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Applications of Fuzzy Differential Equations on Vibrating Spring Mass System

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Abstract. Modelling several real-world issues in the fuzzy world extensively uses ordinary differential equations. In this paper, a mechanical vibration system with the given mass, spring constant, damping and external force is modelled as a second-order ordinary differential equation. Due to measurement errors, the initial displacement of the string is approximate and assumed to be a fuzzy number. A fuzzy version of the Sumudu transform procedure is used to figure out this vibrating spring-mass system with fuzzy initial displacement. The output is displayed as a table at various computational stages. The consequences are visibly presented diagrammatically for different values of r and t. There is a good agreement between the computed results and the analytical solution.

1. Introduction

Most of the real-world situations we encounter on a daily basis are inherently ambiguous and confusing. The limited, ambiguous and unclear information makes it challenging to define any phenomena accurately. Zadeh's [29] fuzzy set theory is capable of handling these kinds of circumstances. The fuzzy hypothesis claims that these words are linguistic terms. An approximated framework of reasoning for addressing these language factors is provided by fuzzy theory. Each word in a linguistic is given a membership value according to this idea, and all of these terms may then be readily placed into a fuzzy set. We commonly utilize fuzzy numbers, also known as terms like nearly 6, almost 6, over 6, and other similar numbers. Fuzzy numbers were first proposed by Chang and Zadeh [8] in 1972. In [9–11], the authors expanded the concept of fuzzy numbers in 1978 and in 1982. A new parametric

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representation of fuzzy numbers and their arithmetic operations was presented by Ming Ma et al. [23] and they discussed fuzzy linear equations and fuzzy calculus.

When using differential equations to represent a dynamic system, the main goals are to forecast and analyze the behaviours associated with that dynamic system. The beginning circumstances and/or system parameters are unknown when the system model is seen as a set of fuzzy differential equations. Fuzzy differential equations make sense as a representation of dynamical systems with uncertainty. This ambiguity results from the blurry nature of borders. Puri et al. [25] expanded the Radstrom embedding theorem, allowing them to create the concept of the differential of a fuzzy valued function. Goetsche et al. [13] laid the foundation for understanding and applying calculus concepts in the context of fuzzy sets, which deal with uncertainty and imprecision in mathematical modelling. Kaleva [17, 18] studied differentiability and integrability properties of fuzzy valued functions and established the existence and uniqueness theorem for a solution of fuzzy differential equations and the validity of Peano's theorem for fuzzy differential equations. On the creation of fuzzy differential equations, a lot of effort has been done by Buckley [5,6]. Researchers have illustrated several engineering applications for various forms of fuzzy differential equations [4]. Laksmikantham [21] discussed the issues between ordinary differential equations and fuzzy differential equations. The authors Chalco et al. [7] examined a particular class of fuzzy differential equations in which a continuous fuzzy mapping created using Zadeh's extension principle provides the dynamics.

The Sumudu transform, introduced by Watugala [26], is an adaptation of the Laplace transform, with a number of appealing advantages over earlier integral transforms, notably the 'unity' feature that can be helpful for solving differential equations. For partial differential equations, Weerakon [27, 28] has expanded the Sumudu transform. Eltayeb et al. [12,20] have investigated the Sumudu transform's ideas and uses. Hassan et al. [14, 15] discussed some properties of the Sumudu transformation and mixed the Sumudu transform with the Adomain decomposition method for solving the Riccati equation of variable fractional order.

Fuzzy differential equations differ from precise differential equations in some aspects [4]. First order linear ordinary differential equations (FOLODE) were characterized in a fuzzy setting by Mondal et al. [24]. In this case, generalized triangular fuzzy numbers are employed to describe the coefficients and/or starting conditions of the FOLODE. The FOLODE's solution procedure is created using the Laplace transform. In [1, 2], the authors extended the classical Sumudu transform into a fuzzy setting. They proposed some new results on linearity, fuzzy derivative, preserving, convolution, shifting and solved fuzzy ordinary and fuzzy partial differential equations under the generalized differentiability concept. Aboodh et al. [3] discussed some of the fundamental properties of Sumudu transform and solved linear ordinary differential equations with constant coefficients. Khan et al. [19] obtained analytical solutions for fuzzy linear differential equations applying fuzzy Sumudu transform under generalized H differentiability. Jafari and Razvarz [16] suggested significant theorems on fuzzy Sumudu transform and solved fuzzy differential equations.

This study applies the fuzzy Sumudu transform method to solve the fuzzy differential equations. Fuzzy differential equations and their initial conditions are expressed in their parametric forms. A new type of arithmetic operation is used on the parametric forms and the fuzzy Sumudu transform method is applied for the solution of a mechanical vibration system with uncertain initial displacement, which is modelled as a second-order fuzzy differential equation. Following that, the response and the exact solution are compared.

2. Preliminary Results

This segment refers to the known basic concepts needed to achieve the obtained results in this study.

Definition 2.1. A fuzzy number can be described as a fuzzy set $\tilde{m}$ that is determined on the set $\mathbb{R}$ of real numbers with membership function $\tilde{m} : \mathbb{R} \to [0, 1]$ such that

- $\tilde{m}$ is normal, (i.e.) height of $\tilde{m} = 1$.
- $\tilde{m}$ is convex, (i.e.) $\tilde{m}(\delta s + (1 - \delta)t) \geq \min\{\tilde{m}(s), \tilde{m}(t)\}$ for all $s, t \in \mathbb{R}, \delta \in [0, 1]$.
- $\tilde{m}$ is piecewise continuous.

Definition 2.2. A fuzzy number $\tilde{m}$ on $\mathbb{R}$ is a triangular fuzzy number if its membership function $\tilde{m} : \mathbb{R} \to [0, 1]$ satisfies :

$$\tilde{m}(x) = \begin{cases} \dfrac{x - m_1}{m_2 - m_1}, & \text{for } m_1 \leq x \leq m_2 \\ \dfrac{m_3 - x}{m_3 - m_2}, & \text{for } m_2 \leq x \leq m_3 \\ 0, & \text{otherwise.} \end{cases}$$

We denote this triangular fuzzy number as $\tilde{m} = (m_1, m_2, m_3)$.

Definition 2.3. [23] A fuzzy number $\tilde{m}$ is a pair $(\underline{m}, \overline{m})$ of functions $\underline{m}(r), \overline{m}(r),\ r \in [0, 1]$ which meet the following necessities:
(i) $\underline{m}(r)$ is a bounded left continuous monotonic increasing function within [0,1],
(ii) $\overline{m}(r)$ is a bounded left continuous monotonic decreasing function within [0,1],
(iii) $\underline{m}(r) \leq \overline{m}(r)$; $r \in [0, 1]$.

Definition 2.4. [23] For a fuzzy number $\tilde{m} = (\underline{m}, \overline{m})$, the number $m_0 = \left(\dfrac{\underline{m}(1) + \overline{m}(1)}{2}\right)$ is called the location index number and $m_* = (m_0 - \underline{m})$, $m^* = (\overline{m} - m_0)$ are called the left and right fuzziness index functions respectively. Hence every fuzzy number $\tilde{m} = (\underline{m}, \overline{m})$ can also be expressed as $\tilde{m} = (m_0, m_*, m^*)$. The collection of all fuzzy numbers on real $\mathbb{R}$ is referred by $F(\mathbb{R})$.

Definition 2.5. [23] We introduce a lattice L as

$$L = \{h \mid h : [0, 1] \to [0, \infty) \text{ is nondecreasing and left continuous}\}.$$

The order in L is the natural order defined by $h \leq g$ if and only if $h(r) \leq g(r)$ for all $r \in [0;1]$. It is easy to show that

$$[h \vee g](r) = \max\{h(r), g(r)\},$$

$$[h \wedge g](r) = \min\{h(r), g(r)\},$$

where $h \vee g$ and $h \wedge g$ are supremum and infimum of h and g.

Definition 2.6. [23] **(Arithmetic Operations)**
For arbitrary fuzzy numbers $\tilde{m} = (m_0, m_*, m^*)$, $\tilde{n} = (n_0, n_*, n^*)$ and $* \in \{+, -, \times, \div\}$, the arithmetic operations are defined by

$$\begin{aligned}\tilde{m} * \tilde{n} =& (m_0 * n_0, m_* \vee n_*, m^* \vee n^*) \\ =& (m_0 * n_0, \max\{m_*, n_*\}, \max\{m^*, n^*\}) .\end{aligned}$$

The fuzziness index functions follows the lattice rule, which is the least upper bound in the lattice L. On the other hand, the location index number following the existing usual arithmetic.

Note: The division $\tilde{m} \div \tilde{n}$ is possible only when $n_0 \neq 0$.

Definition 2.7. [23] **(Comparing Fuzzy Numbers)**
An efficient method for comparing fuzzy numbers is to make use of a ranking function that is determined by the graded means of the fuzzy numbers. For a triangular fuzzy number $\tilde{m} = (m_0, m_*, m^*)$, we define

$$\mathcal{R}(\tilde{m}) = \left(\frac{m^* + 4m_0 - m_*}{4}\right).$$

For any two triangular fuzzy numbers $\tilde{m} = (m_0, m_*, m^*)$ and $\tilde{n} = (n_0, n_*, n^*)$ in $F(\mathbb{R})$, we have

$$(i).\ \mathcal{R}(\tilde{m}) > \mathcal{R}(\tilde{n}) \Leftrightarrow \tilde{m} \succ \tilde{n}$$

$$(ii).\ \mathcal{R}(\tilde{m}) < \mathcal{R}(\tilde{n}) \Leftrightarrow \tilde{m} \prec \tilde{n}$$

$$(iii).\ \mathcal{R}(\tilde{m}) = \mathcal{R}(\tilde{n}) \Leftrightarrow \tilde{m} \approx \tilde{n}$$

Remark 2.1. If $\mathcal{R}(\tilde{m}) \geq \mathcal{R}(\tilde{0})$, i.e. if $\mathcal{R}(\tilde{m}) \geq 0$, then the triangular fuzzy number $\tilde{m} = (m_0, m_*, m^*)$ is said to be non-negative and is referred by $\tilde{m} \succeq \tilde{0}$.

Remark 2.2. $\mathcal{R}(c_1\tilde{m} + c_2\tilde{n}) = c_1\mathcal{R}(\tilde{m}) + c_2\mathcal{R}(\tilde{n})$.

Definition 2.8. [23] For arbitrary fuzzy numbers $\tilde{m} = (\underline{m}, \overline{m})$ and $\tilde{n} = (\underline{n}, \overline{n})$, the measure

$$\mathcal{D}(\tilde{m}, \tilde{n}) = \sup_{0 \leq r \leq 1} \max\{|\ \underline{m}(r) - \underline{n}(r)\ |, |\ \overline{m}(r) - \overline{n}(r)\ |\},$$

is a fuzzy metric which defines the distance between $\tilde{m}$ and $\tilde{n}$. Such metric is comparable to the one employed by Puri and Ralescu [25] and Kaleva [17].

Definition 2.9. [10] A fuzzy-valued function $\tilde{f}:\mathbb{R}\to F(\mathbb{R})$ is said to be continuous at $t_0\in\mathbb{R}$, given $\epsilon>0$, there exist $\delta>0$ such that $D(\tilde{f}(t),\tilde{f}(t_0))<\epsilon$ whenever $\mid t-t_0\mid<\delta$.

Definition 2.10. [23] Let $\tilde{f}:\mathbb{R}\to F(\mathbb{R})$ be a fuzzy-valued function and let $t_0\in\mathbb{R}$. The derivative $\tilde{f}'(t_0)$ of $\tilde{f}$ at the point t_0 is defined by

$$\tilde{f}'(t_0)=\lim_{h\to 0}\frac{\tilde{f}(t_0+h)-\tilde{f}(t_0)}{h},$$

provided that this limit taken with respect to the metric D, exists.

The above definition is different from the definitions in [13, 25], because of the meaning of the difference $\tilde{f}(t+h)-\tilde{f}(t)$. Here we take the difference as in definition (2.6).

Definition 2.11. [23] Let $\tilde{f}:[a,b]\to F(\mathbb{R})$. For any partition $P=\{t_0,t_1,\cdots,t_n\}$ of $[a,b]$ and for arbitrary $\zeta_i: t_{i-1}\le\zeta_i\le t_i, 1\le i\le n$, let

$$R_p=\sum_{i=1}^{n}\tilde{f}(\zeta_i)(t_i-t_{i-1}),$$

The definite integral of $\tilde{f}(t)$ over $[a,b]$ is

$$\int_a^b\tilde{f}(t)dt=lim\ R_p,\quad \max_{1\le i\le n}\mid t_i-t_{i-1}\mid\to 0,$$

provided that this limit exists in the metric D.

Theorem 2.12. *[23] Let $\tilde{f}(t)$ be integrable, then*

(i) $\int_a^b\tilde{f}(t)dt=\left(\int_a^b\tilde{f}_0(t)dt,\sup_{t\in[a,b]}f(t)_*,\sup_{t\in[a,b]}f(t)^*\right)$.

(ii) $\int_a^b[c_1\tilde{f}(t)+c_2\tilde{g}(t)]dt=c_1\int_a^b\tilde{f}(t)dt+c_2\int_a^b\tilde{g}(t)dt$.

3. Fuzzy Sumudu Transformation

Let $\tilde{f}(t)$ be a fuzzy-valued function which is continuous. Let $\tilde{f}(ut)e^{-t}$ be an improper fuzzy Riemann integrable over $[0,\infty)$, then $\int_0^\infty\tilde{f}(ut)e^{-t}dt$ is said to be fuzzy Sumudu transform of $\tilde{f}(t)$ and it is represented by,

$$S[\tilde{f}(t)]=\int_0^\infty\tilde{f}(ut)e^{-t}dt,\ (u\in[-\tau,\tau])=\tilde{G}(u).$$

In the fuzzy valued function argument, in order to factor the t variable, the variable u is used. We have, $\tilde{f}(t,r)$ is defined using a fuzzy parametric form $\tilde{f}(t,r)=(f_0(t),f_*(t),f^*(t))$. Then

$$S[\tilde{f}(t,r)]=\int_0^\infty\tilde{f}(ut,r)e^{-t}dt=\left(\int_0^\infty f_0(ut,r)e^{-t}dt,\sup\{f_*(ut,r)\},\sup\{f^*(ut,r)\}\right).$$

Hence the Sumudu transform of $\tilde{f}(t,r)$ based on the Ming Ma's differentiability (Definition 2.10) is given by

$$S[\tilde{f}(t,r)]=\big(S[f_0(t,r)],f_*(t,r),f^*(t,r)\big)=\tilde{G}(u).$$

Theorem 3.1. *Let $\tilde{f};\tilde{g}: R \to F(R)$ be two continuous fuzzy-valued functions. Suppose that c_1 and c_2 are arbitrary constants, then*

$$S[c_1\tilde{f}(x)+c_2\tilde{g}(x)] \approx c_1 S[\tilde{f}(x)]+c_2 S[\tilde{g}(x)].$$

Proof: From $\mathcal{R}(c_1\tilde{m}+c_2\tilde{n}) = c_1\mathcal{R}(\tilde{m})+c_2\mathcal{R}(\tilde{n})$ and from the result (ii) of theorem (2.12), it is easy to prove

$$\begin{aligned} S[\mathcal{R}(c_1\tilde{f}(x)+c_2\tilde{g}(x))] =& S[c_1\mathcal{R}(\tilde{f}(x))+c_2\mathcal{R}(\tilde{g}(x))] = S[c_1\mathcal{R}(\tilde{f}(x))]+S[c_2\mathcal{R}(\tilde{g}(x))] \\ =& c_1 S[\mathcal{R}(\tilde{f}(x))]+c_2 S[\mathcal{R}(\tilde{g}(x))]. \end{aligned}$$

Hence from definition (2.7), we have

$$S[c_1\tilde{f}(x)+c_2\tilde{g}(x)] \approx c_1 S[\tilde{f}(x)]+c_2 S[\tilde{g}(x)].$$

3.1. **Inverse Sumudu Transform.** The complex inversion formula for Sumudu transform is given by

$$\tilde{f}(t) = S^{-1}(\tilde{G}(u)) = \int_{c-i\infty}^{c+i\infty} \frac{1}{u}\tilde{G}(\frac{1}{u})e^{ut}du.$$

Here s is the radius of the circular region while all singularities of $\frac{1}{u}\tilde{G}(\frac{1}{u})$ lie in $(s) < c$. In this case, the integral of $\frac{1}{u}\tilde{G}(\frac{1}{u})e^{ut}$ around the circular region will approach zero.

3.2. **Sumudu transform and inverse Sumudu transform of some basic functions.**

(i) $S(1) = 1 = G(u)$
Inversion Formula: $S^{-1}(1) = 1 = f(t)$

(ii) $S\left(\frac{t^n}{n!}\right) = u^n = G(u)$
Inversion Formula : $S^{-1}(n!u^n) = t^n = f(t)$

(iii) $S\left(e^{at}\right) = \frac{1}{1-au} = G(u)$
Inversion Formula : $S^{-1}\left(\frac{1}{1-au}\right) = e^{at} = f(t)$

(iv) $S(\sin(at)) = \frac{au}{1+a^2u^2} = G(u)$
Inversion Formula : $S^{-1}\left(\frac{u}{1+a^2u^2}\right) = \frac{\sin(at)}{a} = f(t)$

(v) $S(\cos(at)) = \frac{1}{1+a^2u^2} = G(u)$
Inversion Formula : $S^{-1}\left(\frac{1}{1+a^2u^2}\right) = \cos(at) = f(t)$

Theorem 3.2. *If $\tilde{G}(u)$ is fuzzy Sumudu transform of $\tilde{f}(t) = (f_0(t), f_*(t), f^*(t))$, then the fuzzy Sumudu transform of the derivatives with integer order is given by*

$$S\left[\frac{d^n\tilde{f}(t)}{dt^n}\right] = u^{-n}\left[\tilde{G}(u) - \sum_{k=1}^{n-1} u^k \frac{d^n\tilde{f}(t)}{dt^n}\,|_{t=0}\right]. \tag{3.1}$$

Proof: The fuzzy Sumudu transform of the first derivative of the function $\tilde{f}(t)$, $\frac{d\tilde{f}(t)}{dt} = \tilde{f}'(t)$, is given by

$$\begin{aligned}
S\left[\frac{d\tilde{f}(t)}{dt}\right] &= S[\tilde{f}'(t)] = \int_0^\infty \tilde{f}'(ut)e^{-t}dt = \frac{1}{u}\int_0^\infty \tilde{f}'(x)e^{-x/u}dx \\
&= \frac{1}{u}\int_0^\infty \tilde{f}'(t)e^{-t/u}dt = \frac{1}{u}\lim_{p\to\infty}\int_0^p \tilde{f}'(t)e^{-t/u}dt \\
&= \frac{1}{u}\lim_{p\to\infty}\int_0^p \frac{d\tilde{f}(t)}{dt}e^{-t/u}dt \\
&= \frac{1}{u}\lim_{p\to\infty}\left\{\left[e^{-t/u}\tilde{f}(t)\right]_{t=0}^p + \frac{1}{u}\int_0^p \tilde{f}(t)e^{-t/u}dt\right\} \\
&= \frac{1}{u}\left\{-\tilde{f}(0) + \frac{1}{u}\lim_{p\to\infty}\int_0^p \tilde{f}(t)e^{-t/u}dt\right\} \\
&= \frac{1}{u}\left\{-\tilde{f}(0) + \frac{1}{u}\int_0^\infty \tilde{f}(t)e^{-t/u}dt\right\} \\
&= -\frac{1}{u}\tilde{f}(0) + \frac{1}{u^2}\tilde{G}(u).
\end{aligned}$$

That is

$$S[\tilde{f}'(t)] = -\frac{1}{u}\tilde{f}(0) + \frac{1}{u}\tilde{G}(u). \tag{3.2}$$

Proceeding in the same manner, we get the fuzzy Sumudu transform of the nth order derivative as

$$S\left[\frac{d^n\tilde{f}(t)}{dt^n}\right] = u^{-n}\left[\tilde{G}(u) - \sum_{k=1}^{n-1} u^k \frac{d^n\tilde{f}(t)}{dt^n}\,|_{t=0}\right].$$

4. Solving second order fuzzy differential equations with fuzzy Sumudu transform in fuzzy environment

Consider a second-order general ODE

$$f''(t) = f(t, f(t), f'(t)), \tag{4.1}$$

with the initial conditions $f(t_0) = f_0, f'(t_0) = g_0$, where $f : [t_0, P] \times \mathbb{R} \longrightarrow \mathbb{R}$.

Suppose that the initial conditions are imprecise and are expressed as triangular fuzzy numbers in terms of their parametric forms (location index and fuzziness index functions), the second-order ordinary differential equation (4.1) becomes the subsequent differential equation for a fuzzy environment:

$$\tilde{f}''(t) = f(t, \tilde{f}(t), \tilde{f}'(t)), 0 \le t \le P, \tag{4.2}$$

with fuzzy initial conditions

$$\tilde{f}(t_0) = \tilde{f}_0 = (f_0(0), f_*(0), f^*(0)),$$
$$\tilde{f}'(t_0) = \tilde{g}_0 = (g_0(0), g_*(0), g^*(0)), \quad for \quad all \quad 0 < r \leq 1,$$

where $\tilde{f} : [t_0, P] \times \mathbb{R} \longrightarrow F(\mathbb{R})$.

5. An application on vibrating spring mass system

Consider a vibrating spring mass system discussed by Sahni et al. [22].

A vibrating spring with a mass of $m = 1kg$ is affiliated to the spring along spring constant $k = 9lbs/ft$ and paltry damping. The outside force function that regulates the spring's motion is $3\cos t$ with initial displacement $f(0) = 2, f^{'}(0) = 0$.

This vibrating spring mass system is modelled as a second-order ODE as $f^{''} + 9f = 3\cos t$ with the initial conditions $f(0) = 2, f^{'}(0) = 0$.

Solution:

When all the parameters of this spring mass system including the initial displacements are precise, the governing equation of this system will be a crisp second-order ODE

$$f^{''} + 9f = 3\cos t \quad \text{with} \quad f(0) = 2, f^{'}(0) = 0. \tag{5.1}$$

On solving this equation, the displacement of the spring at any time t is given by

$$f(t) = \frac{1}{8}\left[13\cos 3t + 3\cos t\right]. \tag{5.2}$$

In particular, when $t = 0.001$, the displacement of the spring is $f(t) = 1.9999$.

Fuzzy solution:

Suppose that the initial displacements are imprecise and are modelled as triangular fuzzy numbers, then the governing equation of the given spring mass system is a second order fuzzy differential equation

$$\tilde{f}^{''} + 9\tilde{f} = 3\cos t, \tag{5.3}$$

with the fuzzy initial conditions $\tilde{f}(0) = (2, 1-r, 1-r), \tilde{f}'(0) = (0, 1-r, 1-r), \quad r \in [0, 1]$.

The Fuzzy Sumudu Transform:

Taking fuzzy Sumudu transform on both sides of the above equation (5.3), we have

$$S[\tilde{f}''(t)] + 9S[\tilde{f}(t)] = 3S[\cos t].$$

By using the theorem (3.1) we get,

$$
\begin{aligned}
&\frac{S[\tilde{f}(t)]-\tilde{f}(0)-u\tilde{f}'(0)}{u^2}+9S[\tilde{f}(t)]=3S[\cos t]\\
\Rightarrow\quad & S[\tilde{f}(t)]-\tilde{f}(0)-u\tilde{f}'(0)+9u^2S[\tilde{f}(t)]=3u^2S[\cos t]\\
\Rightarrow\quad & (1+9u^2)S[\tilde{f}(t)]=\tilde{f}(0)+u\tilde{f}'(0)+3u^2S[\cos t]\\
\Rightarrow\quad & S[\tilde{f}(t)]=\frac{3u^2}{(u^2+1)(1+9u^2)}+\frac{\tilde{f}(0)}{(1+9u^2)}+\frac{u\tilde{f}'(0)}{(1+9u^2)}\\
\Rightarrow\quad & S[\tilde{f}(t)]=\frac{3}{8(u^2+1)}+\frac{3}{8(1+9u^2)}+\frac{\tilde{f}(0)}{(1+9u^2)}+\frac{u\tilde{f}'(0)}{(1+9u^2)}\\
\Rightarrow\quad & S[\tilde{f}(t)]=\left(\frac{3}{8(u^2+1)}\right)-\left(\frac{3}{8(1+9u^2)}\right)+\frac{(2,1-r,1-r)}{(1+9u^2)}+\frac{u(0,1-r,1-r)}{(1+9u^2)}.
\end{aligned}
$$

By expressing all the terms interms of their parametric form, we have

$$
\begin{aligned}
S[\tilde{f}(t)]=&\left(\frac{3}{8(u^2+1)},0,0\right)-\left(\frac{3}{8(1+9u^2)},0,0\right)+\frac{(2,1-r,1-r)}{(1+9u^2)}+\frac{u(0,1-r,1-r)}{(1+9u^2)}\\
=&\left(\frac{3}{8(u^2+1)},0,0\right)-\left(\frac{3}{8(1+9u^2)},0,0\right)+\frac{(2,1-r,1-r)}{(1+9u^2)}\\
=&\left(\frac{3}{8(u^2+1)}-\frac{3}{8(1+9u^2)}+\frac{2}{(1+9u^2)},\max\{0,0,1-r\},\max\{0,0,1-r\}\right)\\
=&\left(\frac{3}{8(u^2+1)}-\frac{3}{8(1+9u^2)}+\frac{2}{(1+9u^2)},1-r,1-r\right)\\
S[\tilde{f}(t)]=&\left(\frac{3}{8(u^2+1)}+\frac{13}{8(1+9u^2)},1-r,1-r\right).
\end{aligned}
$$

Taking inverse fuzzy Sumudu transform, the displacement of the spring at any time t is given by,

$$\tilde{f}(t)=\left[\frac{3}{8}\cos t+\frac{13}{8}\cos t,(1-r),(1-r)\right]=(f_0(t),f_*(t),f^*(t)). \tag{5.4}$$

In particular, when $t=0.001$, the displacement of the spring is given by

$$\tilde{f}(t)=(f_0(t),f_*(t),f^*(t))=(1.9999,1-r,1-r).$$

$$\text{That is } \tilde{f}(t)=(f_1(t),f_2(t),f_3(t))=(0.9999+r,1.9999,2.9999-r). \tag{5.5}$$

Also for a particular time $t=0.001$ and for different values of r, the displacement of the spring is illustrated in the following table 1 and Figure 1.

5.1. **Results and Discussion.** By considering the step size of 0.1, the displacement $\tilde{f}(t)=(f_1(t),f_2(t),f_3(t))$ of the spring is obtained for different values of r and at time $t=0.001$. The displacement can also be obtained for different values of r and for different values of time t. The results are displayed in Table 1, Figure 1 and Figure 2. It is observed that the vagueness of the solution decreases with the increase in the values of r. Also, when r approaches to 1.0, the fuzzy solution

Table 1. The displacement $\tilde{f}(t) = (f_1(t), f_2(t), f_3(t)) = (0.9999+r, 1.9999, 2.9999-r)$ for different values of r and at time $t = 0.001$

r-**values**	$\tilde{f}(t) = (0.9999 + r, 1.9999, 2.9999 - r)$	Rank of $\tilde{f}(t)$
0.0	(0.9999, 1.9999, 2.9999)	1.9999
0.1	(1.0999, 1.9999, 2.8999)	1.9999
0.2	(1.1999, 1.9999, 2.7999)	1.9999
0.3	(1.2999, 1.9999, 2.6999)	1.9999
0.4	(1.3999, 1.9999, 2.5999)	1.9999
0.5	(1.4999, 1.9999, 2.4999)	1.9999
0.6	(1.5999, 1.9999, 2.3999)	1.9999
0.7	(1.6999, 1.9999, 2.2999)	1.9999
0.8	(1.7999, 1.9999, 2.1999)	1.9999
0.9	(1.8999, 1.9999, 2.0999)	1.9999
1	(1.9999, 1.9999, 2.9999)	1.9999

coincides with the crisp solution. From Figure 2, it is to be noted that the spring's motion reduces with the change in time.

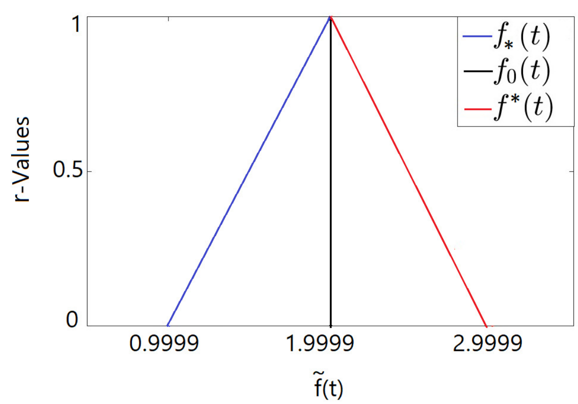

Figure 1. The displacement of the spring for different values of r and at $t = 0.001$

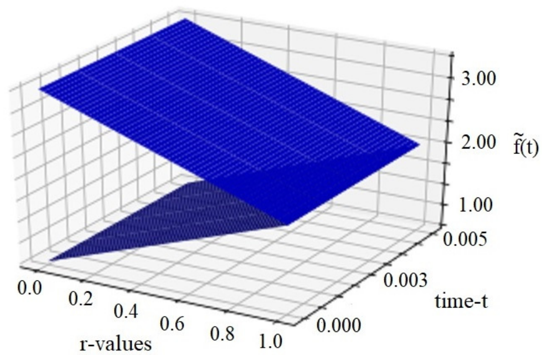

Figure 2. The displacement of the spring for different values of r and t

6. Conclusion

Fuzzy differential equations are mainly used to provide information about uncertain dynamical systems and give more flexible and realistic model. The solution of fuzzy differential equations helps us to understand the behaviour of physical systems in an uncertain environment. In this article, a vibrating spring-mass system with uncertain initial displacement is modeled as a second order fuzzy differential equation. The fuzzy version of the Sumudu transform is applied for the solution of fuzzy differential equation without transforming to its equivalent crisp form. A new fuzzy arithmetic and ranking method is applied on the parametric form of fuzzy numbers, the fuzzy solution is obtained through fuzzy Sumudu transform. A numerical example of the mechanical vibrating system under a fuzzy environment is provided to illustrate the proposed method and the obtained fuzzy solution is compared with the crisp solution. From the example it is observed that fuzzy Sumudu transform is a very effective and reliable tool with less computation in obtaining exact solutions of fuzzy differential equations.

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References

[1] N. Rahman, M. Ahmad, Applications of the Fuzzy Sumudu Transform for the Solution of First Order Fuzzy Differential Equations, Entropy. 17 (2015), 4582–4601. https://doi.org/10.3390/e17074582.

[2] N.A.A. Rahman, M.Z. Ahmad, Fuzzy Sumudu Transform for Solving Fuzzy Partial Differential Equations, J. Nonlinear Sci. Appl. 09 (2016), 3226–3239. https://doi.org/10.22436/jnsa.009.05.111.

[3] K.S. Aboodh, N.E. Mohamed, A.M. Ali, Solving Ordinary Differential Equations With Constant Coefficients Using Sumudu Transform Method, Int. J. Eng. Inf. Syst. 1 (2017), 70–76.

[4] B. Bede, S.G. Gal, Generalizations of the Differentiability of Fuzzy-Number-Valued Functions With Applications to Fuzzy Differential Equations, Fuzzy Sets Syst. 151 (2005), 581–599. https://doi.org/10.1016/j.fss.2004.08.001.

[5] J.J. Buckley, T. Feuring, Fuzzy Differential Equations, Fuzzy Sets Syst. 110 (2000), 43–54. https://doi.org/10.1016/s0165-0114(98)00141-9.

[6] J.J. Buckley, T. Feuring, Fuzzy Initial Value Problem for Nth-Order Linear Differential Equations, Fuzzy Sets Syst. 121 (2001), 247–255. https://doi.org/10.1016/s0165-0114(00)00028-2.

[7] Y. Chalco-Cano, H. Román-Flores, Comparation Between Some Approaches to Solve Fuzzy Differential Equations, Fuzzy Sets Syst. 160 (2009), 1517–1527. https://doi.org/10.1016/j.fss.2008.10.002.

[8] S.S.L. Chang, L.A. Zadeh, On Fuzzy Mapping and Control, IEEE Trans. Syst., Man, Cybern. SMC-2 (1972), 30–34. https://doi.org/10.1109/TSMC.1972.5408553.

[9] D. Dubois, H. Prade, Operations on Fuzzy Numbers, Int. J. Syst. Sci. 9 (1978), 613–626. https://doi.org/10.1080/00207727808941724.

[10] D. Dubois, H. Prade, Towards Fuzzy Differential Calculus Part 2: Integration on Fuzzy Intervals, Fuzzy Sets Syst. 8 (1982), 105–116. https://doi.org/10.1016/0165-0114(82)90001-x.

[11] D. Dubois, H. Prade, Towards Fuzzy Differential Calculus Part 3: Differentiation, Fuzzy Sets Syst. 8 (1982), 225–233. https://doi.org/10.1016/s0165-0114(82)80001-8.

[12] H. Eltayeb, A. Kilicman, A Note on the Sumudu Transforms and Differential Equations, Appl. Math. Sci. 4 (2010), 1089–1098.

[13] R. Goetschel Jr., W. Voxman, Elementary Fuzzy Calculus, Fuzzy Sets Syst. 18 (1986), 31–43. https://doi.org/10.1016/0165-0114(86)90026-6.

[14] H.Z. Mjthap, S.N. Al-Azzawi, Some Properties of Sumudu Transformation, J. Interdiscip. Math. 22 (2019), 1543–1547. https://doi.org/10.1080/09720502.2019.1706853.

[15] H.Z. Mjthap, S.N. Al-Azzawi, Mixing Sumudu Transform and Adomain Decomposition Method for Solving Riccati Equation of Variable Fractional Order, J. Interdiscip. Math. 22 (2019), 1559–1563. https://doi.org/10.1080/09720502.2019.1706858.

[16] R. Jafari, S. Razvarz, Solution of Fuzzy Differential Equations Using Fuzzy Sumudu Transforms, Math. Comput. Appl. 23 (2018), 5. https://doi.org/10.3390/mca23010005.

[17] O. Kaleva, Fuzzy Differential Equations, Fuzzy Sets Syst. 24 (1987), 301–317. https://doi.org/10.1016/0165-0114(87)90029-7.

[18] O. Kaleva, The Cauchy Problem for Fuzzy Differential Equations, Fuzzy Sets Syst. 35 (1990), 389–396. https://doi.org/10.1016/0165-0114(90)90010-4.

[19] N.A. Khan, O.A. Razzaq, M. Ayyaz, On the Solution of Fuzzy Differential Equations by Fuzzy Sumudu Transform, Nonlinear Eng. 4 (2015), 49–60. https://doi.org/10.1515/nleng-2015-0003.

[20] A. Kilicman, H. Eltayeb, M.R. Ismail, A Note on Integral Transforms and Differential Equations, Malays. J. Math. Sci. 6 (2012), 1–18.

[21] V. Laksmikantham, Set Differential Equations Versus Fuzzy Differential Equations, Appl. Math. Comput. 164 (2005), 277–294. https://doi.org/10.1016/j.amc.2004.06.068.

[22] M. Sahni, M. Parikh, R. Sahni, Sumudu Transform for Solving Ordinary Differential Equation in a Fuzzy Environment, J. Interdiscip. Math. 24 (2021), 1565–1577. https://doi.org/10.1080/09720502.2020.1845468.

[23] M. Ma, M. Friedman, A. Kandel, A New Fuzzy Arithmetic, Fuzzy Sets Syst. 108 (1999), 83–90. https://doi.org/10.1016/s0165-0114(97)00310-2.

[24] S.P. Mondal, T.K. Roy, First Order Linear Homogeneous Ordinary Differential Equation in Fuzzy Environment Based On Laplace Transform, J. Fuzzy Set Valued Anal. 2013 (2013), 1-18. https://doi.org/10.5899/2013/jfsva-00174.

[25] M.L. Puri, D.A. Ralescu, Differentials of Fuzzy Functions, J. Math. Anal. Appl. 91 (1983), 552–558. https://doi.org/10.1016/0022-247x(83)90169-5.

[26] G.K. Watugala, Sumudu Transform: A New Integral Transform to Solve Differential Equations and Control Engineering Problems, Int. J. Math. Educ. Sci. Technol. 24 (1993), 35–43. https://doi.org/10.1080/0020739930240105.

[27] S. Weerakoon, Application of Sumudu Transform to Partial Differential Equations, Int. J. Math. Educ. Sci. Technol. 25 (1994), 277–283. https://doi.org/10.1080/0020739940250214.

[28] S. Weerakoon, Complex Inversion Formula for Sumudu Transform, Int. J. Math. Educ. Sci. Technol. 29 (1998), 618–621.

[29] L.A. Zadeh, Fuzzy Sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.