - \mu \hat{G}_t(\boldsymbol{x}), \\ \hat{G}_0(\boldsymbol{x}) = \delta(\boldsymbol{x}). \end{cases} \tag{3.29}$$

Applying the Fourier transform to the last equation, we obtain

$$\begin{cases} \frac{\partial \hat{\mathcal{F}}(\hat{G}_t)}{\partial t} = -\Gamma_{\delta,\mu}(\boldsymbol{x})\hat{\mathcal{F}}(\hat{G}_t), \\ \hat{\mathcal{F}}(\hat{G}_0)(\boldsymbol{x}) = 1, \end{cases} \tag{3.30}$$

which has

$$\hat{\mathcal{F}}(\hat{G}_t)(\boldsymbol{x}) = e^{-t\Gamma_{\delta,\mu}(\boldsymbol{x})} \tag{3.31}$$

as a solution. Equivalently we get

$$\hat{G}_t(\boldsymbol{x}) = \frac{1}{(2\pi)^d}\int_{[0,\frac{2\pi}{\delta}]^d} e^{i\boldsymbol{y}.\boldsymbol{x}}e^{-t\Gamma_{\delta,\mu}(\boldsymbol{y})}d\boldsymbol{y}. \tag{3.32}$$

Equation (3.32) is nothing else than

$$G(t,\boldsymbol{x}) = \theta(t)\hat{G}_t(\boldsymbol{x}), \tag{3.33}$$

for $(t,\boldsymbol{x}) \in \Lambda =]0,+\infty[\times L_\delta$.

Proposition 3.1. *Let $X = \sum_{l=0}^{\infty}(-\lambda)^l X_l(t,\boldsymbol{x})$. Assume that $G \star \eta \in \mathcal{C}$. The perturbative solution of the SPDE (2.1) is given by:*

$$\begin{cases} X_0(t,\boldsymbol{x}) = G\star\eta(t,\boldsymbol{x}) + \hat{G}_t * f(\boldsymbol{x}), \\ X_l(t,\boldsymbol{x}) = G\star F_{l-1}(t,\boldsymbol{x}), \;\; l \ge 1, \end{cases} \tag{3.34}$$

where

$$\begin{aligned} F_l(t,\boldsymbol{x}) =& d\delta^{-2}\sum_{j=0}^{l}\Big[X_j(t,\boldsymbol{x})X_{l-j}(t,\boldsymbol{x})+E\left[X_j(t,\boldsymbol{x}+\boldsymbol{Y})X_{l-j}(t,\boldsymbol{x}+\boldsymbol{Y})\right] \\ & -2X_j(t,\boldsymbol{x})E\left[X_{l-j}(t,\boldsymbol{x}+\boldsymbol{Y})\right]\Big] \end{aligned} \tag{3.35}$$

and $\boldsymbol{Y}$ *is the random variable given in Equation (2.7).*

Proof. Let $X=\sum_{l=0}^{\infty}(-\lambda)^l X_l(t,\boldsymbol{x})$. Substituting it into equation (2.8), we get

$$\begin{aligned} |\nabla X(t,\boldsymbol{x})|^2 =& d\delta^{-2}\sum_{l_1,l_2=0}^{\infty}(-\lambda)^{l_1+l_2}\Big[X_{l_1}(t,\boldsymbol{x})X_{l_2}(t,\boldsymbol{x})+E\left[X_{l_1}(t,\boldsymbol{x}+\boldsymbol{Y})X_{l_2}(t,\boldsymbol{x}+\boldsymbol{Y})\right] \\ & -2X_{l_1}(t,\boldsymbol{x})E\left[X_{l_2}(t,\boldsymbol{x}+\boldsymbol{Y})\right]\Big] \\ =& \sum_{l=0}^{\infty}(-\lambda)^l F_l(t,\boldsymbol{x}). \end{aligned} \tag{3.36}$$

Hence, the solution of equation (2.1) satisfies

$$\begin{aligned} \sum_{l=0}^{\infty}(-\lambda)^l X_l(t,\boldsymbol{x}) &= -\lambda G\star|\nabla X(t,\boldsymbol{x})|^2+G\star\eta(t,\boldsymbol{x})+\hat{G}_t*f(\boldsymbol{x}) \\ &= \sum_{l=0}^{\infty}(-\lambda)^{l+1}G\star F_l(t,\boldsymbol{x})+G\star\eta(t,\boldsymbol{x})+\hat{G}_t*f(\boldsymbol{x}). \end{aligned} \tag{3.37}$$

Equating the coefficients of λ in both sides of equation (3.37), we obtain:

$$\begin{cases} X_0(t,\boldsymbol{x})=G\star\eta(t,\boldsymbol{x})+\hat{G}_t*f(\boldsymbol{x}), \\ X_l(t,\boldsymbol{x})=G\star F_{l-1}(t,\boldsymbol{x}),\ \ l\geq 1. \end{cases} \tag{3.38}$$

Notice that F_{l-1} is given in terms of $X_0,X_1,\dots,X_{l-1}$, which means that X_l is given recursively in terms of $X_0,X_1,\dots,X_{l-1}$. □

Remark 3.1. *The analytic solution of the SPDE (2.1) given by equation (3.34) remains complicated and therefore we cannot extract information about the different coefficients of the equation, mainly the noise* $\eta(t,\boldsymbol{x})$*. The aim of the next section is to introduce a new approach through the Feynman graph formalism to represent graphically the solution as well as its truncated moments.*

4. Graphical Representation of the Solution

The main goal of this section is to introduce a new graph approach to represent the solution of the SPDE (2.1). We define a tree T with root $x\in\Lambda$, n inner vertices and m leaves as a graph without cycles. If we cut the tree T from any vertices (including the root of T), the result are subtrees of T. Let $\Upsilon(n)$, $n\geq 1$, be the set of all rooted trees T with root x, four types of leaves and n inner vertices. $\Upsilon(0)$ is defined to be the set of all rooted trees with 0 inner vertex and two type of leaves 1 and 2. Such a graph is called a tree of order $n\geq 0$.

The following table resumes all types of edges and leaves of a tree $T \in \Upsilon(n)$:

G	Leaf of type 1 $\eta(t, \boldsymbol{x})$	leaf of type 2 $f(\boldsymbol{x})$	Leaf of type 3 $\eta(t, \boldsymbol{x}+\boldsymbol{Y})$	leaf of type 4 $f(\boldsymbol{x}+\boldsymbol{Y})$	Root of a tree T	Inner vertex
——	$\odot$	$\oslash$	$\ddot{\circ}$	$\otimes$	$\times$	$\bullet$

For the reader convenience, the representation of a tree of order 0 and 1 are given below:

Tree of order 0 Tree of order 1

Figure 1

For each tree $T \in \Upsilon(n)$, $n \geq 0$, we associate an analytical value R given by the following definition:

Definition 1. *For $T \in \Upsilon(n)$, $n \geq 0$, the random variable $R(T, \eta, x)$ is defined as follows.*

(1) *Assign $x \in \Lambda$ to the root of T, where Λ is given in Equation (2.1).*
   (a) *Assign $x_1, \ldots, x_n \in \Lambda$ to the inner vertices.*
   (b) *Assign $y_1, \ldots, y_l \in \Lambda$ to the leaves of type 1 and $\mathbf{z}_1, \ldots, \mathbf{z}_k \in L_\delta$ to the leaves of type 2, where $l, k \in \mathbb{N}$.*
   (c) *Assign $u_1, \ldots, u_q \in \Lambda$ to the leaves of type 3 and $\boldsymbol{v_1}, \ldots, \boldsymbol{v_m} \in L_\delta$ to the leaves of type 4, where $q, m \in \mathbb{N}$.*

(2) *For every edge e, assign a value $G(e)$ if e is connected to a leaf of type 1 or 3, or $\hat{G}_t(e)$ if it is connected to a leaf of type 2 or 4. Here, G and $\hat{G}_t$ are the Green functions defined in (3.10) and (3.29). If a vertex v_0 is connected to two leaves one of type 1 or 2 and the other of type 3 or 4, multiply the result by (-2).*

(3) *For the j-th leaf, multiply its corresponding value by $\eta(y_j)$, $f(\mathbf{z}_j)$, $\eta(u_j)$ or $f(\mathbf{v}_j)$ if this leaf is of type 1, 2, 3 or 4, respectively.*

(4) *Multiply the result by a constant $d\delta^{-2}$.*

(5) *Integrate with respect to the Lebesgue measure $dx_1 \cdots dx_n dy_1 \cdots dy_l dz_1 \cdots dz_k$.*

The following examples clarify to the reader, how we get the analytic value of a given graph using Definition 1. For instance, the analytic values of the graphs given in Figure 1 for the tree of order 0 and order 1 are respectively given by:

$$R(T_0, \eta, x) = \int G(x_1 - x)\eta(x_1)dx_1, \ \ x, x_1 \in \Lambda. \tag{4.1}$$

and

$$R(T_1,\eta,x)=\int G(x_1-x)G(x_2-x_1)G(x_3-x_1)\eta(x_2)f(\mathbf{x}_3)\,dx_1dx_2d\mathbf{x}_3,\ \ x,\,x_1,\,x_2\ \in\Lambda,\ \mathbf{x}_3\ \in\ L_\delta. \tag{4.2}$$

For $T\in\Upsilon(n)$, $n\in\mathbb{N}$ and $v\in V(T)$ be a given vertex of the tree T, let

$$C(T):=\{(T_v,\tilde{T}_v):\ \ T_v\in\Upsilon(j)\text{ and }\tilde{T}_v\in\Upsilon(n-j),\ \ j=0,1,\dots,n\}. \tag{4.3}$$

Clearly there is a one-to-one correspondence between $\Upsilon(n)$ and $C(T)$. For instance, given a rooted tree $T\in\Upsilon(n)$ with root x and let x_1 be the first inner vertex of T and $e_1=\{x,x_1\}$ the first edge of T. If we cut the tree T at x_1, we obtain two rooted subtrees $T_{x_1},\tilde{T}_{x_1}$ with root x_1 and j and $n-j$ inner vertices, respectively. The converse of this process gives the tree $T\in\Upsilon(n)$.

Lemma 1. *Let $T\in\Upsilon(n)$ and x_1 the vertex connected to the root x. Then, we have*

$$R(T,\eta,x)=G*\left[\prod_{S\in\mathcal{S}(T)}R(S,\eta,.)\right], \tag{4.4}$$

where $\mathcal{S}(T)$ is the set of all subtrees of T.

Proof. Let $E(S),L_1,L_2,L_3,L_4$ be the sets of all edges, leaves of type 1, leaves of type 2, leaves of type 3 , leaves of type 4 respectively. From Definition 1, we have

$$\begin{aligned}R(T,\eta,x)&=\int G(x-x_0)\prod_{S\in\mathcal{S}(T)}\left[\prod_{e\in E(S)}G(e)\prod_{l\in L_1}\eta(l)\prod_{l\in L_2}f(l)\prod_{\overrightarrow{l}\in L_3}\eta(\overrightarrow{l})\prod_{\overrightarrow{l}\in L_4}f(\overrightarrow{l})\right]dx_1\\&=G*\left[\prod_{S\in\mathcal{S}(T)}R(S,\eta,.)\right].\end{aligned} \tag{4.5}$$

□

Theorem 4.1. *The solution of the SPDE (2.1) in the sense of formal power series is given by the sum over all rooted trees $T\in\Upsilon(l)$, $l\geq 0$, that are evaluated according to the rules fixed in Definition 1. More precisely, we have*

$$X_l(x,\eta)=\sum_{T\in\Upsilon(l)}R(T,x,\eta).$$

Proof. We will proceed the proof by induction on l. For $l=0$, we have

$$X_0(t,\boldsymbol{x})=G\star\eta(t,\boldsymbol{x})+\hat{G}_t*f(\boldsymbol{x}).$$

Clearly, this is a sum of the evaluation of the two trees in $\Upsilon(0)$. Suppose the result true for all $k = 0, \dots, l-1$ and show it for l. Using Equation (3.34), Definition 1 and Lemma 1 yield

$$\begin{aligned} X_l(t,\boldsymbol{x}) =& G \star \sum_{j=0}^{l-1} d\delta^{-2}\Big[X_j(t,\boldsymbol{x})X_{l-j}(t,\boldsymbol{x}) + E\left[X_j(t,\boldsymbol{x}+\boldsymbol{Y})X_{l-j}(t,\boldsymbol{x}+\boldsymbol{Y})\right] - 2X_j(t,\boldsymbol{x})E\left[X_{l-j}(t,\boldsymbol{x}+\boldsymbol{Y})\right]\Big] \\ =& \sum_{j=0}^{l-1} \sum_{T_1\in\Upsilon(j), T_2\in\Upsilon(l-j)} R(T_1,T_2,x,\eta) \\ =& \sum_{T\in\Upsilon(l)} R(T,x,\eta). \end{aligned} \tag{4.6}$$

□

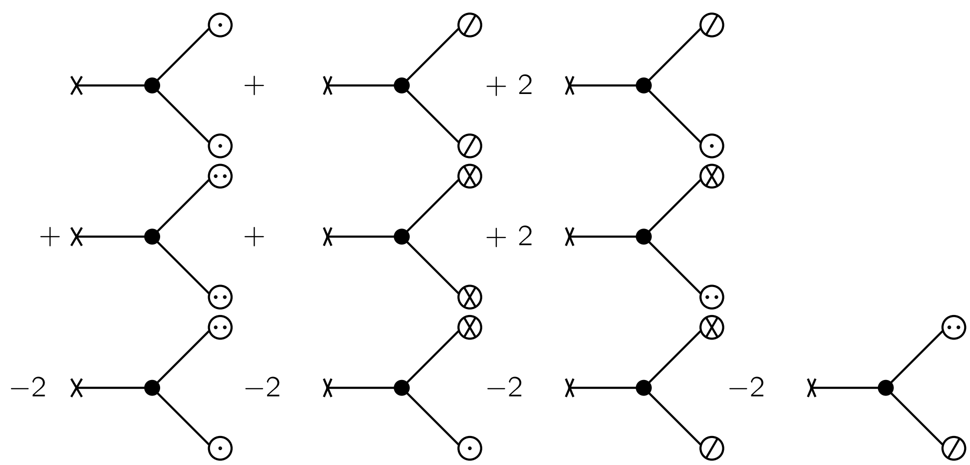
In the following, we give graphical representations of the solutions X_0 and X_1:

$$X_0(t,\boldsymbol{x}) = G \star \eta(t,\boldsymbol{x}) + \hat{G}_t * f(\boldsymbol{x}) \tag{4.7}$$

$=$ 

and

$$\begin{aligned} X_1(t,\boldsymbol{x}) &= G \star F_0(t,\boldsymbol{x}) \\ &= d\delta^{-2} G \star \Big[X_0(t,\boldsymbol{x})^2 + E\left(X_0(t,\boldsymbol{x}+\boldsymbol{Y})^2\right) - 2X_0(t,\boldsymbol{x})E\left(X_0(t,\boldsymbol{x}+\boldsymbol{Y})\right)\Big] \end{aligned} \tag{4.8}$$

$=$ 

5. Truncated Moments of the Solution

The aim of this section is to represent graphically the moments as well as the truncated moments of the solution of the SPDE (1) given in Theorem 4.1.

Proposition 5.1. *Let $T_1 \in \Upsilon(i_1), \dots, T_n \in \Upsilon(i_n)$, $i_1, \dots, i_n \in \mathbb{N}$, be n rooted trees with roots $x_1, \dots, x_n$, respectively. Then, in the sense of formal power series, we have*

$$\left\langle \prod_{i=1}^{n} X(x_i,\eta) \right\rangle = \sum_{m=0}^{\infty} (-\lambda)^m \sum_{\substack{T_1\in\Upsilon(i_1),\dots,T_n\in\Upsilon(i_n) \\ i_1+\dots+i_n=m}} \Big\langle R(T_1,x_1,\eta)\cdots R(T_n,x_n,\eta)\Big\rangle. \tag{5.1}$$

Proof. The proof is a trivial consequence of Theorem 4.1 as

$$\begin{aligned}\Big\langle \prod_{i=1}^{n} X(x_i,\eta)\Big\rangle &= \Big\langle \sum_{l_1=0}^{\infty}(-\lambda)^{l_1}X_{l_1}(x_1,\eta)\cdots\sum_{l_n=0}^{\infty}(-\lambda)^{l_n}X_{l_n}(x_n,\eta)\Big\rangle\\ &= \sum_{m=0}^{\infty}(-\lambda)^m \sum_{\substack{l_1,\ldots,l_n\in\mathbb{N}\\ l_1+\cdots+l_n=m}} \Big\langle X_{l_1}(x_1,\eta)\cdots X_{l_n}(x_n,\eta)\Big\rangle \qquad (5.2)\\ &= \sum_{m=0}^{\infty}(-\lambda)^m \sum_{\substack{T_1\in\Upsilon(i_1),\ldots,T_n\in\Upsilon(i_n)\\ i_1+\cdots+i_n=m}} \Big\langle R(T_1,x_1,\eta)\cdots R(T_n,x_n,\eta)\Big\rangle.\end{aligned}$$

□

Definition 2. *Let $x_1,\ldots,x_k\in\Lambda$, $k\in\mathbb{N}$, and $I=\{I_1,\ldots,I_k\}$ be a partition of the set $\{1,\ldots,n\}$. The truncated moment functions $\langle\eta(x_1)\cdots\eta(x_k)\rangle$ are recursively defined by*

$$\Bigg\langle \prod_{i=1}^{k}\eta(x_i)\Bigg\rangle = \sum_{I=\{I_1,\ldots,I_k\}}\prod_{i=1}^{k}\langle I_l\rangle^T, \qquad (5.3)$$

where

$$\prod_{i=1}^{k}\langle I_l\rangle^T = \Bigg\langle \prod_{i\in I_l}\eta(x_i)\Bigg\rangle^T.$$

Now, It is natural to define a new graph $\mathcal{G}(m,n)$ as a combination of n rooted trees $T_1,\ldots,T_n$ with 4 types of leaves and m inner vertices, the leaves of types 1 and 3 will be connected by empty vertices ○ . The set of such graphs is denoted by $\mathbb{G}(m,n)$.

Definition 3. *Let $\mathcal{G}\in\mathbb{G}(m,n)$ and $x_1,\ldots,x_k\in\Lambda$. We define the analytic value of $\mathcal{G}$ evaluated at $(x_1,\ldots,x_k)$, denoted by $\mathcal{A}(\mathcal{G})(x_1,\ldots,x_k)$, as follows.*

(1) (a) *Assign the values $x_1,\ldots,x_n\in\Lambda$ to the roots of the trees $T_1,\ldots,T_n$.*
  (b) *Assign the values $\hat{x}_1,\ldots,\hat{x}_p\in\Lambda$ to the inner vertices.*
  (c) *Assign $y_1,\ldots,y_l\in\Lambda$ to the leaves of type 1 and $\mathbf{z}_1,\ldots,\mathbf{z}_k\in L_\delta$ to the leaves of type 2, where $l,k\in\mathbb{N}$.*
  (d) *Assign $u_1,\ldots,u_q$ to the leaves of type 3 and $\mathbf{v}_1,\ldots,\mathbf{v}_p\in L_\delta$ to the leaves of type 4, .*

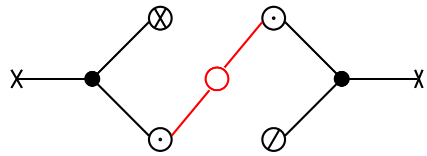
(2) *For every edge e, assign a value $G(e)$ to this edge if it is connected to a leaf of type 1 or 3, or $\hat{G}_t(e)$ if it is connected to a leaf of type 2 or 4. Here, G and $\hat{G}_t$ are the Green functions defined in (3.10) and (3.29)*

(3) *For each empty vertex with i legs connected to the leaves 1 and 3 with arguments $r_1,\ldots,r_i$, multiply with $\langle\eta(r_1)\cdots\eta(r_i)\rangle^T$.*

(4) *For the leaves of type 2 or 4, multiply with $f(\mathbf{y}_j)$ or $f(\mathbf{z}_s)$, respectively, where $j=1,\ldots,k$ and $s=1,\ldots,r$.*

(5) *Integrate with respect to the Lebesgue measure $dy_1\cdots dy_l dz_1\cdots dz_k$.*

For the reader convenience, we represent a graph $\mathcal{G} \in \mathbb{G}(2,2)$



its analytic value is

$$\begin{aligned}\mathcal{A}(\mathcal{G}) &= \int G(x_1-x)G(x_2-x_1)G(x_3-x_1)G(y_1-y)G(y_2-y_1)G(y_3-y_1)\,\langle\eta(x_3)\eta(y_2)\rangle^T \\ &\times \quad f(\mathbf{y}_3)\, f(\mathbf{x}_2+\mathbf{Y})\, dx_1\cdots dx_3 dy_1\cdots dy_3, \quad x_1,\cdots y_3 \in \Lambda,\ \mathbf{y}_3, \mathbf{x}_2 \in L_\delta. \qquad (5.4)\end{aligned}$$

Theorem 5.1. *Let $T_1, \ldots, T_n$, be n rooted trees with roots $x_1, \ldots, x_n$, respectively. The moment of the solution is given by a sum over all graphs $\mathcal{G} \in \mathbb{G}(m,n)$ of m-th order that are evaluated according the rule described in Definition 3, that is*

$$\Big\langle \prod_{i=1}^{n} X(x_i,\eta)\Big\rangle = \sum_{m=0}^{\infty}(-\lambda)^m \sum_{\mathcal{G}\in\mathbb{G}(m,n)} \mathcal{A}(\mathcal{G})(x_1,\ldots,x_n). \qquad (5.5)$$

Proof. The proof is straightforward by using Proposition 5.1 and Definition 3. □

Using Lévy Khinchine theorem, we know that the characteristic function of a Lévy noise η satisfies

$$C_\eta(t) = \int_{\mathbb{R}} e^{ist} d\eta(s) = e^{\psi(t)}, \text{ for } t \in \mathbb{R},$$

where $\psi : \mathbb{R} \to \mathbb{C}$ is the Lévy characteristic function represented by

$$\psi(t) = i\bar{a}t - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left(e^{ist} - 1 - \frac{ist}{1+s^2}\right) dM(s), \qquad (5.6)$$

$\forall t \in \mathbb{R}$, where $\bar{a} \in \mathbb{R}$, $\sigma^2 \geq 0$, and M is a Lévy measure on $\mathbb{R}\setminus\{0\}$ satisfying

$$\int_{\mathbb{R}\setminus\{0\}} \min\{1, s^2\} dM(s) < \infty. \qquad (5.7)$$

In addition, if all moments of M exist, we have

$$\psi(t) = iat - \frac{\sigma^2 t^2}{2} + z\int_{\mathbb{R}\setminus\{0\}} \left(e^{ist} - 1\right) dr(s), \qquad (5.8)$$

where $a = \bar{a} - \int_{\mathbb{R}\setminus\{0\}} \frac{s}{1+s^2} dM(s)$, $z = \int_{\mathbb{R}\setminus.\{0\}} dM(s)$ and $r = \frac{M}{z}$.

The following theorem due to [5] gives a relationship between the truncated moments and the moments of the noise η. This is very useful for our work since it will simplify the statement 3) in Definition (3).

Theorem 5.2. *Let $x_1, \ldots, x_n \in \Lambda$ and η be the general space-time Lévy noise. Then the following holds*

$$\Big\langle \prod_{i=1}^{n} \eta(x_i)\Big\rangle^T = C_n \int \delta(x-x_1)\cdots\delta(x-x_1)dx, \qquad (5.9)$$

where

$$C_n = (-i)^n \frac{d^n \psi(t)}{dt^n}\Big|_{t=0} = \delta_{n,1} a - \delta_{n,2}\sigma^2 + \int_{\mathbb{R}-\{0\}} s^n dr(s) \tag{5.10}$$

and $\delta_{n,m}$ *is the Kronecker symbol.*

Using the previous theorem, the coefficient of statement 3 in Definition 3 can be replaced by $C_n \int \delta(x-x_1)\cdots\delta(x-x_1)dx$ and we get new simplified values $\hat{\mathcal{A}}$ of the graphs $\mathcal{G} \in \mathbb{G}(m,n)$ as it is shown by the following theorem:

Theorem 5.3. *Let* $T_1, \dots, T_n$*, be* n *rooted trees with roots* $x_1, \dots, x_n$*, respectively. The moments of the solution are given by*

$$\Big\langle \prod_{i=1}^{n} X(x_i, \eta) \Big\rangle = \sum_{m=0}^{\infty} (-\lambda)^m \sum_{\mathcal{G}\in\mathbb{G}(m,n)} \hat{\mathcal{A}}(\mathcal{G})(x_1, \dots, x_n). \tag{5.11}$$

Proof. The proof is immediate from Definition (3), and Theorems (5.1) and (5.2). □

6. A Remark on Some Applications

By taking $p = 1$ and $\mu = 0$, equation (1) becomes:

$$\begin{cases} \dfrac{\partial X}{\partial t}(t,\mathbf{x}) = \Delta X(t,\mathbf{x}) - \lambda|\nabla X(t,\mathbf{x})|^2 + \eta(t,\mathbf{x}), \\ X(0,\mathbf{x}) = f(\mathbf{x})\ ;\ (t,\mathbf{x}) \in \Lambda, \end{cases} \tag{6.1}$$

which is nothing else then the KPZ equation studied in [19]. Using the so called Cole-Hopf transformation given by:

$$T^{\lambda}(t,\mathbf{x}) = \exp\Big(-\lambda X(t,\mathbf{x})\Big), \tag{6.2}$$

one can transform equation (6.1) into a linear SPDE called Burger equations. The solution as well as the truncated moments of the solution can be graphically represented as done in this work.

Another application helds by taking $p = 1$, $\lambda = 0$ and $d = 1$. This is the first part of the FitzHugh-Nagumo equation

$$\begin{cases} X_t(t,x) = X_{xx}(t,x) + X(t,x) + \eta_1(t,x) - Y, \\ Y_t(t,x) = \sigma X(t,x) - \gamma Y(t,x) + \eta_2(t,x). \end{cases} \tag{6.3}$$

This is a simple representation of a class of excitable-oscillatory systems (e.g. a neuron). It describes the physiological state of a nerve which can be resting, active, refractory, enhanced, depressed etc. For more details see FitzHugh [13]. This model deals with neural networks which can be considered as a graph with m edges and n vertices. Again our graph formalism in the current work can be applied.

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