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Results on Katugampola Fractional Derivatives and Integrals

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Abstract. In this paper, we introduce and develop a new definitions for Katugampola derivative and Katugampola integral. In particular, we defined a (left) fractional derivative starting from a of a function f of order $\alpha \in (m-1, m]$ and a (right) fractional derivative terminating at b, where $m \in \mathbb{N}$. Then, we give some proprieties in relation to these operators such as linearity, product rule, quotient rule, power rule, chain rule, and vanishing derivatives for constant functions.

1. Introduction

The fractional calculus [1–3] attracted many researches in the last and present centuries. The impact of this fractional calculus in both pure and applied branches of science and engineering started to increase substantially during the last two decades apparently [4–9]. Most of the fractional derivatives are defined via fractional integral, see [10–13]. Two of which are the most popular ones, that is,

(1) Riemann-Liouville definition [3, 14]. For $\alpha \in [n-1, n)$ such that $n \in \mathbb{N}$, the derivative of the function f is given by

$$^{RL}D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t f(x)(t-x)^{n-\alpha-1}\, dx.$$

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(2) Caputo definition [3, 14]. For $\alpha \in [n-1, n)$ such that $n \in \mathbb{N}$, the derivative of the function f is given by

$$^{C}D_{a}^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} f^{(n)}(x)(t-x)^{n-\alpha-1}\,dx.$$

Recently, the authors in [15, 16] defined a new well-behaved simple fractional derivative called "the conformable fractional derivative" and "the Katugampola fractional derivative" depending just on the basic limit definition of the derivative. They then defined the fractional derivative of higher order (i.e. of order $\alpha > 1$) as we will see below in next sections. They also defined the fractional integral of order $0 < \alpha \leq 1$ only. The authors in [17, 18] introduced Katugampola Fourier and Laplace transforms by using these definitions, and they used these transforms to solve some fractional partial differential equations as in [19]. In [20], Abedaljwad proceed on to develop the definitions of left and right conformable fractional derivatives, and also Left and right conformable fractional integrals, see the references [21–25] to get a further overview about some schemes related to the fractional integro-differential operators.

In this paper, we have organized and generalized the basic definition and concepts of the Katugampola fractional derivatives and integrals. We define the left fractional derivative starting from a of a function f of order $\alpha \in (m-1, m]$, and the right fractional derivative terminating at b, where $m \in \mathbb{N}$. As a consequence, we give then some new results. This article is organized as follows: In Section 2, necessary preliminaries of the Katugampola fractional calculus are recalled. In Section 3 and Section 4, the left and right Katugampola fractional derivatives and fractional integrals of higher orders are defined, the fractional chain rule and some proprieties are obtained, and the action of fractional derivatives and integrals to each other are discussed. Finally, Section 5 summarizes the whole contributions of this work.

2. Preliminaries

In this section, we review some necessary definitions and essential results in relation to the Katugampola fractional calculus theory.

Definition 2.1. *Let $f : [0, \infty) \to R$ and $t > 0$. Then the "Katugampola fractional derivative" of the function f of order α is defined by*

$$D_t{}^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t\,e^{\varepsilon\, t^{-\alpha}}) - f(t)}{\varepsilon}, \tag{2.1}$$

for $t > 0$ and $\alpha \in (0, 1]$. If f is an α-differentiable in some $(0, a)$, $a > 0$ and $\lim_{t \to 0^+} D^{\alpha} f(t)$ exists, then $D_t{}^{\alpha} f(0) = \lim_{t \to 0^+} D_t{}^{\alpha} f(t)$.

Definition 2.2. *If $\alpha \in (n, n+1]$ and f is an n-differentiable at $t > 0$, for some $n \in \mathbb{N}$. Then, the α-fractional derivative of f is defined by*

$$D^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f^{(n)}\left(t.e^{\varepsilon.t^{n-\alpha}}\right) - f^{(n)}(t)}{\varepsilon}, \tag{2.2}$$

if the limit exists.

Lemma 2.1. *If $\alpha \in (n, n+1]$ and f is an $(n+1)$-differentiable at $t > 0$, for some $n \in \mathbb{N}$. Then, we have*

$$D^{\alpha} f(t) = \ t^{n+1-\alpha} f^{(n+1)}(t) . \tag{2.3}$$

Also, for $\alpha \in (0,1]$ and $t > 0$, we have

$$D^{\alpha} f(t) = \ t^{1-\alpha} \frac{df}{dt}(t).$$

Lemma 2.2. *Let $\alpha \in (0,1]$ and f, g be two α-differentiable functions at a point $t > 0$. Then, we have*

(1) $D^{\alpha}(af(t) + bg(t)) = aD^{\alpha} f(t) + bD^{\alpha} g(t)$, *for all $a, b \in \mathbb{R}$,*
(2) $D^{\alpha}(fg) = fD^{\alpha} g + gD^{\alpha} f$,
(3) $D^{\alpha}\left(\frac{f}{g}\right) = \frac{gD^{\alpha} f + fD^{\alpha} g}{g^2}$,
(4) $D^{\alpha}(f \circ g)(t) = f'(g(t)) . D^{\alpha} g(t)$ *for f differentiable at $g(t)$.*

Definition 2.3. *Let f be a function defined on $(s, t]$ such that $s \geq 0$ and $t \geq s$. Then, the α-fractional integral of the function f is defined by*

$$I_{\alpha}{}^{s}(f)(t) = \int_{s}^{t} f(x) . (x)^{\alpha-1} \ dx, \tag{2.4}$$

if the Riemann improper integral exists.

Lemma 2.3. *Let $s \geq 0$ and $\alpha \in (0,1)$. Let f be a continuous function such that $I_{\alpha}^{s} f$ exists. Then*

$$D^{\alpha} I_{\alpha}{}^{s}(f)(t) = f(t). \tag{2.5}$$

3. Left and right Katugampola derivatives

In this section, we define left and right Katugampola fractional derivatives for $\alpha \in (0,1]$, and in general for $\alpha \in (m-1, m]$, where $m \in \mathbb{N}$. Consequently, we obtain several results. We prove that the left and right fractional derivatives satisfy the product rule, quotient rule and chain rule. Also, we give the relation between them and the usual derivative.

Definition 3.1. *The left Katugampola fractional derivative starting from a of the function $f : [a, \infty) \to \mathbb{R}$ of order $\alpha \in (0,1]$ is defined by*

$$(D_{\alpha}{}^{a} f)(t) = \lim_{\varepsilon \to 0} \frac{f(t\, e^{\varepsilon (t-a)^{-\alpha}}) - f(t)}{\varepsilon}. \tag{3.1}$$

When $a = 0$, we write $(D_{\alpha} f)(t)$. If $(D_{\alpha}^{a} f)(t)$ exists on (a, b), then

$$(D_{\alpha}^{a} f)(a) = \lim_{t \to a^{+}} (D_{\alpha}^{a} f)(t). \tag{3.2}$$

The right Katugampola fractional derivative of order $\alpha \in (0,1]$ terminating at b is defined by

$$({}_{\alpha}{}^{b}Df)(t) = -\lim_{\varepsilon \to 0} \frac{f(t\,e^{\varepsilon (b-t)^{-\alpha}}) - f(t)}{\varepsilon}. \tag{3.3}$$

If $({}^{b}_{\alpha}Df)(t)$ exists on (a,b), then

$$({}^{b}_{\alpha}Df)(b) = \lim_{t \to b^-} ({}^{b}_{\alpha}Df)(t). \tag{3.4}$$

The following lemma gives some properties for the left and the right fractional derivative such as the product rule, quotient rule, and also gives the relation between them and the usual derivative.

Lemma 3.1. *Let $\alpha \in (0,1]$ and f, g are defined on $[a,\infty)$. Then, we have*

(1) $D^a_\alpha(fg) = fD^a_\alpha g + gD^a_\alpha f$,
(2) ${}^{a}_{\alpha}D(fg) = f{}^{a}_{\alpha}Dg + g{}^{a}_{\alpha}Df$.
(3) $D^a_\alpha\left(\frac{f}{g}\right) = \frac{gD^a_\alpha f + fD^a_\alpha g}{g^2}$,
(4) ${}^{a}_{\alpha}D\left(\frac{f}{g}\right) = \frac{g{}^{a}_{\alpha}Df + f{}^{a}_{\alpha}Dg}{g^2}$,
(5) $D^a_\alpha f(t) = t\,(t-a)^{-\alpha} f'(t)$,
(6) ${}^{b}_{\alpha}Df(t) = -t\,(b-t)^{-\alpha} f'(t)$.

Theorem 3.1. *Assume $f : [a,\infty) \to \mathbb{R}$ is the left α-differentiable function, where $0 < \alpha \leq 1$. Let $h(t) = f(g(t))$. Then, $h(t)$ is the left α-differentiable and for all t with $t \neq a$ and $g(t) \neq 0$, we have*

$$D^a_\alpha (f \circ g)(t) = (D^a_\alpha f)(g(t))\,.\,(D^a_\alpha g)(t)\,.\frac{(g(t)-a)^\alpha}{g(t)}. \tag{3.5}$$

Proof. By setting $u = t + \varepsilon t\,(t-a)^{-\alpha} + \frac{\varepsilon^2 t^2}{2!}(t-a)^{-2\alpha} + \ldots$ in (3.1) and using the continuity of g, we can have

$$\begin{aligned}(D^a_\alpha f)(t) &= \lim_{u \to t} \frac{f(g(u)) - f(g(t))}{u-t}.\,t\,(t-a)^{-\alpha} \\ &= \lim_{u \to t} \frac{f(g(u)) - f(g(t))}{g(u)-g(t)}.\lim_{u \to t} \frac{g(u)-g(t)}{u-t}.\,t\,(t-a)^{-\alpha}.\end{aligned}$$

This consequently implies

$$\begin{aligned}(D^a_\alpha f)(t) &= (D_\alpha{}^a f)(g(t))\,.\frac{(g(t)-a)^\alpha}{g(t)}\,(D_\alpha{}^a g)(t) \\ &= (D_\alpha{}^a f)(g(t))\,.\,(D_\alpha{}^a g)(t)\,.\frac{(g(t)-a)^\alpha}{g(t)}.\end{aligned}$$

$\square$

In a similar manner, we can prove the following theorem for the right α-differentiable functions.

Theorem 3.2. *Assume $f : (-\infty, a] \to \mathbb{R}$ is a right α-differentiable function, where $0 < \alpha \leq 1$. Let $h(t) = f(g(t))$. Then, $h(t)$ is a right α-up differentiable and for all t with $t \neq a$ and $g(t) \neq 0$, we have*

$$ {}^{a}_{\alpha}D\,(f \circ g)(t) = ({}_{\alpha}{}^{a}Df)(g(t))\,.\,({}_{\alpha}{}^{a}Dg)(t)\,.\frac{(g(t) - a)^{\alpha}}{g(t)}. \tag{3.6}$$

Next, we consider the possibility of $\alpha \in (m-1, m]$, where $m \in \mathbb{N}$. We have the following definition.

Definition 3.2. *A left fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, is defined by*

$$(D_{\alpha}{}^{a}f)(t) = \lim_{\varepsilon \to 0} \frac{f^{(m-1)}\left(t\, e^{\varepsilon\,(t-a)^{m-\alpha-1}}\right) - f^{(m-1)}(t)}{\varepsilon}\ ,\ \ t > a, \tag{3.7}$$

$$(D_{\alpha}{}^{a}f)(a) = \lim_{t \to a^{+}} (D_{\alpha}{}^{a}f)(t). \tag{3.8}$$

provided the limits exist and $f(t)$ is $(m-1)$-differentiable at $t > a$.

The right fractional derivative starting from a of a function $f : (-\infty, b] \to \mathbb{R}$ of order $\alpha \in (m-1, m]$, where $m \in \mathbb{N}$, is defined by

$$({}_{\alpha}{}^{b}Df)(t) = (-1)^{m} \lim_{\varepsilon \to 0} \frac{f^{(m-1)}\left(t\, e^{\varepsilon\,(b\,-\,t)^{m-\alpha-1}}\right) - f^{(m-1)}(t)}{\varepsilon}\ ,\ \ b > t, \tag{3.9}$$

$$({}_{\alpha}{}^{b}Df)(b) = \lim_{t \to b^{-}}\ {}_{\alpha}{}^{a}Df(t). \tag{3.10}$$

provided the limits exist and $f(t)$ is $(m-1)$-differentiable at $b > t$.

Note that if $\alpha = m$, then the fractional derivative of f becomes $f^{(m)}(t)$. Also, when $m = 1$, then $\alpha \in (0, 1]$ and the definition coincides with those in Definition 3.1.

Lemma 3.2. *Let $\alpha \in (m-1, m]$ such that $f(t)$ and $g(t)$ are (m)-differentiable at $t > a$ and $b > t$. Then, we have*

(1) $D_{\alpha}{}^{a}(f(t) + g(t)) = D_{\alpha}{}^{a}f(t) + D_{\alpha}{}^{a}g(t)$.
(2) ${}_{\alpha}{}^{b}D(f(t) + g(t)) = {}_{\alpha}{}^{b}Df(t) +_{\alpha}{}^{b}D\,g(t)$.
(3) $D_{\alpha}{}^{a}(\lambda f(t)) = \lambda D_{\alpha}{}^{a}f(t)$, *where λ is a constant.*
(4) ${}_{\alpha}{}^{b}D(\lambda f(t)) = \lambda\, {}_{\alpha}{}^{b}Df(t)$, *where λ is a constant.*
(5) $D_{\alpha}{}^{a}f(t) = t\,(t-a)^{m-\alpha-1}\, f^{(m)}(t)$.
(6) ${}_{\alpha}{}^{b}Df(t) = (-1)^{m}\, t\,(b-t)^{m-\alpha-1}\, f^{(m)}(t)$.
(7) $D_{\alpha}^{a}(t-a)^{\gamma} = \left\{\prod_{k=0}^{m-1} t\,(\gamma - k)\,(t-a)^{\gamma-\alpha-1}\ ,\ \gamma \notin \{0, 1, 2, \cdots, m-1\}\right\}$.
(8) ${}^{b}_{\alpha}D(t-b)^{\gamma} = \left\{\ \prod_{k=0}^{m-1} t\,(\gamma - k)\,(b-t)^{\gamma-\alpha-1}\ ,\ \gamma \notin \{0, 1, 2, \cdots, m-1\}\right\}$.

Proof.
- By Definition 3.2, we can proof parts (1) to (4).

- To prove part (5), wa have

$$
\begin{aligned}
D_\alpha^a f(t) &= \lim_{\varepsilon\to 0}\frac{f^{(m-1)}\left(t\, e^{\varepsilon (t-a)^{m-\alpha-1}}\right)-f^{(m-1)}(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f^{(m-1)}\left[t\left(1+\varepsilon\,(t-a)^{m-\alpha-1}+o\left(\varepsilon^2\right)\right)\right]-f^{(m-1)}(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f^{(m-1)}\left[t+h\right]-f^{(m-1)}(t)}{\frac{h}{t\,(t-a)^{m-\alpha-1}[1+o(\varepsilon)]}}\\
&= t\,(t-a)^{m-\alpha-1}\, f^{(m)}\,(t)\,.
\end{aligned}
$$

- Part (6) is similar to part (5).
- We can use parts (5) and (6) to prove parts (7) and (8).

□

Lemma 3.3. *Let $f:[a,\infty)\to\mathbb{R}$ be twice differentiable function on (a,∞) and $0<\alpha,\beta\le 1$ be such that $1<\alpha+\beta\le 2$. Then*

$$
\left(D_\alpha{}^a D_\beta{}^a f\right)(t)=\frac{t}{t-a}\,D_{\alpha+\beta}^a f\,(t)+(t-a)^{-\beta-1}\,(t\,(1-\beta)-a)\,D_\alpha^a f\,(t)\,. \tag{3.11}
$$

Proof. Since $1<\alpha+\beta\le 2$ and $0<\alpha,\beta\le 1$, we have

$$
\begin{aligned}
\left(D_\alpha^a D_\beta{}^a f\right)(t) &= t\,(t-a)^{-\alpha}\,\frac{d}{dt}\left[t\,(t-a)^{-\beta}\,\frac{df}{dt}\right]\\
&= t\,(t-a)^{-\alpha}\left[t\,(t-a)^{-\beta}\,\frac{d^2 f}{dt^2}+\left((t-a)^{-\beta}-\beta t\,(t-a)^{-\beta-1}\right)\frac{df}{dt}\right]\\
&= t^2\,(t-a)^{-\alpha-\beta}\,\frac{d^2 f}{dt^2}+\left(t\,(t-a)^{-\alpha-\beta}-\beta t^2\,(t-a)^{-\alpha-\beta-1}\right)\frac{df}{dt}\\
&= \frac{t^2}{(t-a)}\,(t-a)^{-\alpha-\beta+1}\,\frac{d^2 f}{dt^2}+\left[(t-a)^{-\beta}-\beta t\,(t-a)^{-\beta-1}\right]t\,(t-a)^{-\alpha}\,\frac{df}{dt}\\
&= \frac{t}{t-a}\,D_{\alpha+\beta}^a f\,(t)+(t-a)^{-\beta-1}\,(t\,(1-\beta)-a)\,D_\alpha^a f\,(t)\,,
\end{aligned}
$$

which completes the proof. □

It should be mentioned here that due to we have

$$
D_\alpha{}^a f(t)=t\,(t-a)^{m-\alpha-1}\,f^{(m)}\,(t)\,,\alpha\in(m-1,m]\,,
$$

then we can observe regarding (3.11) that when $a=0$ and $\alpha,\beta\to 1$, we obtain

$$
\left(D_\alpha{}^a D_\beta{}^a f\right)(t)=D_2^0 f\,(t)=f''\,(t)\,.
$$

Also, it must be noted regarding the same equation that when $\alpha\in\left(0,\frac{1}{2}\right]$ and $a<t$, we have

$$
\left(D_\alpha{}^a D_\beta{}^a f\right)(t)=\frac{t}{t-a}\,D_{2\alpha}^a f\,(t)+(t-a)^{-\alpha-1}\,(t\,(1-\alpha)-a)\,D_\alpha^a f\,(t)\,. \tag{3.12}
$$

4. Left and right Katugampola integrals

In this section, we define the left and right Katugampola fractional integral for $\alpha \in (0,1]$ and for $\alpha \in (m-1, m]$, where $m \in \mathbb{N}$. As a consequence, we obtain several results.

Definition 4.1. *(Left fractional integral) Let $a \geq 0$, $\alpha \in (0,1]$ and $t \geq a$. Let f be a function defined on $(a,t]$ such that $a \in \mathbb{R}$. Then, the α-fractional integral of the function f is defined by*

$$I_\alpha{}^a (f)(t) = \int_a^t \frac{f(x).(x-a)^\alpha}{x}\, dx. \tag{4.1}$$

Definition 4.2. *(Right Fractional Integral) Let $b \geq t$ and $\alpha \in (0,1]$. Also, let f be a function defined on $(t,b]$ and $b \in \mathbb{R}$. Then, the α-fractional integral of the function f is defined by*

$${}_\alpha{}^b I (f)(t) = \int_t^b \frac{f(x).(b-x)^\alpha}{x}\, dx. \tag{4.2}$$

Lemma 4.1. *Assume that $f : [a,\infty) \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then, for all $t > a$, we have*

$$D_\alpha{}^a I_\alpha{}^a (f)(t) = f(t). \tag{4.3}$$

Proof.

$$\begin{aligned} D_\alpha{}^a I_\alpha{}^a (f)(t) &= D_\alpha{}^a \left[\int_a^t \frac{f(x).(x-a)^\alpha}{x}\, dx\right] \\ &= t\,(t-a)^{-\alpha}\, \frac{d}{dt}\left[\int_a^t \frac{f(x).(x-a)^\alpha}{x}\, dx\right] \\ &= t\,(t-a)^{-\alpha}\, .\frac{f(t).(t-a)^\alpha}{t} \\ &= f(t). \end{aligned}$$

□

In a similar manner, we can prove the following lemma in the right case.

Lemma 4.2. *Assume that $f : (-\infty, b] \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then, for all $t < b$, we have*

$${}_\alpha{}^b D_\alpha{}^b I (f)(t) = f(t). \tag{4.4}$$

In what follows, we state a certain that could work as a generalization for definitions 4.1 and 4.2.

Definition 4.3. *A left fractional integral starting from a of a function $f : [a,\infty) \to \mathbb{R}$ of order $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, is defined by*

$$I_\alpha^a (f)(t) = \frac{1}{(m-1)!} \int_a^t \frac{f(x).(t-x)^{m-1}.(x-a)^{\alpha-m+1}}{x}\, dx,\ \alpha > 0,\ t > a, \tag{4.5}$$

with

$$I_\alpha{}^0 (f)(t) = f(x). \tag{4.6}$$

In the same regard, we can define the right fractional integral terminating at b of a function $f : (-\infty, b] \to \mathbb{R}$ of order $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, as follows

$$ {}_{\alpha}{}^{b}I\left(f\right)(t) = \frac{1}{(m-1)!}\int_t^b \frac{f(x).\left(x-t\right)^{m-1}.\left(b-x\right)^{\alpha-m+1}}{x}\,dx,\ \alpha > 0,\ b > t, \tag{4.7}$$

with

$$ {}_{\alpha}{}^{0}I\left(f\right)(t) = f\left(x\right). \tag{4.8}$$

Lemma 4.3. *If $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, and $f : [a, \infty) \to \mathbb{R}$ is $(m-1)$-differentiable. Then for all $t > a$, we have*

(1) $D_{\alpha}{}^{a} I_{\alpha}{}^{a}\left(f\right)(t) = f(t)$,

(2) $I_{\alpha}{}^{a} D_{\alpha}{}^{a}\left(f\right)(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}\left(a\right).\frac{(t-a)^k}{k!}$.

Proof. (1) Herein, we can have

$$\begin{aligned}
D_{\alpha}^{a} I_{\alpha}^{a}\left(f\right)(t) &= D_{\alpha}{}^{a}\left[\frac{1}{(m-1)!}\int_a^t \frac{f(x).\left(t-x\right)^{m-1}.\left(x-a\right)^{\alpha-m+1}}{x}\,dx\right] \\
&= t\left(t-a\right)^{m-\alpha-1}\frac{d^m}{dt^m}\left[\frac{1}{(m-1)!}\int_a^t \frac{f(x).\left(t-x\right)^{m-1}\left(x-a\right)^{\alpha-m+1}}{x}dx\right] \\
&= t\left(t-a\right)^{m-\alpha-1}\frac{d}{dt}\left[\int_a^t \frac{f(x).\left(x-a\right)^{\alpha-m+1}}{x}\,dx\right] \\
&= t\left(t-a\right)^{m-\alpha-1}.\frac{f(t).\left(t-a\right)^{\alpha-m+1}}{t} = f(t).
\end{aligned}$$

(2) To prove this result, we obtain

$$\begin{aligned}
I_{\alpha}^{a} D_{\alpha}^{b}\left(f\right)(t) &= \frac{1}{(m-1)!}\int_a^t \frac{\left(t-x\right)^{m-1}.\left(x-a\right)^{\alpha-m+1}}{x} D_{\alpha}{}^{b}\left(f\right)(x).dx \\
&= \frac{1}{(m-1)!}\int_a^t \frac{\left(t-x\right)^{m-1}.\left(x-a\right)^{\alpha-m+1}}{x}.x.\left(x-a\right)^{m-\alpha-1}\frac{d^m f}{dx^m}.dx \\
&= \frac{1}{(m-1)!}\int_a^t \left(t-x\right)^{m-1}\frac{d^m f}{dx^m}.dx.
\end{aligned}$$

Now, by integral by parts, we have

$$I_{\alpha}^{a} D_{\alpha}^{b}\left(f\right)(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}\left(a\right).\frac{\left(t-a\right)^k}{k!}.$$

□

Similarly, we can proof the following lemma in the right case.

Lemma 4.4. *If $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, and $f : (-\infty, b] \to \mathbb{R}$ is $(m-1)$-differentiable. Then, for all $b > t$, we have*

(1) ${}_{\alpha}{}^{b}D_{\alpha}{}^{b}If\left(t\right) = f\left(t\right)$

(2) ${}_{\alpha}{}^{b}I_{\alpha}{}^{b}D\mathrm{f}\left(t\right) = f\left(t\right) - \sum_{k=0}^{m-1}\left(-1\right)^k f^{(k)}\left(b\right).\frac{(b-t)^k}{k!}$

Lemma 4.5. *Let $f : [0, \infty) \to \mathbb{R}$ be a function and $0 < \alpha, \mu \leq 1$ be such that $1 < \alpha + \mu \leq 2$. Then*

$$(I_\alpha I_\mu f)(t) = \frac{t^\mu}{\mu}(I_\alpha f)(t) + \frac{1}{\mu}(I_{\alpha+\mu} f)(t) - \frac{t}{\mu}\int_0^t s^{\alpha+\mu-2} f(s)\, ds.$$

Proof. Since $0 < \alpha + \mu - 1 \leq 1$ and $0 < \alpha, \mu \leq 1$, we have

$$(I_{\alpha+\mu} f)(t) = \int_0^t f(x).x^{\alpha+\beta-2}\, dx.$$

In addition, we can observe

$$\begin{aligned}
(I_\alpha I_\mu f)(t) &= \int_0^t \left(\int_0^x f(s) s^{\alpha-1} ds\right) x^{\mu-1} dx \\
&= \int_0^t f(s) s^{\alpha-1} \left(\int_s^t x^{\mu-1} dx\right) ds \\
&= \int_0^t f(s) s^{\alpha-1} \left(\frac{t^\mu}{\mu} - \frac{s^\mu}{\mu}\right) ds \\
&= \frac{t^\mu}{\mu}\int_0^t f(s) s^{\alpha-1} ds - \frac{1}{\mu}\left[t\int_0^t s^{\mu+\alpha-2} f(s) ds - \int_0^t s^{\mu+\alpha-2} f(s)\right] \\
&= \frac{t^\mu}{\mu}(I_\alpha f)(t) + \frac{1}{\mu}(I_{\alpha+\mu} f)(t) - \frac{t}{\mu}\int_0^t s^{\alpha+\mu-2} f(s)\, ds.
\end{aligned}$$

□

5. Conclusion

In this paper, we have organized and generalized the basic definition and concepts of the Katugampola fractional derivatives and integrals. We have defined the left fractional derivatives starting from a of a function f of order$\alpha \in (m-1, m]$, the right fractional derivatives terminating at b, and consequently we have provided some useful results in relation to these operators.

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