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Generalized Hyers-Ulam Stability of Additive Functional Inequality in Modular Spaces and β-Homogeneous Banach Spaces

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Abstract. In this work, we investigate the generalised Hyers-Ulam stability of additive functional inequality in modular spaces with Δ_2-conditions and in β-homogeneous Banach spaces.

1. Introduction and Preliminaries

Nakano established the theory of modulars on linear spaces and the related theory of modular linear spaces in 1950 [10]. After a while, many mathematicians have worked hard to develop this theory, for example, Amemiya [1], Yamamuro [15], Orlicz [11], Mazur [8], Musielak [9], Luxemburg [6], and Turpin [14]. The study of interpolation theory [5, 7] and various Orlicz spaces [11] has up till now made extensive use of the notion of modulars and modular spaces.

Now, we will define the modular space and its properties.

Definition 1.1 ([10]). *Let Y be an arbitrary vector space. A functional $\rho : Y \to [0, \infty)$ is called a modular if for arbitrary $x, y \in Y$;*

(1) $\rho(x) = 0$ *if and only if* $x = 0$.

(2) $\rho(\alpha x) = \rho(x)$ *for every scalar* α *with* $|\alpha| = 1$.

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(3) *$\rho(\alpha x+\beta y) \leq \rho(x)+\rho(y)$ if and only if $\alpha+\beta=1$ and $\alpha, \beta \geq 0$.*
If (3) is replaced by:
(4) *$\rho(\alpha x+\beta y) \leq \alpha\rho(x)+\beta\rho(y)$ if and only if $\alpha+\beta=1$ and $\alpha, \beta \geq 0$, then we say that ρ is a convex modular.*

A modular ρ defines a corresponding modular space, i.e., the vector space Y_ρ given by:

$$Y_\rho=\{x \in Y : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

A function modular is said to be satisfy the Δ_2-condition if there exist $\tau>0$ such that $\rho(2x) \leq \tau\rho(x)$ for all $x \in Y_\rho$.

Definition 1.2. *Let $\{x_n\}$ and x be in Y_ρ. Then:*

(1) *The sequence $\{x_n\}$, with $x_n \in Y_\rho$, is ρ-convergent to x and write: $x_n \to x$ if $\rho(x_n-x) \to 0$ as $n \to \infty$.*
(2) *The sequence $\{x_n\}$, with $x_n \in Y_\rho$, is called ρ-Cauchy if $\rho(x_n-x_m) \to 0$ as $n : m \to \infty$.*
(3) *Y_ρ is called ρ-complete if every ρ-Cauchy sequence in Y_ρ is ρ-convergent.*

Proposition 1.1. *In modular space,*

- *If $x_n \overset{\rho}{\to} x$ and a is a constant vector, then $x_n+a \overset{\rho}{\to} x+a$.*
- *If $x_n \overset{\rho}{\to} x$ and $y_n \overset{\rho}{\to} y$ then $\alpha x_n+\beta y_n \overset{\rho}{\to} \alpha x+\beta y$, where $\alpha+\beta \leq 1$ and $\alpha, \beta \geq 1$.*

Remark 1.1. *Note that $\rho(x)$ is an increasing function, for all $x \in X$. Suppose $0<a<b$, then property (2.3) of Definition 1.1 with $y=0$ shows that $\rho(ax)=\rho\left(\frac{a}{b}bx\right) \leq \rho(bx)$ for all $x \in Y$. Morever, if ρ is a convexe modular on X and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$.*

In general, if $\lambda_i \geq 0$, $i=1,\dots,n$ and $\lambda_1, \lambda_2, \dots, \lambda_n \leq 1$ then $\rho(\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_n x_n) \leq \lambda_1\rho(x_1)+\lambda_2\rho(x_2)+\cdots+\lambda_n\rho(x_n)$.

If $\{x_n\}$ is ρ-convergent to x, then $\{cx_n\}$ is ρ-convergent to cx, where $|c| \leq 1$. But the ρ-convergent of a sequence $\{x_n\}$ to x does not imply that $\{\alpha x_n\}$ is ρ-convergent to αx_n for scalars α with $|\alpha|>1$.

If ρ is a convex modular satisfying Δ_2 condition with $\tau=2$, then $\rho(x) \leq \tau\rho(\frac{1}{2}x) \leq \frac{\tau}{2}\rho(x)$ for all x. Hence $\rho=0$. Consequently, we must have $\tau \geq 2$ if ρ is convex modular.

In 1940, Ulam [12] raised the first stability problem concerning the existence of an exact solution near to the function satisfiyng the equation or inequation approximattely . He proposed a question, if there exists an exact homomorphism near an approximate homomorphism. Hyers [3] found an answer in Banach space and then many authors have investigated the stability problems.

This paper consist of 4 sections. In section 2, we show the stability of the following inequation in modular space satisfying Δ_2-condition with $\tau=2$.

$$\rho(f(x+y)-f(x)-f(y)) \leq \rho\left(f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right) \text{ for all } x, y \in X.$$

In section 3, we obtain a like result in β-homogeneous complex Banch space of the following inequation, using the control of Gavruta

$$\|f(x+y)-f(x)-f(y)\| \leq \left\| f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right\|. \tag{1.1}$$

In section 4, we show the stability of the following inequation associated with the Jordan triple derivation in fuzzy Banach algebra

$$N(f(x+y)-f(x)-f(y)) \geq N\left(f(\frac{x+y}{2})-\frac{1}{2}f(x)-\frac{1}{2}f(y), t\right). \tag{1.2}$$

2. Additive Functional Inequalities in Modular Space

Throughout this section, assume that X is a linear space, and that Y_ρ is a ρ-complete modular sapace.

Lemma 2.1. *Let $f : X \to Y_\rho$ be a mapping such that*

$$\rho(f(x+y)-f(x)-f(y)) \leq \rho\left(f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right) \text{ for all } x, y \in X. \tag{2.1}$$

Then f is additive.

Proof. Letting $x = y = 0$ in (2.1), we get:

$$\rho(f(0)) \leq 0.$$

So

$$f(0) = 0.$$

Letting $y = -x$ in (2.1), we get:

$$\begin{aligned}\rho(f(x)+f(-x)) &\leq \rho\left(\frac{1}{2}(f(x)+f(-x))\right)\\ &\leq \frac{1}{2}\rho(f(x)+f(-x)) \text{ for all } x \in X.\end{aligned}$$

Hence $f(-x) = -f(x)$ for all $x \in X$.
Letting $x = y$ in (2.1), we get: $\rho(f(2x)-2f(x)) \leq 0$, and so $f(2x) = 2f(x)$ for all $x \in X$. Thus $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$. It follows from (2.1) that:

$$\begin{aligned}\rho(f(x+y)-f(x)-f(y)) &\leq \rho\left(\frac{1}{2}f(x+y)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right)\\ &\leq \frac{1}{2}\rho(f(x+y)-f(x)-f(y))\end{aligned}$$

and so

$$f(x+y) = f(x)+f(y) \text{ for all } x, y \in X.$$

Now, we prove the Hyers-Ulam stability of the additive functional inequality (2.1) in modular spaces.

□

Theorem 2.1. *Let X be a linear space, ρ be a convexe modular satisfying Δ_2-condition with $\tau = 2$ and Y_ρ be a ρ-complete modular space. Let $\varphi : X^2 \to [0, \infty)$ be a function with:*

$$\psi(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi\left(2^{j-1}x, 2^{j-1}y\right) < \infty, \tag{2.2}$$

and

$$\rho(f(x+y) - f(x) - f(y)) \leq \rho\left(f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\right) + \varphi(x, y) \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping: $h : X \to Y_\rho$ such that:

$$\rho(f(x) - h(x)) \leq \psi(x, x). \tag{2.4}$$

Proof. Letting $y = x$ in (2.3), we get: $\rho(f(2x) - 2f(x)) \leq \varphi(x, y)$ for all $x \in X$. So

$$\rho\left(\frac{1}{2}f(2x) - f(x)\right) \leq \frac{1}{2}\varphi(x, x). \tag{2.5}$$

Then by induction, we write:

$$\rho\left(\frac{f\left(2^k x\right)}{2^k} - f(x)\right) \leq \sum_{j=1}^{k} \frac{1}{2^j} \varphi\left(2^{j-1}x, 2^{j-1}x\right) \tag{2.6}$$

for all $x \in X$ and all positif integer k. Indeed, the case $k = 1$ follows from (2.5). Assume that (2.6) holds for $k \in \mathbb{N}$. Then we have the following inequality

$$\begin{aligned}
\rho\left(\frac{f\left(2^{k+1}x\right)}{2^{k+1}} - f(x)\right) &= \rho\left(\frac{1}{2}\left(\frac{f\left(2^k \cdot 2x\right)}{2^k} - f(2x)\right) + \frac{1}{2}f(2x) - f(x)\right) \\
&\leq \frac{1}{2}\rho\left(\frac{f\left(2^k \cdot 2x\right)}{2^k} - f(2x)\right) + \frac{1}{2}\rho(f(2x) - 2f(x)) \\
&\leq \frac{1}{2}\sum_{j=1}^{k} \frac{1}{2^j}\left(2^j x, 2^j x\right) + \frac{1}{2}\varphi(x, x) \\
&= \sum_{j=1}^{k+1} \frac{1}{2^j}\varphi\left(2^{j-1}x, 2^{j-1}x\right).
\end{aligned}$$

Hence (2.6) holds for every $k \in \mathbb{N}$.

Let m and n be nonnegative integers with $n > m$. By (2.6), we have

$$\begin{aligned}
\rho\left(\frac{f\left(2^n x\right)}{2^n} - \frac{f\left(2^m x\right)}{2^m}\right) &= \rho\left(\frac{1}{2^m}\left(\frac{f\left(2^{n-m} \cdot 2^m x\right)}{2^{n-m}} - f\left(2^m x\right)\right)\right) \\
&\leq \frac{1}{2^m} \cdot \sum_{j=1}^{n-m} \frac{1}{2^j}\varphi\left(2^{j-1} \cdot 2^m x, 2^{j-1} \cdot 2^m x\right)
\end{aligned}$$

$$= \sum_{j=1}^{n-m} \frac{1}{2^{j+m}} \varphi\left(2^{m+j-1}x, 2^{m+j-1}x\right)$$

$$= \sum_{k=m+1}^{n} \frac{1}{2^{k}} \varphi\left(2^{k-1}x, 2^{k-1}x\right). \tag{2.7}$$

Then by (2.2) and (2.7) we conclude that $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a ρ-Cauchy sequence in Y_ρ. The ρ-completeness of Y_ρ guarantees its ρ-convergence. Hence, there exists a mapping $h : X \to Y_\rho$ defined by:

$$h(x) = \rho - \text{limit } \frac{f(2^n x)}{2^n}; \qquad x \in X. \tag{2.8}$$

Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.7), we get (2.4).

Now, we prove that h is additive. We note that:

$$\begin{aligned}
\rho\left(\frac{f(2^n(x+y))}{2^{n+2}} - \frac{f(2^n x)}{2^{n+2}} - \frac{f(2^n y)}{2^{n+2}}\right) &\leq \frac{1}{2^{n+2}} \rho\left(f(2^n(x+y)) - f(2^n x) - f(2^n y)\right) \\
&\leq \frac{1}{2^{n+2}} \rho\left(f\left(\frac{2^n(x+y)}{2}\right) - \frac{1}{2} f(2^n x) - \frac{1}{2} f(2^n y)\right) \\
&\quad + \frac{1}{2^{n+2}} \varphi(2^n x, 2^n y) \\
&\leq \frac{1}{2} \rho\left(\frac{1}{2}\left(\frac{f\left(\frac{2^n(x+y)}{2}\right)}{2^n}\right) - \frac{1}{4} \times \frac{f(2^n x)}{2^n} - \frac{1}{4} \times \frac{f(2^n y)}{2^n}\right) \\
&\quad + \frac{1}{2^{n+2}} \varphi(2^n x, 2^n y).
\end{aligned}$$

Hence

$$\begin{aligned}
\rho\left(\frac{1}{4} h((x+y)) - \frac{1}{4} h(x) - \frac{1}{4} f(y)\right) &\leq \frac{1}{2} \rho\left(\frac{1}{2}\left(h\left(\frac{x+y}{2}\right)\right) - \frac{1}{4} h(x) - \frac{1}{4} h(y)\right) \\
&\leq \frac{1}{4} \rho\left(h\left(\frac{x+y}{2}\right) - \frac{1}{2} h(x) - \frac{1}{2} h(y)\right).
\end{aligned}$$

And so

$$\begin{aligned}
\rho(h((x+y)) - h(x) - h(y)) &\leq 4\rho\left(\frac{1}{4} h(x+y) - \frac{1}{4} h(x) - \frac{1}{4} h(y)\right) \\
&\leq \rho\left(h\left(\frac{x+y}{2}\right) - \frac{1}{2} h(x) - \frac{1}{2} h(x)\right).
\end{aligned}$$

Then by Lemma 2.1, h is additive.

We see that:

$$\begin{aligned}
\rho\left(\frac{h(2x) - 2h(x)}{2^2}\right) &= \rho\left(\frac{1}{2^2}\left(h(2x) - \frac{f(2^{n+1}x)}{2^n}\right) + \frac{1}{2}\left(\frac{f(2^{n+1}x)}{2^{n+1}} - h(x)\right)\right) \\
&\leq \frac{1}{2^2} \rho\left(h(2x) - \frac{f(2^{n+1}x)}{2^n}\right) + \frac{1}{2} \rho\left(\frac{f(2^{n+1}x)}{2^{n+1}} - h(x)\right)
\end{aligned} \tag{2.9}$$

for all $x, y \in X$. By (2.8), the right hand side of (2.9) tends to 0 as $n \to \infty$. Therefore, it follows that

$$h(2x) = 2h(x), \qquad x \in X.$$

Finally, to show the uniqueness of h, assume that h_1 and h_2 are additive mapping satisfying (2.4). □

Then we write:

$$\begin{aligned}
\rho\left(\frac{h_1(x) - h_2(x)}{2}\right) &= \rho\left(\frac{1}{2}\left(\frac{h_1\left(2^k x\right)}{2^k} - \frac{f\left(2^k x\right)}{2^k}\right) + \frac{1}{2}\left(\frac{f(2^k x)}{2^k} - \frac{h_2\left(2^k x\right)}{2^k}\right)\right) \\
&\leq \frac{1}{2}\rho\left(\frac{h_1\left(2^k x\right)}{2^k} - \frac{f(2^k x)}{2^k}\right) + \frac{1}{2}\rho\left(\frac{f\left(2^k x\right)}{2^k} - \frac{h_2\left(2^k x\right)}{2^k}\right) \\
&\leq \frac{1}{2} \cdot \frac{1}{2^k}\left\{\rho\left(h_1\left(2^k x\right) - f\left(2^k x\right)\right) + \rho\left(h_2\left(2^k x\right) - f\left(2^k x\right)\right)\right\} \\
&\leq \frac{1}{2^k}\psi\left(2^k x, 2^k y\right) \longrightarrow 0 \text{ as } k \to \infty.
\end{aligned}$$

This implies that $h_1 = h_2$.

Now, we have the classical Ulam stability of (2.1) by putting $\varphi = \epsilon > 0$.

Corollary 2.1. *Let X be a linear space, ρ be a convexe modular and Y_ρ be a ρ-complete modular space satisfying Δ_2-condition with $\tau = 2$. Assume $f : X \to Y_\rho$ is a mapping such that $f(0) = 0$ and:*

$$\rho(f(x+y) - f(x) - f(y)) \leq \rho\left(f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\right) + \varepsilon$$

for all $x, y \in X$. Then there exists a unique additive mapping $h : X \to Y_\rho$ such that

$$\rho(f(x) - h(x)) \leq \varepsilon, \qquad x \in X.$$

Corollary 2.2. *Let X be a normed linear space, ρ be a convex modular and Y_ρ be a ρ-complete modular space. Let $\theta > 0$ and $0 < p < 1$ real numbers. Assume that $f : X \to Y_\rho$ is a mapping ratifying:*

$$\rho(f(x+y) - f(x) - f(y)) \leq \rho\left(f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\right) + \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \to Y_\rho$ such that:

$$\rho(f(x) - h(x)) \leq \frac{2\theta\|x\|^p}{2 - 2^p}. \tag{2.11}$$

Proof. Replacing (x, y) with (x, x) in (2.10), we have:

$$\rho(f(2x) - 2f(x)) \leq 2\theta\|x\|^p.$$

Hence

$$\rho\left(\frac{1}{2}f(2x) - f(x)\right) \leq \theta\|x\|^p. \tag{2.12}$$

Then by induction, we write:

$$\rho\left(\frac{f\left(2^k x\right)}{2^k}-f(x)\right) \leq \sum_{j=1}^{k} \frac{1}{2^{j-1}}\left(2^{j-1}\right)^p \theta\|x\|^p$$
$$=\sum_{j=1}^{k} 2^{(p-1)(j-1)} \theta\|x\|^p \tag{2.13}$$

for all $x \in X$, and all positive integer k.

Indeed, the case $k=1$ follows from (2.12). Assume that (2.13) holds for $h \in \mathbb{N}$. Then we have the following inequality

$$\begin{aligned}
\rho\left(\frac{f\left(2^{k+1} x\right)}{2^{k+1}}-f(x)\right) &= \rho\left(\frac{1}{2}\left(\frac{f\left(2^k \cdot 2x\right)}{2^k}-f(2x)\right)+\frac{1}{2} f(2x)-f(x)\right) \\
&\leq \frac{1}{2} \rho\left(\frac{f\left(2^k \cdot 2x\right)}{2^k}-f(2x)\right)+\frac{1}{2} \rho(f(2x)-2f(x)) \\
&\leq \frac{1}{2} \sum_{j=1}^{k} 2^{(p-1)(j-1)} \theta \cdot 2^p\|x\|^p+\theta\|x\|^p \\
&= \sum_{j=1}^{k} 2^{(p-1)j} \theta\|x\|+\theta\|x\|^p \\
&= \sum_{j=1}^{k+1} 2^{(p-1)(j-1)} \theta\|x\|^p
\end{aligned}$$

Hence (2.13) holds for every $k \in \mathbb{N}$. Let m and n be nonnegative integers with $n>m$. By (2.10), we have:

$$\rho\left(\frac{f\left(2^n x\right)}{2^n}-\frac{f\left(2^m x\right)}{2^m}\right) = \rho\left(\frac{1}{2^m}\left(\frac{f\left(2^{n-m} \cdot 2^m x\right)}{2^{n-m}}-f\left(2^m x\right)\right)\right)$$
$$\leq \frac{1}{2^m} \sum_{j=1}^{n-m} 2^{(p-1)(j-1)} \theta\left\|2^m x\right\|^p$$
$$=2^{m(p-1)} \theta\|x\|^p \frac{1-2^{(p-1)(n-m)}}{1-2^{p-1}} \tag{2.14}$$

It follows from (2.14) that the sequence $\left\{\frac{f\left(2^n x\right)}{2^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y_ρ is ρ-complete modular space, the sequence $\left\{\frac{f\left(2^n x\right)}{2^n}\right\}$ converges. So one can define the mapping $h: X \rightarrow Y_\rho$ by:

$$h(x)=\rho-\operatorname{limit}\left\{\frac{f\left(2^n x\right)}{2^n}\right\} \text{ for all } x \in X.$$

Moreover, letting $m=0$ and passing to the limit $n \rightarrow \infty$ in (2.14), we get (2.11). The rest of the proof is similar to the proof of Theorem 2.1. □

3. Stability of (2.1) in β-Homogeneous Spaces

In 2016, C. Park [13] proved the generalised Hyer-Ulam-Rassias stability of additive ρ-functional inequalities in β-homogeneous complex Banach space.

In this section, we prove the generalised Hyers-Ulam stability of (1.1) from linear space to β-homogeneous complex Banach space, using the control of Gavruta.

Definition 3.1. *Let X be a linear space over $\mathbb{C}$. An F-norm is a function $\|\cdot\| : X \to [0, \infty)$ such that :*

1. $\|x\| = 0$ *if and only if* $x = 0$,
2. $\|\lambda x\| = \|\|\|x\|$ *for every* $x \in X$ *and every* λ *with* $|\lambda| = 1$,
3. $\|x + y\| \leq \|x\| + \|y\|$ *for every* $x, y \in X$,
4. $\|\lambda_n x\| \to 0$ *provided* $\lambda_n \to 0$,
5. $\|\lambda x_n\| \to 0$ *provided* $x_n \to 0$.

(X, d) is a metric space by letting $d(x, y) = \|x - y\|$. It is called an F-space if d is complete. If, in addition, $\|tx\| = t^\beta \|x\|$ for all $x \in X$ and $t \in \mathbb{C}$, then $\|\cdot\|$ is called β-homogeneous $(\beta > 0)$. A β-homogeneous F-space is called a β-homogeneous complex Banach space.

Remark 3.1. *For an s-convex modular ρ, if we define*

$$\|x\|_p = \inf \left\{ \alpha^s > 0; \rho\left(\frac{x}{\alpha}\right) \leq 1 \right\}, x \in Y_\rho.$$

Then $\|\cdot\|_p$ is an F-norm on Y_ρ such that $\|\lambda x\|_\rho = |\lambda|^s \|x\|_\rho$. Hence, $\|\cdot\|_\rho$ is s-homogeneous. For $s = 1$, this norm is called the luxemburg norm.

Now, we prove the generalised Hyers-Ulam Gavruta stability of (1.1) from linear spaces to β-homogeneous Banach spaces.

Theorem 3.1. *Let X be a linear space, Y be a β-homogeneous complex Banach space $(0 < \beta \leq 1)$, and $\varphi : X^2 \to [0, \infty)$ be function with*

$$\psi(x, y) = \frac{1}{2^\beta} \sum_{j=1}^{n} \frac{1}{2^{(j-1)\beta}} \varphi\left(2^{j-1}x, 2^{j-1}y\right) < \infty \tag{3.1}$$

for all $x, y \in X$. Assume that $f : X \to X$ is a mapping satisfying $f(0) = 0$ and

$$\|f(x + y) - f(x) - f(y)\| \leq \left\| f\left(\frac{x + y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \right\| + \varphi(x, y) \tag{3.2}$$

for all $x, y \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that:

$$\|f(x) - h(x)\| \leq \psi(x, x) \tag{3.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (3.2), we get: $\|f(2x) - 2f(x)\| \leq \varphi(x, x)$ and so

$$\left\|\frac{1}{2}f(2x) - f(x)\right\| \leq \frac{1}{2^\beta}\varphi(x, x). \tag{3.4}$$

By induction on $k \in \mathbb{N}$, using (3.4) it is easy to see that:

$$\left\|\frac{f\left(2^k x\right)}{2^k} - f(x)\right\| \leq \frac{1}{2^\beta}\sum_{j=1}^{k}\frac{1}{2^{(j-1)\beta}}\varphi\left(2^{j-1}x, 2^{j-1}x\right) \qquad x \in X. \tag{3.5}$$

for all $k \in \mathbb{N}$. Let m and n be nonnegative integers with $n > m$. Then by (3.5), we have

$$\begin{aligned}\left\|\frac{f\left(2^n x\right)}{2^n} - \frac{f\left(2^m x\right)}{2^m}\right\| &= \left\|\frac{1}{2^m}\left(\frac{f\left(2^n x\right)}{2^{n-m}} - f\left(2^m x\right)\right)\right\| \\ &\leq \frac{1}{2^{m\beta}}\cdot\frac{1}{2^\beta}\sum_{j=1}^{n-m}\frac{1}{2^{(j-1)\beta}}\varphi\left(2^{j+m-1}x, 2^{j+m-1}x\right) \\ &= \frac{1}{2^\beta}\sum_{j=1}^{n-m}\frac{1}{2^{(j+m-1)\beta}}\varphi\left(2^{j+m-1}x, 2^{j+m-1}x\right) \\ &= \frac{1}{2^\beta}\sum_{k=m+1}^{n}\frac{1}{2^{(k-1)\beta}}\varphi\left(2^{k-1}x, 2^{k-1}x\right). \end{aligned} \tag{3.6}$$

Since the last expression (3.6) goes to 0 by (3.1), it follows that, for every $x \in X$, the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence in X.

Since X is complete, we know that the sequence is convergent. Hence, there exists a mapping: $h : X \to Y$ defined by

$$h(x) = \lim_{n\to\infty}\frac{f\left(2^n x\right)}{2^n}, \qquad x \in X.$$

Letting $m = 0$ and passing the limit $n \to \infty$ in (3.6), we obtain (3.3). In order to show that T is additive, we write

$$\begin{aligned}\|h(x+y) - h(x) - h(y)\| &= \lim_{n\to\infty}\left\|\frac{f\left(2^n(x+y)\right)}{2^n} - \frac{f\left(2^n x\right)}{2^n} - \frac{f\left(2^n y\right)}{2^n}\right\| \\ &= \lim_{n\to\infty}\frac{1}{2^{n\beta}}\left\|f\left(2^n(x+y)\right) - f\left(2^n x\right) - f\left(2^n y\right)\right\| \\ &\leq \lim_{n\to\infty}\frac{1}{2^{n\beta}}\left\|f\left(\frac{2^n(x+y)}{2}\right) - \frac{1}{2}f\left(2^n x\right) - \frac{1}{2}f\left(2^n y\right)\right\| \\ &+ \frac{1}{2^{n\beta}}\varphi\left(2^n x, 2^n y\right) \\ &\leq \left\|h\left(\frac{x+y}{2}\right) - \frac{1}{2}h(x) - \frac{1}{2}h(y)\right\|. \end{aligned}$$

Then by [13, Lemma 2.1.], T is additive.

Now, let $h : X \to X$ be another additive mapping satisfying (3.2). Then we have:

$$\begin{aligned}
\|h_1(x) - h_2(x)\| &= \frac{1}{2^{\beta n}} \|h_1(2^n x) - h_2(2^n x)\| \\
&\le \frac{1}{2^{\beta n}} \left(\|h_1(2^n x) - f(2^n x)\| + \|h_2(2^n x) - f(2^n x)\|\right) \\
&\le \frac{2}{2^{\beta n}} \psi(2^n x, 2^n x) \\
&\le \frac{2}{2^{\beta n}} \cdot \frac{1}{2^{\beta}} \sum_{=1}^{\infty} \frac{1}{2^{j-1}\beta} \varphi\left(2^{j+n-1}, 2^{j+n-1} x\right) \\
&\le 2^{1-\beta} \sum_{j=1}^{\infty} \frac{1}{2^{\beta(j+n-1)}} \varphi\left(2^{j+n-1} x, 2^{j+n-1} x\right) \\
&= 2^{1-\beta} \sum_{k=n+1}^{\infty} \frac{1}{2^{\beta(k-1)}} \varphi\left(2^{k-1} x, 2^{k-1} x\right) \longrightarrow 0 \text{ as } k \to \infty,
\end{aligned}$$

for all $x \in X$, from which it follows that $h_1 = h_2$. □

Letting $\varphi = \varepsilon > 0$ in Theorem 3.1, we obtain a result on classical Ulam stability of the additive functional inequality.

Corollary 3.1. *Let X be a linear space and X be a β-homogeneous complete Banach space with $0 < \beta \le 1$.*
If $f : X \to X$ is a mapping satisfying $f(0) = 0$ and

$$\|f(x+y) - f(x) - f(y)\| \le \left\| f\left(\frac{x+y}{2}\right) - \frac{1}{2} f(x) - \frac{1}{2} f(y) \right\| + \varepsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $h : X \to Y$ such that:

$$\|f(x) - h(x)\| \le \frac{\epsilon}{2^{\beta} - 1}.$$

4. Stability of (1.2) in Fuzzy Banach Algebras

Let X be a real algebra, and $D : X \to X$ is an additive mapping:

(1) D is called a derivation if

$$D(xy) = D(x)y + xD(y), \qquad x, y \in X$$

(2) D is called a Jordan derivation if

$$D(x^2) = D(x)x + xD(x), \qquad x \in X$$

(3) In addition, D is called a Jordan triple derivation in the sens from [2] if

$$D(xyx) = D(x)yx + xD(y)x + xyD(x), \qquad x, y \in X$$

if an additive mapping is a derivation, so it is a Jordan derivation, and if D is a Jordan derivation, so it is a Jordan triple derivation.

However, the converse implication is note true in general.

Theorem 4.1. *Let (X, N) be a fuzzy Banach algebra, and $\varphi : X^2 \to [0, \infty)$ be a function such that $\varphi(0, 0) = 0$ and there exists an $0 < L < 1$ satisfying*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \text{ for all } x, y \in X.$$

Assume $f : X \to X$ is a mapping satisfies:

(a) $N(f(x+y) - f(x) - f(y)) \geq \min\left\{N\left(f(\frac{x+y}{2}) - \frac{1}{2}f(x) - \frac{1}{2}f(y), t\right), \frac{t}{t+\varphi(x,y)}\right\}$

(b)
$$N(f(xyx) - f(x)yx - xf(y)x - xyf(x), t) \geq \frac{t}{t+\varphi(x,y)} \tag{4.1}$$
for all $x, y \in X,\ t > 0$.

Then there exists a unique jordan triple derivation $h : X \to X$ such that:

$$N(f(x) - h(x), t) \geq \frac{(2-2L)t}{(2-2L)t + \varphi(x,x)}, \qquad x \in X,\ t > 0.$$

The mapping T is defined by

$$h(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f(2^n x), \qquad x \in X.$$

Proof. By [4, Theorem 2.4], the mapping h is additive. Replace (x, y) with $(2^n x, 2^n y)$ in (4.1), we get

$$\begin{aligned}
N&\left(\frac{1}{2^{3n}}f(2^{3n}xyx) - \frac{1}{2^{3n}}2^{2n}f(2^n x)yx - \frac{1}{2^{3n}}2^{2n}xf(2^n y)x - \frac{1}{2^{3n}}2^{2n}xyf(2^n x), t\right) \\
&= N(f(2^{3n}xyx) - 2^{2n}f(2^n x)yx - 2^{2n}xf(2^n y)x - 2^{2n}xyf(2^n x), 2^{3n}t) \\
&\geq \frac{2^{3n}t}{2^{3n}t + \varphi(2^n x, 2^n y)} \\
&\geq \frac{2^{3n}t}{2^{3n}t + (2L)^n\varphi(x,y)} \\
&= \frac{t}{t + \left(\frac{L}{4}\right)^n \varphi(x,y)}
\end{aligned}$$

Then

$$h(xyx) = h(x)yx + xh(y)x + xyh(x), \qquad x, y \in X. \tag{4.2}$$

Therefore, h is a Jordan triple derivation. □

Let A an algebra. If whenever $aAa = \{a\}$ for $a \in A$, implies $a = 0$, then A is called semiprime. All C^*-Algebra are examples of semiprime algebras. Let R be a ring. If $2r = 0$ implies $r = 0$ for $r \in \mathbb{R}$, then R is said to be 2-torsion free. Now, we show that the mapping f in Theorem 4.1 is a derivation if the algebra is semiprime.

Theorem 4.2. *Let (X, N) be a unital 2-torsion free semiprime fuzzy Banach algebra. Let $\varphi : X^2 \to [0, \infty)$ be a function such that $\varphi(0, 0) = 0$ and there exists an $0 < L < 1$ satisfying:*

(a) $\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{x}{2}\right)$,

(b) $\left\{\frac{1}{2^n}\varphi\left(x, \frac{y}{2^n}\right) \backslash n \in \mathbb{N}\right\}$ *is bounded,*

Assume $f : X \to X$ is a mapping such that

(c) $N(f(x+y) - f(x) - f(y)) \geq \min\left\{N\left(f(\frac{x+y}{2}) - \frac{1}{2}f(x) - \frac{1}{2}f(y), t\right), \frac{t}{t + \varphi(x, y)}\right\}$,

(d)

$$N(f(xyx) - f(x)yx - xf(y)x - xyf(x), t) \geq \frac{t}{t + \varphi(x, y)}. \tag{4.3}$$

Then f is an additive derivation.

Proof. We know that: $h(x) = N - \lim_{n\to\infty} \frac{1}{2^n}f(2^n x),\ x \in X$ is an additive Jordan triple derivation. Replacing (x, y) with $(2^n x, y)$ in (4.3), we get

$$\begin{aligned} N\left(\frac{1}{2^{2n}}f(2^{2n}xyx) - \frac{1}{2^{2n}}2^n f(2^n x)yx - \frac{1}{2^{2n}}2^{2n}xf(y)x - \frac{1}{2^{2n}}2^n xyf(2^n x), t\right) \\ = N(f(2^{2n}xyx) - 2^n f(2^n x)yx - 2^{2n}xf(y)x - 2^n xyf(2^n x), 2^{2n}t) \\ \geq \frac{2^{2n}t}{2^{2n}t + \varphi\left(2^n x, y\right)} \\ \geq \frac{2^{2n}t}{2^{2n}t + (2L)^n\varphi(x, \frac{y}{2^n})} \\ = \frac{t}{t + \left(\frac{L}{2}\right)^n \varphi(x, \frac{y}{2^n})} \end{aligned}$$

from wich we have:

$$h(xyx) = h(x)yx + xf(y)x + xyh(x) \tag{4.4}$$

for all $x, y \in X$. Comparing (4.4) and (4.2), we get:

$$xh(y)x = xf(y)x \text{ for all } x \in X.$$

Letting $x = 1$, we conclude thqt $T = f$. Then f is a Jordan triple derivation. By [2, Theorem 4.3], we conclude that f is an additive derivation (Every Jordan triple derivation on a 2-torsion free semiprime ring is a derivation.) □

5. Conclution

In this work, we have proved the Hyers-Ulam stability of additive functional inequality, using the direct method, ftrom linear spaces to modular spaces satisfuing Δ_2-condition with $\tau = 2$.

We have also proved the same result for β-homogeneous Banach spaces.

Finally, we have shown the stability of the functional equation associated with the Jordan triple derivation in fuzzy Banach algebra by a fixed point method.

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References

[1] I. Amemiya, On the Representation of Complemented Modular Lattices, J. Math. Soc. Japan. 9 (1957), 263–279. https://doi.org/10.2969/jmsj/00920263.

[2] M. Brešar, Jordan Mappings of Semiprime Rings, J. Algebra. 127 (1989), 218–228. https://doi.org/10.1016/0021-8693(89)90285-8.

[3] D.H. Hyers, On the Stability of the Linear Functional Equation, Proc. Natl. Acad. Sci. U.S.A. 27 (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222.

[4] J. Kim, G.A. Anastassiou, C. Park, Additive ρ-Functional Inequalities in Fuzzy Normed Spaces, J. Comput. Anal. Appl. 21 (2016), 1115–1126.

[5] M. Krbec, Modular Interpolation Spaces I, Z. Anal. Anwend. 1 (1982), 25–40. https://doi.org/10.4171/zaa/3.

[6] W.A.J. Luxemburg, Banach Function Spaces, Phd Thesis, Delft University of Technology, Delft, The Netherlands, (1959).

[7] L. Maligranda, Orlicz Spaces and Interpolation, In: Seminars in Mathematics, Departamento de Matemática, Universidade Estadual de Campinas, Brasil, (1989).

[8] B. Mazur, Modular curves and the eisenstein ideal, Publ. Math. L'Inst. Hautes Sci. 47 (1977), 33–186. https://doi.org/10.1007/bf02684339.

[9] J. Musielak, Orlicz Spaces and Modular Spaces, Springer, Berlin, Heidelberg, 1983. https://doi.org/10.1007/BFb0072210.

[10] H. Nakano, Modulared Semi-Ordered Linear Spaces, Maruzen, Tokyo, (1950).

[11] W. Orlicz, Collected Papers. I, II, PWN-Polish Scientific Publishers, Warszawa, (1988).

[12] S.M. Ulam, Problems in Modern Mathematics, Wiley, New York, (1964).

[13] C. Park, Additive ρ-Functional Inequalities in β-Homogeneous Normed Spaces, Filomat, 30 (2016), 1651–1658. https://www.jstor.org/stable/24898739.

[14] P. Turpin, Fubini Inequalities and Bounded Multiplier Property in Generalized Modular Spaces, Comment. Math. 1 (1978), 331–353.

[15] S. Yamamuro, On Conjugate Spaces of Nakano Spaces, Trans. Amer. Math. 5oc. 90 (1959), 291–311.