

Purities of Ordered Ideals in Ordered Ternary Semirings**Jatuporn Sanborisoot^{1,2,*}, Pakorn Palakawong na Ayutthaya²**¹*Algebra and Applications Research Unit, Department of Mathematics, Mahasarakham University,
44150, Mahasarakham, Thailand*²*Department of Mathematics, Khon Kaen University, 40002, Khon Kaen, Thailand***Corresponding author: jatuporn.san@msu.ac.th*

Abstract. We introduce the notions of left pure, lateral pure, right pure and quasi-pure ordered ideals of ordered ternary semirings and characterize several regularity conditions on ordered ternary semirings using the purities of their ordered ideals.

1. INTRODUCTION

A semiring is a notable generalization of a ring and appears in applications to automata theory, formal languages, optimization theory, and other branches of applied mathematics. Clearly, we are always able to construct a ternary semiring from a given semiring but there is a ternary semiring such that it cannot be considered a semiring. Similarly, we can consider a given ternary semiring as an ordered ternary semiring together with the equality relation. For ordered semigroups, Changphas and Sanborisoot [1] defined the notions of left pure and right pure ideals and characterized left weakly regular and right weakly regular ordered semigroups in terms of left pure and right pure ideals, respectively. They also studied these concepts in the context of ternary semigroups in [5]. Moreover, the concepts of pure fuzzy ideals and weakly pure fuzzy ideals in ordered ternary semigroups were introduced in [6]. Palakawong na Ayutthaya and Pibajjommee [3] defined the notions of left pure, right pure, quasi-pure, and bi-pure ordered ideals and used them to characterize several kinds of regularities on ordered semirings.

Motivated by [1, 3, 5], we introduce the notions of left pure, lateral pure, right pure and quasi-pure ordered ideals in ordered ternary semirings and characterize several regularity conditions by means of these purities.

Received: Aug. 23, 2023.

2020 *Mathematics Subject Classification.* 16Y60, 20N10, 06F05.

Key words and phrases. semiring; ordered ternary semiring; ordered ideal; pure ideal.

2. PRELIMINARIES

A *semiring* is an algebraic system $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups satisfying the distributive property, i.e., $(a + b) \cdot c = a \cdot c + b \cdot c$ and $c \cdot (a + b) = c \cdot a + c \cdot b$ for all $a, b, c \in S$.

A nonempty set S together with a ternary operation $f : S \times S \times S \rightarrow S$ is called a *ternary groupoid*. If a ternary groupoid (S, f) satisfies the associative property, i.e., $f(f(a, b, c), d, e) = f(a, f(b, c, d), e) = f(a, b, f(c, d, e))$ for all $a, b, c, d, e \in S$, then it is called a *ternary semigroup*.

An algebraic system $(S, +, f)$ is said to be a *ternary semiring* if $(S, +)$ is a semigroup, (S, f) is a ternary semigroup and the following conditions are satisfied: $f(x + y, a, b) = f(x, a, b) + f(y, a, b)$, $f(a, x + y, b) = f(a, x, b) + f(a, y, b)$, and $f(a, b, x + y) = f(a, b, x) + f(a, b, y)$ for all $a, b, x, y \in S$.

It is easy to construct a ternary semiring $(S, +, f)$ from the given semiring $(S, +, \cdot)$ by defining $f(a, b, c) = (a \cdot b) \cdot c$ for all $a, b, c \in S$. However, there is a ternary semiring such that it cannot be considered as a semiring.

Example 2.1. Let $\mathbb{Z}^-$ be the set of all negative integers. Then $(\mathbb{Z}^-, +, \cdot)$ is a ternary semiring where $+$ is the usual addition and $\cdot$ is the usual multiplication. However, $(\mathbb{Z}^-, +, \cdot)$ is not a semiring because $a \cdot b \notin \mathbb{Z}^-$ for any $a, b \in \mathbb{Z}^-$.

An *ordered ternary semiring* is an algebraic structure $(S, +, f, \leq)$ such that $(S, +, f)$ is a ternary semiring and $(S, \leq)$ is a partially ordered set satisfying the following properties; for any $a, b, x, y \in S$, if $a \leq b$, then $a + x \leq b + x$, $x + a \leq x + b$, $f(a, x, y) \leq f(b, x, y)$, $f(x, a, y) \leq f(x, b, y)$, and $f(x, y, a) \leq f(x, y, b)$.

An ordered ternary semiring $(S, +, f, \leq)$ is called *additively commutative* if $a + b = b + a$ for all $a, b \in S$. An element 0 of $(S, +, f, \leq)$ is said to be an *absorbing zero* if $x + 0 = x = 0 + x$ and $f(0, x, y) = f(x, 0, y) = f(x, y, 0) = 0$ for all $x, y \in S$.

Throughout this work, we simply write S instead of an ordered ternary semiring $(S, +, f, \leq)$ and always assume that it is additively commutative and contains an absorbing zero. For the convenience of writing, we also shortly write the juxtaposition abc instead of $f(a, b, c)$ for any $a, b, c \in S$.

Let A, B , and C be nonempty subsets of an ordered ternary semiring S . We define the following notations:

$$A + B = \{a + b \mid a \in A, b \in B\}, \quad (2.1)$$

$$ABC = \{abc \mid a \in A, b \in B, c \in C\}, \quad (2.2)$$

$$\Sigma A = \left\{ \sum_{i \in I} a_i \mid a_i \in A \text{ and } I \text{ is a finite set} \right\}, \text{ and} \quad (2.3)$$

$$[A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}. \quad (2.4)$$

In a particular case of $a \in S$, we write $aAB, AaB, ABa, \Sigma a$, and $[a]$ instead of $\{a\}AB, A\{a\}B, AB\{a\}, \Sigma\{a\}$, and $(\{a\})$, respectively.

If $I = \emptyset$, then we set $\sum_{i \in I} a_i = 0$ where 0 is an absorbing zero of S for all $a_i \in S$.

The following lemma contains some basic properties of the finite sums and the operator $(\]$ of nonempty subsets of an ordered ternary semiring and their proofs are routine.

Lemma 2.1. *Let A , B , and C be nonempty subsets of an ordered ternary semiring S . Then the following statements hold.*

- (1) $A \subseteq \Sigma A$ and $\Sigma A = \Sigma(\Sigma A)$.
- (2) If $A \subseteq B$, then $\Sigma A \subseteq \Sigma B$.
- (3) $\Sigma(A + B) \subseteq \Sigma A + \Sigma B$.
- (4) $(\Sigma A)BC \subseteq \Sigma ABC$, $A(\Sigma B)C \subseteq \Sigma ABC$, and $AB(\Sigma C) \subseteq \Sigma ABC$.
- (5) $\Sigma(A \cap B) \subseteq \Sigma A \cap \Sigma B$.
- (6) $\Sigma A \cup \Sigma B \subseteq \Sigma(A \cup B)$.
- (7) $\Sigma(A] \subseteq (\Sigma A]$.
- (8) $A \subseteq (A]$ and $(A] = ((A])$.
- (9) If $A \subseteq B$, then $(A] \subseteq (B]$.
- (10) $(A] + (B] \subseteq (A + B]$.
- (11) $(A]BC \subseteq (ABC]$, $A(B]C \subseteq (ABC]$, and $AB(C] \subseteq (ABC]$.
- (12) $(A \cap B] \subseteq (A] \cap (B]$.
- (13) $(A \cup B] = (A] \cup (B]$.

3. ORDERED IDEALS OF ORDERED TERNARY SEMIRINGS

A nonempty set T of an ordered ternary semiring S is called a *ternary subsemiring* of S if $T + T \subseteq T$ and $TTT \subseteq T$. The following definition contains several special kinds of ternary subsemirings of an ordered ternary semiring.

Definition 3.1. *A nonempty subset A of an ordered ternary semiring S is called an ordered left ideal (resp. ordered lateral ideal, ordered right ideal) of S if $A + A \subseteq A$, $A = (A]$, and $SSA \subseteq A$ (resp. $SAS \subseteq A$, $ASS \subseteq A$). If A is an ordered left ideal, an ordered lateral ideal, and an ordered right ideal of S , then it is called an ordered ideal of S .*

Example 3.1. *The set $2\mathbb{Z}_0^- = \{0, -2, -4, -6, \dots\}$ is an ordered ideal of the ordered ternary semiring $(\mathbb{Z}_0^-, +, \cdot, \leq)$, where $\mathbb{Z}_0^-$ is the set of all negative integers together with zero, $+$ is the usual addition, $\cdot$ is the usual multiplication, and $\leq$ denotes the equality relation.*

Example 3.2. *Let $S = \{0, a, b, c\}$. Define a binary operation $+$ on S as the following table.*

$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	b
b	b	b	b	b
c	c	b	b	c

Define a ternary operation f on S by

$$\begin{aligned} f(0, x, y) &= f(x, 0, y) = f(x, y, 0) = 0 \text{ for all } x, y \in S, \\ f(a, a, a) &= a, f(a, b, a) = f(a, c, a) = b, \\ f(b, x, a) &= f(c, x, a) = b \text{ for all } x \in S \setminus \{0\}, \\ f(x, y, b) &= b, \text{ and } f(x, y, c) = c \text{ for all } x, y \in S \setminus \{0\}. \end{aligned}$$

Define a binary relation on S by $\leq := \{(0, 0), (a, a), (b, b), (c, c), (b, c)\}$. Then $(S, +, f, \leq)$ is an ordered ternary semiring. Since $\{0, b\} + \{0, b\} \subseteq \{0, b\}$, $\{0, b\} = (\{0, b\})$, and $SS\{0, b\} \subseteq \{0, b\}$, we get that $\{0, b\}$ is an ordered left ideal of S . However, $\{0, b\}$ is not an ordered right ideal and not an ordered lateral ideal of S because $S\{0, b\}S = \{0, b\}SS = \{0, b, c\} \not\subseteq \{0, b\}$.

Let a be an element of an ordered ternary semiring S . The ordered left ideal (resp. ordered lateral ideal, ordered right ideal, ordered ideal) of S generated by a is the smallest ordered left ideal (resp. ordered lateral ideal, ordered right ideal, ordered ideal) of S containing a denoted by $L(a)$ (resp. $M(a)$, $R(a)$, $J(a)$).

Theorem 3.1. Let a be an element of an ordered ternary semiring S . The following statements hold.

- (i) $L(a) = (\Sigma a + \Sigma SSa)$.
- (ii) $M(a) = (\Sigma a + \Sigma SaS + \Sigma SSaSS)$.
- (iii) $R(a) = (\Sigma a + \Sigma aSS)$.
- (iv) $J(a) = (\Sigma a + \Sigma SSa + \Sigma aSS + \Sigma SaS + \Sigma SSaSS)$.

Proof. Let $a \in S$. The proofs of (i), (ii), and (iii) can be obtained similarly as special cases of (iv).

(iv) Let $J = (\Sigma a + \Sigma SSa + \Sigma aSS + \Sigma SaS + \Sigma SSaSS)$. Clearly, $J + J \subseteq J$, $J = (J)$, and $a = a + 0 + 0 + 0 + 0 \in J$. Next, we obtain

$$\begin{aligned} SSJ &= SS(\Sigma a + \Sigma SSa + \Sigma aSS + \Sigma SaS + \Sigma SSaSS) \\ &\subseteq (\Sigma SSa + \Sigma SaS + \Sigma SSaSS) \\ &= (\{0\} + \Sigma SSa + \{0\} + \Sigma SaS + \Sigma SSaSS) \\ &\subseteq J. \end{aligned}$$

Analogously, we get $JSS \subseteq J$. Consider

$$\begin{aligned} SJS &= S(\Sigma a + \Sigma SSa + \Sigma aSS + \Sigma SaS + \Sigma SSaSS)S \\ &\subseteq (\Sigma SaS + \Sigma SSaSS) \\ &= (\{0\} + \{0\} + \{0\} + \Sigma SaS + \Sigma SSaSS) \\ &\subseteq J. \end{aligned}$$

Now, J is an ordered ideal of S containing a . Let K be an ordered ideal of S containing a . Then $J = (\Sigma a + \Sigma SSa + \Sigma aSS + \Sigma SaS + \Sigma SSaSS) \subseteq (\Sigma K + \Sigma SSK + \Sigma KSS + \Sigma SKS + \Sigma SSKSS) \subseteq (\Sigma K + \Sigma K + \Sigma K + \Sigma K) = (\Sigma K) = K$. Therefore, J is the smallest ordered ideal of S containing a . $\square$

Now, we introduce the notion of an ordered quasi-ideal of an ordered ternary semiring as a generalization of an ordered left ideal, an ordered lateral ideal, and an ordered right ideal as the following definition.

Definition 3.2. A nonempty subset A of an ordered ternary semiring S is called an ordered quasi-ideal of S if $A + A \subseteq A$, $A = (A]$ and $(\Sigma SSA) \cap (\Sigma SAS + \Sigma SSASS) \cap (\Sigma ASS) \subseteq A$.

It is obvious that if A is an ordered left ideal, an ordered lateral ideal, or an ordered right ideal of an ordered ternary semiring S , then A is an ordered quasi-ideal of S .

The following example is an example of an ordered quasi-ideal of an ordered ternary semiring which is not an ordered left ideal and not an ordered lateral ideal.

Example 3.3. Let $S = \{0, a, b, c\}$. Define a binary operation $+$ on S by the following table.

$+$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	c
c	c	a	c	c

Define a ternary operation f on S by

$$\begin{aligned} f(0, x, y) &= f(x, 0, y) = f(x, y, 0) = 0 \text{ for all } x, y \in S, \\ f(a, x, y) &= f(c, x, y) = a, \text{ and } f(b, x, y) = b \text{ for all } x, y \in S \setminus \{0\}. \end{aligned}$$

Then $(S, +, f, =)$ is an ordered ternary semiring where $=$ is the relation $\{(x, x) \in S \times S\}$. Let $A = \{0, a\}$. Observe that $A + A \subseteq A$, $A = (A]$, and

$$SSA = SAS = \{0, a, b\}, \quad ASS = \{0, a\}, \quad SSASS = \{0, a, b\}.$$

Hence,

$$(\Sigma SSA) \cap (\Sigma SAS + \Sigma SSASS) \cap (\Sigma ASS) = \{0, a, b\} \cap \{0, a, b\} \cap \{0, a\} = \{0, a\}.$$

Therefore, A is an ordered quasi-ideal of S . However, A is not an ordered left ideal and not an ordered lateral ideal of S because $SAS = SSA = \{0, a, b\} \not\subseteq A$.

Let a be an element of an ordered ternary semiring S . The ordered quasi-ideal of S generated by a is the smallest ordered quasi-ideal of S containing a , denoted by $Q(a)$.

Theorem 3.2. Let a be an element of an ordered ternary semiring S . Then

$$Q(a) = (\Sigma a + ((\Sigma Ssa) \cap (\Sigma SaS + \Sigma SSaSS) \cap (\Sigma aSS))).$$

Proof. Let $a \in S$ and $Q = (\Sigma a + ((\Sigma Ssa) \cap (\Sigma SaS + \Sigma SSaSS) \cap (\Sigma aSS)))$. Clearly, $Q + Q \subseteq Q$ and $a = a + 0 \in Q$. We obtain that

$$\begin{aligned} (\Sigma SSQ) &= (\Sigma SS(\Sigma a + ((\Sigma Ssa) \cap (\Sigma SaS + \Sigma SSaSS) \cap (\Sigma aSS)))) \\ &\subseteq (\Sigma SS(\Sigma a + (\Sigma Ssa))) \end{aligned}$$

$$\begin{aligned}
&\subseteq ((\Sigma SSa + \Sigma SSSSa]) \\
&\subseteq (\Sigma SSa + \Sigma SSa] \\
&= (\Sigma SSa].
\end{aligned}$$

Similarly, we can prove that $(\Sigma QSS] \subseteq (\Sigma aSS]$. We get that

$$\begin{aligned}
&(\Sigma SQS + \Sigma SSQS]) \\
&= (\Sigma S(\Sigma a + ((\Sigma SSa] \cap (\Sigma SaS + \Sigma SSaSS] \cap (\Sigma aSS]))S \\
&\quad + \Sigma SS(\Sigma a + ((\Sigma SSa] \cap (\Sigma SaS + \Sigma SSaSS] \cap (\Sigma aSS]))SS] \\
&= (\Sigma S(\Sigma a + (\Sigma SaS + \Sigma SSaSS])S \\
&\quad + \Sigma SS(\Sigma a + (\Sigma SaS + \Sigma SSaSS])SS] \\
&\subseteq (\Sigma SaS + \Sigma SSaSS].
\end{aligned}$$

It follows that

$$\begin{aligned}
(\Sigma SSQ] \cap (\Sigma SQS + \Sigma SSQS]) \cap (\Sigma QSS] &\subseteq (\Sigma SSa] \cap (\Sigma SaS + \Sigma SSaSS] \cap (\Sigma aSS] \\
&= \{0\} + ((\Sigma SSa] \cap (\Sigma SaS + \Sigma SSaSS] \cap (\Sigma aSS]) \\
&\subseteq Q.
\end{aligned}$$

Now, Q is an ordered quasi-ideal of S containing a . Let K be an ordered quasi-ideal of S containing a . Then $Q = (\Sigma a + ((\Sigma SSa] \cap (\Sigma SaS + \Sigma SSaSS] \cap (\Sigma aSS])) \subseteq (\Sigma K + ((\Sigma SSK] \cap (\Sigma SKS + \Sigma SSKSS] \cap (\Sigma KSS])) \subseteq (\Sigma K] = K$. Therefore, Q is the smallest ordered quasi-ideal of S containing a . $\square$

Remark 3.1. The intersection of an ordered left ideal, an ordered lateral ideal, and an ordered right ideal of an ordered ternary semiring is an ordered quasi-ideal.

Proof. Let L , M , and R be an ordered left ideal, an ordered lateral ideal, and an ordered right ideal of S . It is not difficult to show that $L \cap M \cap R$ is closed under addition. We consider

$$\begin{aligned}
&(\Sigma SS(L \cap M \cap R]) \cap ((\Sigma S(L \cap M \cap R)S \\
&\quad + \Sigma SS(L \cap M \cap R)SS]) \cap (\Sigma (L \cap M \cap R)SS] \\
&\subseteq (\Sigma SSL] \cap (\Sigma SMS + \Sigma SSMSS] \cap (\Sigma RSS] \\
&\subseteq (\Sigma L] \cap (\Sigma M + \Sigma M] \cap (\Sigma R] \\
&= (\Sigma L] \cap (\Sigma M] \cap (\Sigma R] \\
&= L \cap M \cap R.
\end{aligned}$$

Hence, $L \cap M \cap R$ is an ordered quasi-ideal of S . $\square$

Definition 3.3. An ordered ternary semiring S is called **regular** if $a \in (\Sigma aSaSa]$ for all $a \in S$.

The following remark follows directly from Definition 3.3.

Remark 3.2. An ordered ternary semiring S is regular if and only if $A \subseteq (\Sigma ASASA]$ for all $\emptyset \neq A \subseteq S$.

Let A be a nonempty subset of an ordered ternary semiring S . The *ordered left ideal* (resp. *ordered lateral ideal*, *ordered right ideal*) of S generated by A is the smallest ordered left ideal (resp. ordered lateral ideal, ordered right ideal) of S containing A , denoted by $L(A)$ (resp. $M(A)$, $R(A)$). The following lemma can be obtained analogously to Theorem 3.1.

Lemma 3.1. Let A be a nonempty subset of an ordered ternary semiring S . The following statements hold.

- (i) $L(A) = (\Sigma A + \Sigma SSA]$.
- (ii) $M(A) = (\Sigma A + \Sigma SAS + \Sigma SSASS]$.
- (iii) $R(A) = (\Sigma A + \Sigma ASS]$.

Lemma 3.2. Let A be a nonempty subset of a regular ordered ternary semiring S . The following statements hold.

- (i) $L(A) = (\Sigma SSA]$.
- (ii) $M(A) = (\Sigma SAS + \Sigma SSASS]$.
- (iii) $R(A) = (\Sigma ASS]$.

Proof. Let $\emptyset \neq A \subseteq S$. Assume that S is regular.

(i) Using Remark 3.2, we get that

$$\Sigma A \subseteq \Sigma(\Sigma ASASA] \subseteq (\Sigma ASASA] \subseteq (\Sigma SSSSA] \subseteq (\Sigma SSA].$$

By Lemma 3.1 it turns out that

$$L(A) = (\Sigma A + \Sigma SSA] \subseteq ((\Sigma SSA] + \Sigma SSA] \subseteq (\Sigma SSA + \Sigma SSA] = (\Sigma SSA] = (\{0\} + \Sigma SSA] \subseteq L(A).$$

Hence, $L(A) = (\Sigma SSA]$.

(ii) Using Remark 3.2, we get

$$\Sigma A \subseteq \Sigma(\Sigma ASASA] \subseteq (\Sigma ASASA] \subseteq (\Sigma SSASS].$$

By Lemma 3.1, it turns out that

$$\begin{aligned} M(A) &= (\Sigma A + \Sigma SAS + \Sigma SSASS] \\ &\subseteq ((\Sigma SSASS] + \Sigma SAS + \Sigma SSASS] \\ &\subseteq (\Sigma SAS + \Sigma SSASS] \\ &= (\{0\} + \Sigma SAS + \Sigma SSASS] \\ &\subseteq M(A). \end{aligned}$$

Hence, $M(A) = (\Sigma SAS + \Sigma SSASS]$.

(iii) It can be proved in a similar way to (i). □

Theorem 3.3. Let S be a regular ordered ternary semiring. Every ordered quasi-ideal Q of S can be written in the form

$$Q = L \cap M \cap R,$$

where L , M , and R are an ordered left ideal, an ordered lateral ideal, and an ordered right ideal of S , respectively.

Proof. Let Q be an ordered quasi-ideal of S . It is clear that $Q \subseteq L(Q) \cap M(Q) \cap R(Q)$. Using Lemma 3.2, we get that $L(Q) \cap M(Q) \cap R(Q) = (\Sigma SSQ] \cap (\Sigma SQS + \Sigma SSQSS] \cap (\Sigma QSS] \subseteq Q$. Therefore, $Q = L(Q) \cap M(Q) \cap R(Q)$. $\square$

4. PURE ORDERED IDEALS OF ORDERED TERNARY SEMIRINGS

In this section, we give the notions of several kinds of pure ordered ideals on ordered ternary semirings and use them to characterize regularities of ordered ternary semirings.

Definition 4.1. An ordered ideal A of an ordered ternary semiring S is called left pure (resp. lateral pure, right pure) if for each $x \in A$, $x \in (\Sigma AAx]$ (resp. $x \in (\Sigma Ax A + \Sigma A A x A A]$, $x \in (\Sigma x A A]$).

We describe a connection between a left pure ordered ideal (resp. lateral pure ordered ideal, right pure ordered ideal) and an ordered left ideal (resp. an ordered lateral ideal, an ordered right ideal) of an ordered ternary semiring as the following theorem.

Theorem 4.1. Let A be an ordered ideal of an ordered ternary semiring S . The following statements hold.

- (i) A is left pure if and only if $A \cap L = (\Sigma A A L]$ for each ordered left ideal L of S .
- (ii) A is lateral pure if and only if $A \cap M = (\Sigma A A M A A]$ for each ordered lateral ideal M of S .
- (iii) A is right pure if and only if $A \cap R = (\Sigma R A A]$ for each ordered right ideal R of S .

Proof. Let A be an ordered ideal of S .

(i) Let L be an ordered left ideal of S . Assume that A is left pure. If $x \in A \cap L$, then $x \in (\Sigma A A x] \subseteq (\Sigma A A L]$. So, $A \cap L \subseteq (\Sigma A A L]$. Clearly, $(\Sigma A A L] \subseteq (\Sigma A] = A$ and $(\Sigma A A L] \subseteq (\Sigma L] = L$. Thus, $(\Sigma A A L] \subseteq A \cap L$. Hence, $A \cap L = (\Sigma A A L]$.

Conversely, let $a \in A$. Using Theorem 3.1(i), we get that

$$\begin{aligned} a \in A \cap L(a) &= (\Sigma A A L(a)] = (\Sigma A A (\Sigma a + \Sigma S S a)] \\ &\subseteq (\Sigma A A a + \Sigma A A S S a] \subseteq (\Sigma A A a + \Sigma A A a] = (\Sigma A A a]. \end{aligned}$$

Therefore, A is left pure.

(ii) Let M be an ordered lateral ideal of S . Assume that A is lateral pure. If $x \in A \cap M$, then

$$\begin{aligned} x &\in (\Sigma A x A + \Sigma A A x A A] \\ &\subseteq (\Sigma A (\Sigma A x A + \Sigma A A x A A] A + \Sigma A A x A A] \\ &\subseteq ((\Sigma A A x A A + \Sigma A A A x A A A] + \Sigma A A x A A] \\ &\subseteq (\Sigma A A x A A + \Sigma A A A x A A A + \Sigma A A x A A] \end{aligned}$$

$$\begin{aligned}
&= (\Sigma AAxAA + \Sigma AAAxAAA] \\
&\subseteq (\Sigma AAMAA + \Sigma AAAMAAA] \\
&\subseteq (\Sigma AAMAA + \Sigma AAMAA] \\
&= (\Sigma AAMAA].
\end{aligned}$$

So, $A \cap M \subseteq (\Sigma AAMAA]$. Clearly, $(\Sigma AAMAA] \subseteq (\Sigma AAA] \subseteq (\Sigma A] = A$ and $(\Sigma AAMAA] \subseteq (\Sigma AMA] \subseteq (\Sigma M] = M$. Thus, $(\Sigma AAMAA] \subseteq A \cap M$. Hence, $A \cap M = (\Sigma AAMAA]$.

Conversely, let $a \in A$. Using Theorem 3.1(ii), we get that

$$\begin{aligned}
a \in A \cap M(a) &= (\Sigma AAM(a)AA] \\
&= (\Sigma AA(\Sigma a + \Sigma SaS + \Sigma SSaSS]AA] \\
&\subseteq (\Sigma AAaAAA + \Sigma AASaSAA + \Sigma AASSaSSAA] \\
&\subseteq (\Sigma AaA + \Sigma AAaAA].
\end{aligned}$$

Therefore, A is lateral pure.

(iii) It can be proved similarly to (i). □

Definition 4.2. An ordered ideal A of an ordered ternary semiring S is called quasi-pure if A is left pure, lateral pure, and right pure.

The following remark follows directly from Definition 4.2.

Remark 4.1. An ordered ideal A of an ordered ternary semiring S is quasi-pure if and only if for each $x \in A$, $x \in (\Sigma AAx] \cap (\Sigma AxA + \Sigma AAxAA] \cap (\Sigma xAA]$.

We define the notion of a left (resp. right) lightly regular ordered ternary semiring in a similar way to that of a left (resp. right) lightly regular ordered ternary semigroup [4] as follows.

Definition 4.3. An ordered ternary semiring S is said to be left lightly regular (resp. right lightly regular) if $a \in (\Sigma SSaSa]$ (resp. $a \in (\Sigma aSaSS]$) for all $a \in S$.

The following theorem is a characterization of a left (resp. right) lightly regular ordered ternary semiring by its left (resp. right) pure ideals.

Theorem 4.2. An ordered ternary semiring S is left lightly regular (resp. right lightly regular) if and only if every ordered ideal of S is left pure (resp. right pure).

Proof. Assume that S is left lightly regular. Let A be an ordered ideal of S and $a \in A$. By assumption, we get that $a \in (\Sigma SSaSa] \subseteq (\Sigma SSaS(\Sigma SSaSa)] \subseteq (\Sigma SSaSSSaSa] \subseteq (\Sigma SSaSSSSaSa] \subseteq (\Sigma AAa]$. Hence, A is left pure.

Conversely, let $a \in S$. Assume that every ordered ideal of S is left pure. Using Theorem 3.1 and 4.1, we obtain that

$$a \in J(a) \cap L(a)$$

$$\begin{aligned}
&= (\Sigma J(a)J(a)L(a)] \\
&\subseteq (\Sigma J(a)SL(a)] \\
&= (\Sigma(\Sigma a + \Sigma SSa + \Sigma aSS + \Sigma SaS + \Sigma SSaSS)S(\Sigma a + \Sigma SSa)] \\
&\subseteq (\Sigma aSa + \Sigma SSaSa].
\end{aligned}$$

Using $a \in (\Sigma aSa + \Sigma SSaSa]$, it turns out that

$$\begin{aligned}
a &\in (\Sigma aSa + \Sigma SSaSa] \\
&\subseteq (\Sigma aS(\Sigma aSa + \Sigma SSaSa) + \Sigma SSaSa] \\
&\subseteq ((\Sigma SSaSa) + \Sigma SSaSa] \\
&\subseteq (\Sigma SSaSa].
\end{aligned}$$

Therefore, S is left lightly regular. $\square$

We define the notion of a softly intra-regular ordered ternary semiring in a similar way to that of a softly intra-regular n -ary semigroup [2], where $n = 3$.

Definition 4.4. An ordered ternary semiring S is said to be softly intra-regular if $a \in (\Sigma SSaSaSS]$ for all $a \in S$.

The following theorem is a characterization of a softly intra-regular ordered ternary semiring by its lateral pure ordered ideals.

Theorem 4.3. An ordered ternary semiring S is softly intra-regular if and only if every ordered ideal of S is lateral pure.

Proof. Assume that S is softly intra-regular. Let A be an ordered ideal of S and $a \in A$. By assumption, we get that

$$\begin{aligned}
a &\in (\Sigma SSaSaSS] \\
&\subseteq (\Sigma SS(\Sigma SSaSaSS)S(\Sigma SSaSaSS)SS] \\
&\subseteq (\Sigma SSSSaSaSSSSSaSaSSSS] \\
&\subseteq (\Sigma SSaSaSaSaSS] \\
&\subseteq (\Sigma SSaSaS(\Sigma SSaSaSS)SaSS] \\
&\subseteq (\Sigma SSaSaSSSaSaSSSSaSS] \\
&\subseteq (\Sigma SSASASSSASASSSASS] \\
&\subseteq (\Sigma AAaAA] \\
&= (\{0\} + \Sigma AAaAA] \\
&\subseteq (\Sigma AaA + \Sigma AAaAA].
\end{aligned}$$

Hence, A is lateral pure.

Conversely, let $a \in S$. Assume that every ordered ideal of S is lateral pure. Using Theorem 3.1 and 4.1, we obtain that

$$\begin{aligned} a &\in J(a) \cap M(a) \\ &= (\Sigma J(a)J(a)M(a)J(a)J(a)) \\ &\subseteq (\Sigma J(a)SM(a)SS) \\ &= (\Sigma(\Sigma a + \Sigma SSa + \Sigma aSS + \Sigma SaS + \Sigma SSaSS) \\ &\quad S(\Sigma a + \Sigma SaS + \Sigma SSaSS)SS) \\ &\subseteq (\Sigma aSaSS + \Sigma SSaSaSS). \end{aligned}$$

Using $a \in (\Sigma aSaSS + \Sigma SSaSaSS)$, it turns out that

$$\begin{aligned} a &\in (\Sigma aSaSS + \Sigma SSaSaSS) \\ &\subseteq (\Sigma(\Sigma aSaSS + \Sigma SSaSaSS)SaSS + \Sigma SSaSaSS) \\ &\subseteq ((\Sigma SSaSaSS) + \Sigma SSaSaSS) \\ &\subseteq (\Sigma SSaSaSS). \end{aligned}$$

Therefore, S is softly intra-regular. □

The following corollary follows immediately from Definition 4.2, Theorem 4.2, and Theorem 4.3.

Corollary 4.1. *An ordered ternary semiring S is left lightly regular, right lightly regular, and softly intra-regular if and only if every ordered ideal of S is quasi-pure.*

Acknowledgments: This research project was financially supported by Mahasarakham University.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] T. Changphas, J. Sanborisoot, Pure Ideals in Ordered Semigroups, Kyungpook Math. J. 54 (2014), 123–129. <https://doi.org/10.5666/KMJ.2014.54.1.123>.
- [2] J. Daengsaen, S. Leeratanavalee, B. Davvaz, On Minimal and Maximal Hyperideals in N-Ary Semihypergroups, Mathematics 8 (2020), 1656. <https://doi.org/10.3390/math8101656>.
- [3] P.P.N. Ayutthaya, B. Pibajommee, Purities of Ordered Ideals of Ordered Semirings, Kyungpook Math. J. 60 (2020), 455–465. <https://doi.org/10.5666/KMJ.2020.60.3.455>.
- [4] Patchara Pornsurat, Pakorn Palakawong Na Ayutthaya, Bundit Pibajommee, Linear Inequations with 5-Digit and Regularity Condition on Ordered Ternary Semigroups, Songklanakarin J. Sci. Technol. 43 (2021), 1550–1555. <https://doi.org/10.14456/sjst-psu.2021.202>.
- [5] J. Sanborisoot, T. Changphas, On Pure Ideals in Ordered Ternary Semigroups, Thai J. Math. 12 (2014), 455–464.
- [6] J. Sanborisoot, On Pure Fuzzy Ideals in Ordered Ternary Semigroups, ICIC Express Lett. 16 (2022), 1279–1286.