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Some Algebraic Characteristics of Bipolar-Valued Fuzzy Subgroups over a Certain Averaging Operator and Its Application in Multi-Criteria Decision Making

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Abstract. In this manuscript, we introduce the concepts of ψ-bipolar-valued fuzzy set (ψ-BVFS), ψ-bipolar-valued fuzzy normal subgroup (ψ-BVFNSG), cut sets $M_{\psi_{(v,\chi)}}\left(C_{v,\chi}M_{\psi}\right)$ of ψ-BVFS and ψ-BVFSG, and bipolar-valued fuzzy cosets (BVF cosets). Further, we explore some algebraic properties of newly defined ψ-BVFSG. In addition, we present some new results of homomorphism and quotient group of ψ-BVFSG. At the end, we provide an application of ψ-BVFS in decision making by using topsis method.

1. Introduction

The ideas of mathematics are gripping by the thought of set theory(set hypothesis). In crisp set, we have two possibilities either yes (element exists) or not. If the entries just enrolled from 0 and 1, then it is a crisp set. For example, a wolf is considered as wild animal but a fish is not. Crisp set is mainly used for digital designing. Its concept is also used for flow of current where one or more switched are attached. Fuzzy logic theory is based on the concept of grade membership function in which the value are from unit interval, that is, [0,1] instead of 0 and 1. Fuzzy Sets (FSs) have been proven to be useful mathematical structures where the data represents a vague collection of objects. Fuzzy sets/logic enable the integration of ambiguous human evaluations in computational issues. It also gives an excellent method for resolving multiple criterion conflicts and better evaluating solutions. In the creation of intelligent systems for decision making, classification,

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analytical thinking, evaluation, and control, new computing techniques rely on Fuzzy sets/logic can be applied. Engineers, mathematicians, computer software developers and researchers, natural scientists, medical researchers, social scientists, and others engaged in research and innovation will find the fuzzy logic essential. There are various types of FSs in the Fuzzy literature, such as Intuitionistic Fuzzy Sets (IFS), valued fuzzy sets, vague sets, complex fuzzy sets (CFS) etc. Bipolar-valued fuzzy sets (BVFS) is an important extension of fuzzy set theory which serves noteworthy in modern mathematics. It has also been used in medical diagnosis and building sciences. BVFSs are the sets in which the entries are enrolled in the form of positive and negative membership functions. The entries of negative membership function is from -1 to 0 and entries of positive membership function is from 0 to 1.

In 1965, Zadeh [1] presented the concept of fuzzy sets (FSs) in his research paper. In 1968, Zadeh [2] initiated the concept of level set or cut set of FS. Fuzzy subgroups (FSGs) based on FSs were introduced by Rosenfeld [3] in 1971. In 1981, Das [4] presented the level subgroups. Mukherjee et al. [5] presented the idea of normal fuzzy subgroups (NFSGs) and fuzzy cosets in 1984. In 1986, Atanassov [6] presented the idea of IFSs which gives a new direction to fuzzy logic. Biswas [7] extended the theory of IFSs to intuitionistic fuzzy subgroups (IFSGs) in 1989. In 1989, Buckley [8] introduced the concept of CFS.

Zhang [9] initiated bipolar fuzzy sets (BFS) in 1998. In 2000, Lee [10] initiated BVFSs by a further work on FSs in which the membership function taken between $[0, 1]$ and $[-1, 1]$. Manemaran et al. [11] applied BFSs to groups and introduced bipolar fuzzy groups in 2010. Saeid and Rafsanjani [12] worked on BVF bck/bci algebras. Al-Husban et al. [13] initiated bipolar complex sets and their properties. Fuzzy Averaging operator is discussed by Dubois in [14]. For more details, see [15]. Shuaib et al. [16] discussed some properties of λ-FSGs in 2019. Shuaib et al. [17] discussed averaging operator on IFSG and proved its properties in 2020. Alghazzawi et al. [18] worked on algebraic characteristics of anti-IFSGs over averaging operator.

Bipolar-valued fuzzy sets (BVFSs) are used in decision making problem widely. Decision making entails selecting best choice from a list of alternatives. The decision is made in some cases based on previous experience. The circumstances and decision making in these situations are analysed using previous experience. For different strategies about decision making, see [19], for group decision making [20], multi-criteria decision making [21–23], composition of BVFSs [24], TOPSIS Method [21]. In particular, averaging operator on bipolar-valued fuzzy sets are widely used in multi-attribute group decision making problems. Multi-attribute group decision making is a strategy in which panel co-operatively takes decision from collection of different alternatives. Riaz and Tehrim [20] discussed cubic bipolar fuzzy information based multi-attribute group decision making by using averaging aggregation operator. BVFSG is widely used in image processing and pattern analysis like computer vision. It is also used with fuzzy graph theory which have a wide range of applications. For example, data arrangements, network communications, google map. In google maps it is used to find shortest path from one place to another.

The main objective of this research is to obtain a class of ψ-BVFSG that corresponded to a given BVFSG. ψ-BVFSG's concept is based on ψ-BVFS in which we apply averaging operator to BVFS. We explore the idea of cut set over ψ-BVFS, ψ-BVF cosets and ψ-BVNFSG with some of their characteristics. In addition, we extended this theory by presenting the concept of homomorphism and quotient group over ψ-BVFSGs. Many of the most important uses may be seen here of ψ-BVFSGs in real world problems like:

- These sets are utilized in project appraisal, which is an essential application of ψ-BVFS. By selecting an appropriate metric ψ, our technique offers a multitude of options for assessing a project.
- In daily life decision making problems, we can apply this concept for managing the budget by choosing a suitable value of ψ.

The rest of paper is arranged as follows, in section 2, BVFS and their properties are discussed. Also, we study the concept of BVFSG, BVNFSG, cosets and some properties of BVFSG. In section 3, we assemble averaging operator of BVFSs, ψ-BVFS, its application in decision making and cut set over ψ-BVFS with some properties related to operations like union and intersection over BVFS. In section 4, we introduce the concept of ψ-BVFSG, ψ-BVF cosets and ψ-BVNFSG with properties related to this theory. In section 5, we discuss homomorphism of ψ-BVFSGs and we give some results. We also present the concept of quotient group with some related to it.

2. Preliminaries

This section carries some basics of bipolar-valued fuzzy subgroup.

Definition 2.1. *[10] A bipolar-valued fuzzy set (BVFS) M in the universal set X is defined as:*

$$M = \{< p, \omega_M^+(p), \omega_M^-(p) >| \, p \in X\},$$

where $\omega_M^+ : X \to [0,1]$ and $\omega_M^- : X \to [-1,0]$ are positive and negative membership mappings.

Definition 2.2. *[10] If S and T are two BVFSs then S is said to be bipolar-valued fuzzy subset of T if $\omega_T^+(p) \geq \omega_S^+(p)$ and $\omega_T^-(p) \leq \omega_S^-(p)$, denoted as $S \subseteq T$.*

Definition 2.3. *[25] For every two BVFSs $M = (\omega_M^+, \omega_M^-)$ and $N = (\omega_N^+, \omega_N^-)$ on G, then intersection and union can be defined as:*

$$(M \cap N)(p) = (min(\omega_M^+(p), \omega_N^+(p)), max(\omega_M^-(p), \omega_N^-(p))),$$

$$(M \cup N)(p) = (max(\omega_M^+(p), \omega_N^+(p)), min(\omega_M^-(p), \omega_N^-(p))).$$

Definition 2.4. *[26] Let $(G, .)$ be a groupoid and N and M are BVFSs of G then the bipolar-valued fuzzy product of M and N is denoted by $M \circ N$ and defined as follow:*

$$M \circ N(g) = (\omega_{M \circ N}^+(g), \omega_{M \circ N}^-(g)),$$

$$(M \circ N)(g) = \vee_{pq=g}[\omega_M^+(p) \wedge \omega_N^+(q)], \wedge_{pq=g}[\omega_M^-(p) \vee \omega_N^-(q)],$$

for every $g \in X$

Definition 2.5. *[27] The separation measure for BVFS $D^+(M_i, M^+)$ and $D^-(M_i, M^-)$ of each pair from positive and negative ideal solution, respectively, are defined:*

$$D^+(M_i, M^+) = \sqrt{\frac{1}{2}\sum_{j=1}^{n}((\omega_{ij}^+ - \xi_j^+)^2 + (\omega_{ij}^- - \overline{\xi}_j^+)^2)}$$

and

$$D^-(M_i, M^-) = \sqrt{\frac{1}{2}\sum_{j=1}^{n}((\omega_{ij}^+ - \xi_j^-)^2 + (\omega_{ij}^- - \overline{\xi}_j^-)^2)},$$

where $\xi_j = \omega_j^+$ and $\overline{\xi}_j = \omega_j^-$, $\xi_j^+ = max_i\{\omega_{ij}^+\}$, $\overline{\xi}_j^+ = max_i\{\omega_{ij}^-\}$, $\xi_j^- = min_i\{\omega_{ij}^+\}$ and $\overline{\xi}_j^- = min_i\{\omega_{ij}^-\}$.

Definition 2.6. *[27] The relative-closeness with respect to positive ideal solution for BVFS is:*

$$R_i(M_i) = \frac{D^-(M_i, M^-)}{D^+(M_i, M^+) + D^-(M_i, M^-)} \qquad i = 1, 2, \cdots n.$$

Example 2.1. *Let us consider two random resturants S_1 and S_2 on the basis of services of R_1, R_2 and R_3. Firstly the decision matrix is given in Table 1.*

Services	S_1	S_2
R_1	(0.2,-0.5)	(0.3,-0.1)
R_2	(0.5,-0.3)	(0.6,-0.9)
R_3	(0.1,-0.9)	(0.5,-0.7)

Table 1: Decision Matrix

Now we will compute separation measures $D^+(R_i, R^+)$ and $D^-(R_i, R^-)$ by using Definition 2.5.

$D^+(R_i, R^+)$	Values	$D^-(R_i, R^-)$	Values
$D^+(R_1, {R_1}^+)$	0.655	$D^-(R_1, {R_1}^-)$	0.158
$D^+(R_2, {R_2}^+)$	0.707	$D^-(R_2, {R_2}^-)$	0.522
$D^+(R_{31}, {R_3}^+)$	0.667	$D^-(R_{31}, {R_3}^-)$	0.200

Table 2: Separation measures

By applying Definition 2.6, we will get relative closeness of each pair with respect to positive ideal solution R_i^+.

Relative closeness	Values
R_1	0.194
R_2	0.575
R_3	0.230

Table 3: Relative closeness

R_2 is the best service of resturants.

Definition 2.7. *[28] Let G be a group then BVFS $M = (\omega_M^+ , \omega_M^-)$ is said to be bipolar-valued fuzzy subgroup (BVFSG) of G if, for all $p, q \in G$*

(1) $\omega_M^+(pq) \geq min\{ \omega_M^+(p), \omega_M^+(q) \}$,
(2) $\omega_M^+(p^{-1}) = \omega_M^+(p)$,
(3) $\omega_M^-(pq) \leq max\{ \omega_M^-(p), \omega_M^-(q) \}$,
(4) $\omega_M^-(p^{-1}) = \omega_M^-(p)$.

Definition 2.8. *[28] A BVFSG M of G is said to be bipolar-valued normal fuzzy subgroup (BVNFSG) if it satisfy the following axioms for all $p, q \in G$*

(1) $\omega_M^+(pq) = \omega_M^+(qp)$,
(2) $\omega_M^-(pq) = \omega_M^-(qp)$.

Definition 2.9. *[28] Let M be a BVFSG of G and $p, g \in G$. An BVFS Mp of G is called bipolar-valued fuzzy (BVF) right cosets of M in G if,*

$$(Mp)(g) = (\omega_{Mp}^+(g), \omega_{Mp}^-(g)),$$

where $\omega_{Mp}^+(g) = \omega_M^+(gp^{-1})$ and $\omega_{Mp}^-(g) = \omega_M^-(gp^{-1})$.

The BVF left coset of M can be defined in the similar manner.

3. ψ-BVFS and its Application in Decision Making

In this section, we discuss study of ψ-BVFS along the application in decision making of this theory. Before this averaging operator is applied on intuitionistic and anti-intuitionistic fuzzy sets in [12] and [18]. Firstly, we applied averaging operator on BVFSs, then we defined ψ-BVFS in which we applied averaging operators on BVFS and proved some properties of it. Then we discuss application of this theory.

Definition 3.1. *Let P a non-empty set and U and V be any two BVFSs of universal set P, then averaging operator U@V is defined as:*

$$U@V = \{< p, \frac{\omega_U^+(p) + \omega_V^+(p)}{2}, \frac{\omega_U^-(p) + \omega_V^-(p)}{2} >: p \in P\}.$$

Example 3.1. *Consider a cyclic group, $C_4 = \{\pm 1, \pm i\}$ as a universal set, let U and V be two BVFSs of C_4 are:*

$$U = \{< 1, 0.40, -0.50 >, < -1, 0.50, -0.20 >, < i, 0.40, -0.60 >, < -i, 0.10, -0.80 >\},$$

$$V = \{< 1, 0.70, -0.30 >, < -1, 0.40, -0.50 >, < i, 0.20, -0.70 >, < -i, 0.80, -0.30 >\}.$$

Then averaging operator of U and V is

$$U@V = \{< 1, 0.55, -0.40 >, < -1, 0.45, -0.35 >, < i, 0.3, -0.65 >, < -i, 0.45, -0.55 >\}.$$

Definition 3.2. *Let M be BVFS of universal set G and $\psi \in [0,1]$, then the ψ-BVFS of G with respect to M is a set in which averaging operator is applied on single BVFS can be defined as:*

$$M_\psi = \{< p, \omega^+_{M_\psi}(p), \omega^-_{M_\psi}(p) >: p \in G\},$$

where $\omega^+_{M_\psi}(p) = \varrho\{\omega^+_M(p), \psi\}$ and $\omega^-_{M_\psi}(p) = \varrho\{\omega^-_M(p), \psi - 1\}$, for all $p \in G$ where ϱ is averaging operator.

Example 3.2. *Consider the cyclic group, $C_4 = \{\pm 1, \pm i\}$ as a universal set, then the BVFSs of C_4 are:*

$$M = \{< 1, 0.40, -0.50 >, < -1, 0.50, -0.70 >, < i, 0.40, -0.60 >, < -i, 0.10, -0.80 >\}.$$

For ψ=0.4 ψ-BVFS M_ψ is defined as:

$$M_{0.4} = \{< 1, 0.40, -0.55 >, < -1, 0.45, -0.65 >, < i, 0.40, -0.60 >, < -i, 0.25, -0.75 >\}.$$

Definition 3.3. *The separation measure $D^+(M_{\psi_i}, M_\psi{}^+)$ and $D^-(M_{\psi_i}, M_\psi{}^-)$ of each pair from positive and negative ideal solution, respectively, for ψ-BVFS are:*

$$D^+(M_{\psi_i}, M_\psi{}^+) = \sqrt{\frac{1}{2}\sum_{j=1}^{n}((\omega^+_{M_{\psi_{ij}}} - \xi^+_j)^2 + (\omega^-_{M_{\psi_{ij}}} - \overline{\xi}^+_j)^2)}$$

and

$$D^-(M_{\psi_i}, M_\psi{}^-) = \sqrt{\frac{1}{2}\sum_{j=1}^{n}((\omega^+_{M_{\psi_{ij}}} - \xi^-_j)^2 + (\omega^-_{M_{\psi_{ij}}} - \overline{\xi}^-_j)^2)},$$

where $\xi = \omega^+$ and $\overline{\xi} = \omega^-_{M_\psi}$, $\xi^+ = max_i\{\omega^+_{M_{\psi_{ij}}}\}$, $\overline{\xi}^+ = max_i\{\omega^-_{M_{\psi_{ij}}}\}$, $\xi^- = min_i\{\omega^+_{M_{\psi_{ij}}}\}$ and $\overline{\xi}^- = min_i\{\omega^-_{M_{\psi_{ij}}}\}$.

In next definition we define relative-closeness with respect to positive ideal solution.

Definition 3.4. *The relative-closeness with respect to positive ideal solution for ψ-BVFS is:*

$$R_i(M_i) = \frac{D^-(M_{\psi_i}, M_\psi{}^-)}{D^+(M_{\psi_i}, M_\psi{}^+) + D^-(M_{\psi_i}, M_\psi{}^-)} \qquad i = 1, 2, \cdots m.$$

Example 3.3. *Suppose an insurance company collect information/feedback from its 5 prime customers $\{A_1, A_2, A_3, A_4, A_5\}$ about its policies and financial benefits related to these policies. Let U_1, U_2 are policies and V_1, V_2 are financial benefits. Here we will analyse that which customer is most satisfied.*

Customers	U_1	V_1
A_1	(0.4,-0.7)	(0.7,-0.9)
A_2	(0.2,-0.1)	(0.8,-0.8)
A_3	(0.5,-0.6)	(0.2,-0.1)
A_4	(0.9,-0.1)	(0.6,-0.3)
A_5	(0.5,-0.3)	(0.2,-0.4)

Table 4: Relative feedback of U_1 and V_1

Customers	U_2	V_2
A_1	*(0.7,-0.1)*	*(0.5,-0.3)*
A_2	*(0.6,-0.3)*	*(0.2,-0.6)*
A_3	*(0.2,-0.6)*	*(0.6,-0.6)*
A_4	*(0.4,-0.2)*	*(0.8,-0.9)*
A_5	*(0.7,-0.4)*	*(0.8,-0.1)*

Table 5: Relative feedback of U_2 and V_2

First we calculate the averaging operators of U_1, V_1 and U_2, V_2. Let $M_1 = U_1@V_1$ and $M_2 = U_2@V_2$.

Customers	$M_1 = U_1@V_1$	$M_2 = U_2@V_2$
A_1	*(0.55,-0.4)*	*(0.6,-0.6)*
A_2	*(0.4,-0.2)*	*(0.5,-0.7)*
A_3	*(0.35,-0.6)*	*(0.4,-0.35)*
A_4	*(0.65,-0.15)*	*(0.7,-0.6)*
A_5	*(0.6,-0.35)*	*(0.5,-0.25)*

Table 6: Calculating M_1 and M_2

Now we will find ψ-BVFS from above table and let $\psi = 0.1$.

Customers	$M_{1\psi}$	$M_{2\psi}$
A_{ψ_1}	*(0.32,-0.65)*	*(0.35,-0.75)*
A_{ψ_2}	*(0.25,-0.55)*	*(0.3,-0.8)*
A_{ψ_3}	*(0.12,-0.75)*	*(0.25,-0.62)*
A_{ψ_4}	*(0.37,-0.52)*	*(0.4,-0.75)*
A_{ψ_5}	*(0.35,-0.62)*	*(0.3,-0.57)*

Table 7: Calculating $M_{1\psi}$ and $M_{2\psi}$

Now we will compute separation measures $D^+(M_{\psi_i}, M_\psi{}^+)$ and $D^+(M_{\psi_i}, M_\psi{}^-)$ by using Definition 3.3.

$D^+(A_{\psi_i}, A_\psi{}^+)$	*Values*	$D^-(A_{\psi_i}, A_\psi{}^-)$	*Values*
$D^+(A_{\psi_1}, A_\psi{}^+)$	*0.165*	$D^-(A_{\psi_1}, A_\psi{}^-)$	*0.176*
$D^+(A_{\psi_2}, A_\psi{}^+)$	*0.196*	$D^-(A_{\psi_2}, A_\psi{}^-)$	*0.172*
$D^+(A_{\psi_3}, A_\psi{}^+)$	*0.252*	$D^-(A_{\psi_3}, A_\psi{}^-)$	*0.127*
$D^+(A_{\psi_4}, A_\psi{}^+)$	*0.129*	$D^-(A_{\psi_4}, A_\psi{}^-)$	*0.264*
$D^+(A_{\psi_5}, A_\psi{}^+)$	*0.087*	$D^-(A_{\psi_5}, A_\psi{}^-)$	*0.250*

Table 8: Separation measures w.r.t. +ve and -ve ideal solution

Now, we can find relative closeness of each pair with respect to the positive ideal solution A^+_ψ by applying Definition 3.4.

Customer	R_i
A_{ψ_1}	*0.516*
A_{ψ_2}	*0.467*
A_{ψ_3}	*0.335*
A_{ψ_4}	*0.671*
A_{ψ_5}	*0.741*

Table 9: Relative closeness

Table 9 shows that A_5 customer is most satisfied with policies and financial benefits.

Definition 3.5. *Let υ and χ belongs to [0,1] such that $\upsilon + \chi \leq 1$. The (υ , χ)- cut set of a ψ-BVFS , M_ψ is denoted as $M_{\psi_{(\upsilon,\chi)}}$ or $C_{\upsilon,\chi}(M_\psi)$ and defined as:*

$$C_{\upsilon,\chi}(M_\psi) = M_{\psi_{(\upsilon,\chi)}} = \{p \in P : \omega^+_{M^\psi}(p) \geq \upsilon, \omega^-_{M^\psi}(p) \leq \chi\}.$$

Example 3.4. *Let $P = \{a, b, c\}$ is universal set and M_ψ={(a, 0.6, -0.4), (b, 0.5, -0.7), (c, 0.8, -1)} then $M_{\psi_{(\upsilon,\chi)}}$={a, b} is cut set of M_ψ where $\upsilon = 0.7$ and $\chi = -0.4$.*

Definition 3.6. *For every two ψ-BVFSs M_ψ=($\omega^+_{M_\psi}, \omega^-_{M_\psi}$) and N_ψ=($\omega^+_{N_\psi}, \omega^-_{N_\psi}$) on G, the intersection and union can be defined as:*

$$(M_\psi \cap N_\psi)(p) = (min(\omega^+_{M_\psi}(p), \omega^+_{N_\psi}(p)), max(\omega^-_{M_\psi}(p), \omega^-_{N_\psi}(p)))$$

and

$$(M_\psi \cup N_\psi)(p) = (max(\omega^+_{M_\psi}(p), \omega^+_{N_\psi}(p)), min(\omega^-_{M_\psi}(p), \omega^-_{N_\psi}(p))).$$

Theorem 3.1. *Intersection of two ψ-BVFSs is also ψ-BVFS.*

Proof: *For any two ψ-BVFSs M_ψ and N_ψ of a group G, we have:*

$$\begin{aligned}
\omega^+_{(M\cap N)_\psi}(p) &= \varrho\{\omega^+_{M\cap N}(p), \psi\} = \varrho\{min\{\omega^+_M(p), \omega^+_N(p)\}, \psi\} \\
&= \varrho\{min\{\omega^+_M(p), \psi\}, min\{\omega^+_N(p), \psi\}\} \\
&= min\{\varrho\{\omega^+_M(p), \psi\}, \varrho\{\omega^+_N(p), \psi\}\} \\
&= min\ \{\ \omega^+_{M^\psi}(p), \omega^+_{M^\psi}(p)\ \} \\
&= \omega^+_{M_\psi \cap N_\psi}(p).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\omega^-_{(M\cap N)_\psi}(p) &= \varrho\{\omega^-_{M\cap N}(p), \psi - 1\} \\
&= \varrho\{max\{\omega^-_M(p), \omega^-_N(p)\}, \psi - 1\} \\
&= \varrho\{max\{\omega^-_M(p), \psi - 1\}, max\{\omega^-_N(p), \psi - 1\}\} \\
&= max\{\varrho\{\omega^-_M(p), \psi - 1\}, \varrho\{\omega^-_N(p), \psi - 1\}\} \\
&= max\ \{\ \omega^-_{M^\psi}(p), \omega^-_{M^\psi}(p)\ \} \\
&= \omega^-_{M_\psi \cap N_\psi}(p).
\end{aligned}$$

Hence, $(M \cap N)_\psi = M_\psi \cap N_\psi$.

Theorem 3.2. *Union of two ψ-BVFSs is also ψ-BVFS.*

Proof: *For any two ψ-BVFSs M_ψ and N_ψ of a group G, we have:*

$$\omega^+_{(M\cup N)_\psi}(p) = \varrho\{\omega^+_{M\cup N}(p), \psi\} = \varrho\{max\{\omega^+_M(p), \omega^+_N(p)\}, \psi\}$$

$= \varrho\{max\{\omega^+_M(p), \psi\}, max\{\omega^+_N(p), \psi\}\}$
$=max\{\varrho\{\omega^+_M(p), \psi\}, \varrho\{\omega^+_N(p), \psi\}\}$
$= max\ \{\ \omega^+_{M^\psi}(p), \omega^+_{M^\psi}(p)\ \}$
$= \omega^+_{M_\psi \cup N_\psi}(p).$

Similarly , $\omega^-_{(M\cup N)_\psi}(p) = \varrho\{\omega^-_{M\cup N}(p), \psi - 1\}$
$= \varrho\{min\{\omega^-_M(p), \omega^+_N(p)\}, \psi - 1\}$
$= \varrho\{min\{\omega^-_M(p), \psi - 1\}, min\{\omega^-_N(p), \psi - 1\}\}$
$=min\{\varrho\{\omega^-_M(p), \psi - 1\}, \varrho\{\omega^-_N(p), \psi - 1\}\}$
$= min\ \{\ \omega^-_{M^\psi}(p), \omega^-_{M^\psi}(p)\ \}$
$= \omega^-_{M_\psi \cup N_\psi}(p).$

Hence $(M \cup N)_\psi = M_\psi \cup N_\psi$.

4. Algebraic Characteristics of ψ-BVFSG

In this section, we study ψ-BVFSG along various algebraic characteristics of this theory. Before this averaging operator is applied on IFSG and anti-IFSGs in [12,18]. Firstly, we defined ψ-BVFSG , ψ-BVF cosets, ψ-BVNFSG and then describe some results related to this situation.

Definition 4.1. *Let G be a group, then* M_ψ *is said to be* ψ*-BVFSG of G if for all* $p, q \in G$

(1) $\omega^+_{M_\psi}(pq) \geq min\ \{\ \omega^+_{M_\psi}(p)\ ,\ \omega^+_{M_\psi}(q)\ \}$,
(2) $\omega^+_{M_\psi}(p^{-1}) = \omega^+_{M_\psi}(p)$,
(3) $\omega^-_{M_\psi}(pq) \leq max\ \{\ \omega^-_{M_\psi}(p)\ \omega^-_{M_\psi}(q)\ \}$,
(4) $\omega^-_{M_\psi}(p^{-1}) = \omega^-_{M_\psi}(p)$.

Example 4.1. *Let* C_4=$\{\pm 1, \pm \iota\}$ *be a group under multiplication and* M_ψ*={(1, 0.7, -0.83), (-1, 0.55, -0.63), (ι, 0.25, -0.19), (ι, 0.25, -0.19)} be a* ψ*-BVFS. Here,* $\omega^+(1) = 0.7$, $\omega^+(-1) = 0.55$, $\omega^+(\iota) = 0.25$, $\omega^+(-\iota) = 0.25$, $\omega^-(1) = -0.83$, $\omega^-(-1) = -0.63$, $\omega^-(\iota) = -0.19$, $\omega^+(-\iota) = -0.19$. *By Definition 4.1,* M_ψ *is* ψ*-BVFSG of* C_4.

Theorem 4.1. *Let G be a group, then every BVFSG is* ψ*-BVFSG.*

Proof: *For any BVFSG M and elements p and q of a group G, we have:*

$\omega^+_{M_\psi}(pq) = \varrho\{\omega^+_M(pq), \psi\} \geq \varrho\{min\{\omega^+_M(p), \omega^+_M(q)\}, \psi\}$
$= \varrho\{min\{\omega^+_M(p), \psi\}, min\{\omega^+_M(q), \psi\}\}$
$=min\{\varrho\{\omega^+_M(p), \psi\}, \varrho\{\omega^+_M(q), \psi\}\}$
$= min\ \{\omega^+_{M_\psi}(p), \omega^+_{M_\psi}(q)\}.$

Similarly,

$\omega^-_{M_\psi}(pq) = \varrho\{\omega^-_M(pq), \psi - 1\} \leq \varrho\{max\{\omega^-_M(p), \omega^-_M(q)\}, \psi - 1\}$
$= \varrho\{max\{\omega^-_M(p), \psi - 1\}, max\{\omega^-_M(q), \psi - 1\}\}$
$=max\{\varrho\{\omega^+_M(p), \psi - 1\}, \varrho\{\omega^+_M(q), \psi - 1\}\}$
$= max\ \{\omega^-_{M_\psi}(p), \omega^-_{M_\psi}(q)\}.$

Now, $\omega^+_{M_\psi}(p^{-1}) = \varrho\{\omega^+_M(p^{-1}), \psi\}$

$= \varrho\{\omega^+_M(p), \psi\}$
$= \omega^+_{M_\psi}(p).$

Similarly, $\omega^-_{M_\psi}(p^{-1}) = \varrho\{\omega^-_M(p^{-1}), \psi - 1\}$
$= \varrho\{\omega^-_M(p), \psi - 1\}$
$= \omega^-_{M_\psi}(p).$

This shows that, every BVFSG is ψ-BVFSG.

Theorem 4.2. *Let M_ψ is ψ-BVFSG of G, then $C_{\upsilon,\chi}(M_\psi)$ is a subgroup of G, where $\omega^+_{M_\psi}(e) \geq \upsilon$ and $\omega^-_{M_\psi}(e) \leq \chi$ and e is identity element of G.*

Proof: *Let $p, q \in C_{\psi_{\upsilon,\chi}}(M)$ be any two elements then,*

$\omega^+_{M_\psi}(p) \geq \upsilon$, $\omega^-_{M_\psi}(p) \leq \chi$ *and* $\omega^+_{M_\psi}(q) \geq \upsilon$ *and* $\omega^-_{M_\psi}(q) \leq \chi$.

Then $\omega^+_{M_\psi}(p) \wedge \omega^+_{M_\psi}(q) \geq \upsilon$ *and* $\omega^-_{M_\psi}(p) \vee \omega^-_{M_\psi}(q) \leq \chi$.

As M_ψ is bipolar - Valued Fuzzy Subgroup of G therefore,

$\omega^+_{M_\psi}(pq^{-1}) \geq \omega^+_{M_\psi}(p) \wedge \omega^+_{M_\psi}(q) \geq \upsilon$, *and*

$\omega^-_{M_\psi}(pq^{-1}) \leq \omega^-_{M_\psi}(q) \vee \omega^-_{M_\psi}(q) \leq \chi$.

Thus $pq^{-1} \in C_{\upsilon,\chi}(M_\psi)$. Hence, $C_{\upsilon,\chi}(M_\psi)$ is subgroup of G.

Theorem 4.3. *Let G be a group, then intersection of two ψ-BVFSGs is also ψ-BVFSG.*

Proof: *For any elements p and q of G . we have,*

$\omega^+_{(M\cap N)_\psi}(pq) = \varrho\{\omega^+_{M\cap N}(pq), \psi\}$
$= \varrho\{min\{\omega^+_M(pq), \omega^+_N(pq)\}, \psi\}$
$= min\{\varrho\{\omega^+_M(pq), \psi\}, \varrho\{\omega^+_N(pq), \psi\}\}$
$= min\ \{\omega^+_{M_\psi}(pq), \omega^+_{N_\psi}(pq)\}$
$\geq min\ \{min\{\omega^+_{M_\psi}(p), \omega^+_{M_\psi}(q)\}, min\{\omega^+_{N_\psi}(p), \omega^+_{N_\psi}(q)\}\}$
$= min\ \{min\{\omega^+_{M_\psi}(p), \omega^+_{N_\psi}(p)\}, min\{\omega^+_{M_\psi}(q), \omega^+_{N_\psi}(q)\}\}$
$= min\ \{\omega^+_{(M\cap N)_\psi}(p), \omega^+_{(M\cap N)_\psi}(q)\}.$

Similarly, $\omega^-_{(M\cap N)_\psi}(pq) = \varrho\{\omega^-_{M\cap N}(pq), \psi - 1\}$
$= \varrho\{max\{\omega^-_M(pq), \omega^-_N(pq)\}, \psi - 1\}$
$= max\{\varrho\{\omega^-_M(pq), \psi - 1\}, \varrho\{\omega^-_N(pq), \psi - 1\}\}$
$= max\ \{\omega^-_{M_\psi}(pq), \omega^-_{N_\psi}(pq)\}$
$\leq max\ \{max\{\omega^-_{M_\psi}(p), \omega^-_{M_\psi}(q)\}, max\{\omega^-_{N_\psi}(p), \omega^-_{N_\psi}(q)\}\}$
$= max\ \{max\{\omega^-_{M_\psi}(p), \omega^-_{N_\psi}(p)\}, max\{\omega^-_{M_\psi}(q), \omega^-_{N_\psi}(q)\}\}$
$= max\ \{\omega^-_{(M\cap N)_\psi}(p), \omega^-_{(M\cap N)_\psi}(q)\}.$

Now, $\omega^+_{(M\cap N)_\psi}(p^{-1}) = \varrho\{\omega^+_{M\cap N}(p^{-1}), \psi\}$
$= \varrho\{min\{\omega^+_M(p^{-1}), \omega^+_N(p^{-1})\}, \psi\}$
$= \varrho\{min\{\omega^+_M(p^{-1}), \psi\}, min\{\omega^+_N(p^{-1}), \psi\}\}$
$= min\{\varrho\{\omega^+_M(p^{-1}), \psi\}, \varrho\{\omega^+_N(p^{-1}), \psi\}\}$
$= min\{\omega^+_{M_\psi}(p^{-1}), \omega^+_{N_\psi}(p^{-1})\}$
$= min\{\omega^+_{M_\psi}(p), \omega^+_{N_\psi}(p)\} = \omega^+_{(M\cap N)_\psi}(p).$

Similarly, $\omega^{-}_{(M\cap N)_\psi}(p^{-1}) = \varrho\{\omega^{-}_{M\cap N}(p^{-1}), \psi - 1\}$
$= \varrho\{max\{\omega^{-}_{M}(p^{-1}), \omega^{-}_{N}(p^{-1})\}, \psi - 1\}$
$= \varrho\{max\{\omega^{-}_{M}(p^{-1}), \psi - 1\}, max\{\omega^{-}_{N}(p^{-1}), \psi - 1\}\}$
$= max\{\varrho\{\omega^{-}_{M}(p^{-1}), \psi - 1\}, \varrho\{\omega^{-}_{N}(p^{-1}), \psi - 1\}\}$
$= max\{\omega^{-}_{M_\psi}(p^{-1}), \omega^{-}_{N_\psi}(p^{-1})\}$
$= max\{\omega^{-}_{M_\psi}(p), \omega^{-}_{N_\psi}(p)\} = \omega^{-}_{(M\cap N)_\psi}(p).$

Remark 4.1. *The union of two ψ-BVFSGs may not be a ψ-BVFSG of G.*

Example 4.2. *Let M_ψ and N_ψ be two BVFSGs of modular groups $2\mathbb{Z}$ and $3\mathbb{Z}$ respectively.*

$$\omega^{+}_{M_\psi} = \begin{cases} 0.5 & if\, a \in 2\mathbb{Z} \\ 1 & otherwise, \end{cases}$$

$$\omega^{-}_{M_\psi} = \begin{cases} -0.8 & if\, a \in 2\mathbb{Z} \\ -0.3 & otherwise, \end{cases}$$

$$\omega^{+}_{N_\psi} = \begin{cases} 0.4 & if\, a \in 3\mathbb{Z} \\ 0.55 & otherwise, \end{cases}$$

$$\omega^{-}_{N_\psi} = \begin{cases} -0.4 & if\, a \in 3\mathbb{Z} \\ -0.65 & otherwise, \end{cases}$$

$$\omega^{+}_{M_\psi\cup N_\psi} = \begin{cases} 0.5 & if\, a \in 2\mathbb{Z} \\ 0.4 & if\, a \in 3\mathbb{Z} - 2\mathbb{Z} \\ 0.55 & otherwise, \end{cases}$$

$$\omega^{-}_{M_\psi\cup N_\psi} = \begin{cases} -0.8 & if\, a \in 2\mathbb{Z} \\ -0.4 & if\, a \in 3\mathbb{Z} \\ -0.3 & otherwise. \end{cases}$$

Let $a = 12$ and $b = 3$, then $\omega^{+}_{M_\psi\cup N_\psi}(a)$=0.5, $\omega^{+}_{M_\psi\cup N_\psi}(b)$=0.4 and $\omega^{+}_{M_\psi\cup N_\psi}(a+b)$=0.55. This shows that $\omega^{+}_{M_\psi\cup N_\psi}(a+b) \geq min\{\omega^{+}_{M_\psi\cup N_\psi}(a), \omega^{+}_{M_\psi\cup N_\psi}(b)\}$. $\omega^{-}_{M_\psi\cup N_\psi}(a)$=-0.8, $\omega^{+}_{M_\psi\cup N_\psi}(b)$=-0.4 and $\omega^{+}_{M_\psi\cup N_\psi}(a+b)$=-0.3. This implies that $\omega^{-}_{M_\psi\cup N_\psi}(a+b) > max\{\omega^{-}_{M_\psi\cup N_\psi}(a), \omega^{-}_{M_\psi\cup N_\psi}(b)\}$. Hence union of ψ-BVFSGs may not be ψ-BVFSG.

Definition 4.2. *Let M_ψ be a ψ-BVFSG of G and $p, g \in G$. An ψ-BVFS $M_\psi p$ of G is called ψ-BVF right coset of M_ψ in G if, $(M_\psi p)(g) = (\omega^{+}_{M_\psi p}(g)\ ,\ \omega^{-}_{M_\psi p}(g))$ where $\omega^{+}_{M_\psi p}(g) = \omega^{+}_{M_\psi}(gp^{-1})$ and $\omega^{-}_{M_\psi p}(g) = \omega^{-}_{M_\psi}(gp^{-1})$.*

The ψ-BVF left coset of M_ψ can be defined in the similar manner.

Definition 4.3. *A ψ-BVFSG M_ψ is said to be ψ- BVNFSG if it satisfy the following axioms for all $p, q \in G$.*

(1) $\omega^{+}_{M_\psi}(pq) = \omega^{+}_{M_\psi}(qp)$,
(2) $\omega^{-}_{M_\psi}(pq) = \omega^{-}_{M_\psi}(qp)$.

Example 4.3. *By Example 4.1, M_ψ=={(1, 0.7, -0.83), (-1, 0.55, -0.63), (ι, 0.25, -0.19), (ι, 0.25, -0.19)} be a ψ-BVFSG. By Definition 4.3,*

$$\omega^+((1)(-1)) = \omega^+((-1)(1))$$
$$\omega^+((1)(\iota)) = \omega^+((\iota)(1))$$
$$\omega^+((\iota)(-1)) = \omega^+((-1)(\iota))$$
$$\omega^+((1)(-\iota)) = \omega^+((-\iota)(1))$$
$$\omega^+((-\iota)(-1)) = \omega^+((-1)(-\iota))$$
$$\omega^+((-\iota)(\iota)) = \omega^+((\iota)(-\iota)).$$

Hence, M_ψ is $\psi - BVNFSG$.

Theorem 4.4. *Every BVNFSG of G is also ψ- BVNFSG of G.*

Proof: *For any BVNFSG M_ψ of a group G, we have*

$$\omega^+_{M_\psi}(pq) = \varrho\{\omega^+_M(pq), \psi\}$$
$$= \varrho\{\omega^+_M(qp), \psi\}$$
$$= \omega^+_{M_\psi}(qp).$$

Similarly, $\omega^-_{M_\psi}(pq) = \varrho\{\omega^-_M(pq), \psi - 1\}$
$$= \varrho\{\omega^-_M(qp), \psi - 1\}$$
$$= \omega^-_{M_\psi}(qp).$$

which show that M_ψ is BVNFSG of G.

It means that every BVNFSG of G is also ψ- BVNFSG of G.

Theorem 4.5. *Let M_ψ be ψ-BVNFSG of group G, then $C_{\upsilon,\chi}(M_\psi)$ is a normal Subgroup of G, where $\omega^+_{M_\psi}(e) \geq \upsilon$ and $\omega^-_{M_\psi}(e) \leq \chi$ and e is identity element of G.*

Proof: *Let $p \in C_{\upsilon,\chi}(M_\psi)$ and $g \in G$ be any element. Then $\omega^+_{M_\psi}(p) \geq \upsilon$, $\omega^-_{M_\psi}(p) \leq \chi$. Also M_ψ is ψ-BVNFSG of group G.*

Therefore, $\omega^+_{M_\psi}(gpg^{-1}) = \omega^+_{M_\psi}(p) \geq \upsilon$ *and* $\omega^-_{M_\psi}(gpg^{-1}) = \omega^-_{M_\psi}(p) \leq \chi$.

if and only if $\omega^+_{M_\psi}(gpg^{-1}) \geq \upsilon$ *and* $\omega^-_{M_\psi}(gpg^{-1}) \leq \chi$.

So, $gpg^{-1} \in C_{\upsilon,\chi}(M_\psi)$. Hence, $C_{\upsilon,\chi}(M_\psi)$ is normal subgroup of G.

Theorem 4.6. *Let M_ψ be ψ-BVFSG of a group G and $p, q \in G$. Then,*

(1) $\omega^+_{M_\psi}(e) \geq \omega^+_{M_\psi}(p)$ *and* $\omega^-_{M_\psi}(e) \leq \omega^-_{M_\psi}(p)$.

(2) $M_\psi(pq^{-1}) = M_\psi(e)$ *implies that* $M_\psi(p) = M_\psi(q)$.

Proof:

(1) *Since* $\omega^+_{M_\psi}(pp^{-1}) = \omega^+_{M_\psi}(e)$ *and,*
$$\omega^+_{M_\psi}(pp^{-1}) \geq min\{\omega^+_{M_\psi}(p), \omega^+_{M_\psi}(p^{-1})\}$$
$$= min\{\omega^+_{M_\psi}(p), \omega^+_{M_\psi}(p)\}$$
$$= \omega^+_{M_\psi}(p).$$
It means, $\omega^+_{M_\psi}(e) \geq \omega^+_{M_\psi}(p)$.
Similarly , $\omega^-_{M_\psi}(pp^{-1}) = \omega^-_{M_\psi}(e)$
$$\omega^-_{M_\psi}(pp^{-1}) \leq max\{\omega^-_{M_\psi}(p), \omega^-_{M_\psi}(p^{-1})\}$$
$$= max\{\omega^-_{M_\psi}(p), \omega^-_{M_\psi}(p)\}$$

$= \omega^-_{M_\psi}(p).$

It means, $\omega^-_{M_\psi}(e) \leq \omega^-_{M_\psi}(p).$

(2) *Let* $\omega^+_{M_\psi}(p) = \omega^+_{M_\psi}(pq^{-1}q) = \omega^+_{M_\psi}((pq^{-1})q)$

$\geq min\{\omega^+_{M_\psi}(pq^{-1}), \omega^+_{M_\psi}(q)\}.$

By using given condition, the inequality transforms into

$\geq min\{\omega^+_{M_\psi}(e), \omega^+_{M_\psi}(q)\}.$

From the part (1), *we have*

$\omega^+_{M_\psi}(p) \geq \omega^+_{M_\psi}(q).$

Similarly,

$\omega^+_{M_\psi}(q) \geq \omega^+_{M_\psi}(p).$

This shows that

$\omega^+_{M_\psi}(p) = \omega^+_{M_\psi}(q).$

Similarly, we get

$\omega^-_{M_\psi}(p) = \omega^-_{M_\psi}(q).$

From above, we obtain

$M_\psi(p) = M_\psi(q).$

Remark 4.2. *Let M_ψ be a ψ-BVNFSG of a group G. The set $G_{M_\psi} = \{p \in G : M_\psi(p) = M_\psi(e)\}$ is a normal subgroup.*

Theorem 4.7. *Every ψ-BVNFSG of a group G satisfies the following axioms:*

(1) $pM_\psi = qM_\psi$ *if and only if* $p^{-1}q \in G_{M_\psi}$,

(2) $M_\psi p = M_\psi q$ *if and only if* $pq^{-1} \in G_{M_\psi}$ *for all* $p, q \in G$.

Proof:

(1) *Let* $pM_\psi = qM_\psi$. *In view of Definition 4.3, we have*

$$\omega^+_{M_\psi}(p^{-1}q) = \omega^+_{pM_\psi}(q) = \omega^+_{qM_\psi}(q) = \omega^+_{M_\psi}(q^{-1}q) = \omega^+_{M_\psi}(e),$$

which implies that

$$\omega^+_{M_\psi}(p^{-1}q) = \omega^+_{M_\psi}(e).$$

Similarly, we have

$$\omega^-_{M_\psi}(p^{-1}q) = \omega^-_{M_\psi}(e).$$

Thus $p^{-1}q \in G_{M_\psi}$.

Conversely, suppose $p^{-1}q \in G_{M_\psi}$. *By applying Definition 4.3 for a fixed element p and any element g of G, we have*

$$\begin{aligned}\omega^+_{pM_\psi}(g) &= \omega^+_{M_\psi}(p^{-1}g) = \omega^+_{M_\psi}(p^{-1}qq^{-1}g)\\ &= \omega^+_{M_\psi}((p^{-1}q)(q^{-1}g))\\ &\leq min\{\omega^+_{M_\psi}(p^{-1}q), \omega^+_{M_\psi}(q^{-1}g)\}\\ &= min\{\omega^+_{M_\psi}(e), \omega^+_{M_\psi}(q^{-1}g)\}\end{aligned}$$

$\leq \omega^{+}_{M_\psi}(q^{-1}g).$

This implies that

$$\omega^{+}_{pM_\psi}(g) \leq \omega^{+}_{qM_\psi}(g).$$

Similarly,

$$\omega^{+}_{qM_\psi}(g) \leq \omega^{+}_{pM_\psi}(g).$$

From above equations, we have

$$\omega^{+}_{pM_\psi}(g) = \omega^{+}_{qM_\psi}(g).$$

Similarly, $\omega^{-}_{pM_\psi}(g) = \omega^{-}_{qM_\psi}(g)$. *Consequently,* $pM_\psi = qM_\psi$.

(2) *It can be proven within the context of the preceding results.*

5. Homomorphism of ψ-BVFSGs

In this section, we provide the concepts related to ψ-BVF homomorphism of quotient group of ψ-BVFSG and we give some results, related to this theory.

Definition 5.1. *Let M_ψ and N_ψ be two ψ-BVFSGs of G and H resp., ϑ be a group homomorphism from G to H, then ϑ is called ψ-BVF homomorphism from M_ψ to N_ψ if $\vartheta(M_\psi) = N_\psi$ is satisfied.*

Theorem 5.1. *Let M_ψ be ψ-BVFSG of group G and ϑ be surjective homomorphism from G to H, then $\vartheta(M_\psi)$ is ψ-BVFSG of H.*

Proof: *Since ϑ is surjective homomorphismm, for any two elements $q_1, q_2 \in H$, there exists $p_1, p_2 \in G$ such that $\vartheta(p_1) = q_1$ and $\vartheta(p_2) = q_2$. Consider,*

$\vartheta(M_\psi)(q_1q_2) = (\omega^{+}_{\vartheta(M_\psi)}(q_1q_2)\ , \omega^{-}_{\vartheta(M_\psi)}(q_1q_2)).$

$$\begin{aligned}
\omega^{+}_{\vartheta(M_\psi)}(q_1q_2) &= \omega^{+}_{(\vartheta(M))_\psi}(q_1q_2)\\
&= \varrho\{\omega^{+}_{\vartheta(M)}(\vartheta(p_1)\vartheta(p_2)), \psi\}\\
&= \varrho\{\omega^{+}_{\vartheta(M)}(\vartheta(p_1p_2)), \psi\}\\
&= \varrho\{\omega^{+}_{M}(p_1p_2), \psi\}\\
&= \omega^{+}_{M_\psi}(p_1p_2)\\
&\geq min\{\omega^{+}_{M_\psi}(p_1), \omega^{+}_{M_\psi}(p_2)\} \text{ for all } p_1, p_2 \in G\\
&= min[\vee\{\omega^{+}_{M_\psi}(p_1), \vartheta(p_1) = q_1\}, \vee\{\omega^{+}_{M_\psi}(p_2), \vartheta(p_2) = q_2\}]\\
&= min\{\omega^{+}_{M_\psi}(\vartheta^{-1}(q_1)), \omega^{+}_{M_\psi}(\vartheta^{-1}(q_2))\}\\
&= min\{\omega^{+}_{\vartheta(M_\psi)}(q_1), \omega^{+}_{\vartheta(M_\psi)}(q_2)\}.
\end{aligned}$$

Similarly , $\omega^{-}_{\vartheta(M_\psi)}(q_1q_2) = \omega^{-}_{(\vartheta(M))_\psi}(q_1q_2)$

$$\begin{aligned}
&= \varrho\{\omega^{-}_{\vartheta(M)}(\vartheta(p_1)\vartheta(p_2)), \psi - 1\}\\
&= \varrho\{\omega^{-}_{\vartheta(M)}(\vartheta(p_1p_2)), \psi - 1\}\\
&= \omega^{-}_{M_\psi}(p_1p_2)\\
&\leq max\{\omega^{-}_{M_\psi}(p_1), \omega^{-}_{M_\psi}(p_2)\}\ \forall p_1, p_2 \in G\\
&= max[\wedge\{\omega^{-}_{M_\psi}(p_1), \vartheta(p_1) = q_1\}, \wedge\{\omega^{-}_{M_\psi}(p_2), \vartheta(p_2) = q_2\}]\\
&= max\{\omega^{-}_{M_\psi}(\vartheta^{-1}(q_1)), \omega^{-}_{M_\psi}(\vartheta^{-1}(q_2))\}
\end{aligned}$$

$= max\{\omega^-_{\vartheta(M_\psi)}(q_1), \omega^-_{\vartheta(M_\psi)}(q_2)\}.$

Now, $\omega^+_{\vartheta(M_\psi)}(q^{-1}) = \vee\{\omega^+_{M_\psi}(p^{-1}) \mid \vartheta(p^{-1}) = q^{-1}\}$

$= \vee\{\omega^+_{M_\psi}(p) \mid \vartheta(p) = q\} = \omega^+_{\vartheta(M_\psi)}(q).$

Similarly, $\omega^-_{\vartheta(M_\psi)}(q^{-1}) = \wedge\{\omega^-_{M_\psi}(p^{-1}) \mid \vartheta(p^{-1}) = q^{-1}\}$

$= \wedge\{\omega^-_{M_\psi}(p) \mid \vartheta(p) = q\}$

$= \omega^-_{\vartheta(M_\psi)}(q).$

This shows that $\vartheta(M_\psi)$ *is a* ψ*-BVFSG of H.*

Theorem 5.2. *Let* M_ψ *be* ψ*- BVNFSG of group G and* $\vartheta : G \longrightarrow H$ *be a bijective homomorphism, then* $\vartheta(M_\psi)$ *be a* ψ*- BVNFSG of a group H.*

Proof: *Consider,* $(\vartheta(M))_\psi(q_1q_2) = (\omega^+_{(\vartheta(M))_\psi}(q_1q_2), \omega^-_{(\vartheta(M))_\psi}(q_1q_2))$

$\omega^+_{(\vartheta(M))_\psi}(q_1q_2) = \varrho\{\omega^+_{\vartheta(M)}(\vartheta(p_1)\vartheta(p_2)), \psi\}$

$= \varrho\{\omega^+_{\vartheta(M)}(\vartheta(p_1p_2)), \psi\}.$

As $\omega^+_{M_\psi}(p_1p_2) = \omega^+_{M_\psi}(p_2p_1),$

we have

$\varrho\{\omega^+_{\vartheta(M)}(\vartheta(p_2p_1)), \psi\} = \varrho\{\omega^+_{\vartheta(M)}(\vartheta(p_2)\vartheta(p_1)), \psi\}$

$= \omega^+_{(\vartheta(M))_\psi}(q_2q_1).$

Similarly, $\omega^-_{(\vartheta(M))_\psi}(q_1q_2) = \varrho\{\omega^-_{\vartheta(M)}(\vartheta(p_1)\vartheta(p_2)), \psi - 1\}$

$= \varrho\{\omega^-_{\vartheta(M)}(\vartheta(p_1p_2)), \psi - 1\}.$

As, $\omega^-_{M_\psi}(p_1p_2) = \omega^-_{M_\psi}(p_2p_1),$

we have

$\varrho\{\omega^-_{\vartheta(M)}(\vartheta(p_2p_1)), \psi - 1\} = \varrho\{\omega^-_{\vartheta(M)}(\vartheta(p_2)\vartheta(p_1)), \psi - 1\}$

$= \omega^-_{(\vartheta(M))_\psi}(q_2q_1).$

Hence, $\vartheta(M_\psi)$ *is a* ψ*-BVNFSG of a group H.*

Theorem 5.3. *Let* N_ψ *be a* ψ*-BVFSG of of a group H and* $\vartheta : G \longrightarrow H$ *be a group homomorphism, then* $\vartheta^{-1}(N_\psi)$ *is* ψ*-BVFSG of a group of G.*

Proof: *Since* N_ψ *be a* ψ*-BVFSG of of a group H then there exists elements* $p, q \in G$ *, we have.*

$\vartheta^{-1}(N_\psi)(pq) = (\omega^+_{\vartheta^{-1}(N_\psi)}(pq) , \omega^-_{\vartheta^{-1}(N_\psi)}(pq)).$

As ϑ *is a group homomorphism from G to H then* $\vartheta(pq) = \vartheta(p)\ \vartheta(q).$

Now, $\omega^+_{\vartheta^{-1}(N_\psi)}(pq) = \omega^+_{N_\psi}(\vartheta(pq))$

$= \omega^+_{N_\psi}(\vartheta(p)\vartheta(q))$

$\geq min\{\omega^+_{N_\psi}(\vartheta(p)) , \omega^+_{N_\psi}(\vartheta(q))\}$

$= min\{\omega^+_{\vartheta^{-1}(N_\psi)}(p) , \omega^+_{\vartheta^{-1}(N_\psi)}(q)\}.$

Similarly, $\omega^-_{\vartheta^{-1}(N_\psi)}(pq) = \omega^-_{N_\psi}(\vartheta(pq))$

$= \omega^-_{N_\psi}(\vartheta(p)\vartheta(q))$

$\leq max\{\omega^-_{N_\psi}(\vartheta(p)) , \omega^-_{N_\psi}(\vartheta(q))\}$

$= max\{\omega^{-}_{\vartheta^{-1}(N_\psi)}(p) , \omega^{-}_{\vartheta^{-1}(N_\psi)}(q)\}.$

Now, $\omega^{+}_{\vartheta^{-1}(N_\psi)}(p^{-1}) = \omega^{+}_{N_\psi}(\vartheta(p^{-1}))$
$= \omega^{+}_{N_\psi}((\vartheta(p))^{-1})$
$= \omega^{+}_{N_\psi}(\vartheta(p))$
$= \omega^{+}_{\vartheta^{-1}(N_\psi)}(p).$

Similarly , $\omega^{-}_{\vartheta^{-1}(N_\psi)}(p^{-1}) = \omega^{-}_{N_\psi}(\vartheta(p^{-1}))$
$= \omega^{-}_{N_\psi}((\vartheta(p))^{-1})$
$= \omega^{-}_{N_\psi}(\vartheta(p))$
$= \omega^{-}_{\vartheta^{-1}(N_\psi)}(p).$

This shows that, $\vartheta^{-1}(N_\psi)$ *is* ψ*-BVFSG of a group of G.*

Theorem 5.4. *Let* N_ψ *be a* ψ*-BVNFSG of group H and* $\vartheta : G \longrightarrow H$ *be a group homomorphism , then* $\vartheta^{-1}(N_\psi)$ *is* ψ*-BVNFSG of a group of G.*

Proof: *Since* N_ψ *be a* ψ*-BVNFSG of a group H, then there exist elements* $p, q \in G$*, we have*

$\vartheta^{-1}(N_\psi)(pq) = (\omega^{+}_{\vartheta^{-1}(N_\psi)}(pq), \omega^{-}_{\vartheta^{-1}(N_\psi)}(pq)).$

We have to show that: $\omega^{+}_{\vartheta^{-1}(N_\psi)}(pq) = \omega^{+}_{\vartheta^{-1}(N_\psi)}(qp)$ *and* $\omega^{-}_{\vartheta^{-1}(N_\psi)}(pq) = \omega^{-}_{\vartheta^{-1}(N_\psi)}(qp).$

As $\vartheta : G \longrightarrow H$ *be a group homomorphism then* $\vartheta(pq) = \vartheta(p)\ \vartheta(q).$

Now , $\omega^{+}_{\vartheta^{-1}(N_\psi)}(pq) = \omega^{+}_{N_\psi}(\vartheta(pq))$
$= \omega^{+}_{N_\psi}(\vartheta(p)\vartheta(q))$
$= \omega^{+}_{N_\psi}(\vartheta(q)\vartheta(p))$
$= \omega^{+}_{N_\psi}(\vartheta(qp))$
$= \omega^{+}_{\vartheta^{-1}(N_\psi)}(qp).$

Similarly , $\omega^{-}_{\vartheta^{-1}(N_\psi)}(pq) = \omega^{-}_{N_\psi}(\vartheta(pq))$
$= \omega^{-}_{N_\psi}(\vartheta(p)\vartheta(q))$
$= \omega^{-}_{N_\psi}(\vartheta(q)\vartheta(p))$
$= \omega^{-}_{N_\psi}(\vartheta(qp))$
$= \omega^{-}_{\vartheta^{-1}(N_\psi)}(qp).$

This shows that, $\vartheta^{-1}(N_\psi)$ *is* ψ*-BVNFSG of a group of G.*

Definition 5.2. *For* ψ*-BVNFSG* M_ψ *the quotient group is denoted by* G/M_ψ *and it can be defined as:*

G/M_ψ=$\{M_\psi : p \in G.M_\psi p * M_\psi q = M_\psi pq$ *for all* $p, q \in G$ *and* $*$ *is binary operation*$\}$.

Theorem 5.5. *Let* G/M_ψ *be a quotient group of G induced by* ψ*-BVNFSG and* $p \in G$*. Then this is a natural epimorphism* $\vartheta : p \to M_\psi p$ *between group G and* G/M_ψ *with* $ker(\vartheta) = G_{M_\psi}$.

Proof: *Consider*

$$\vartheta(pq) = M_\psi pq, \qquad p, q \in G.$$
$$=M_\psi p * M_\psi q = \vartheta(p)\vartheta(q).$$

Thus, ϑ *is natural homomorphism.*

The surjective property for ϑ can be simply proved.
That means that ϑ is epimorphism. Now consider

$$ker(\vartheta) = \{p \in G : \vartheta(p) = M_{\psi}e\}$$
$$= \{p \in G : M_{\psi}p = M_{\psi}e\}.$$

We get

$$ker(\vartheta) = \{p \in G : p \in G_{M_{\psi}}\} = G_{M_{\psi}}.$$

6. Conclusion

In this paper, we extended the theory of BVFS by introducing ψ-BVFS and by taking into account the averaging operator on BVFS. Also an application related to this theory has been presented. We proved that intersection of two ψ-BVFSs is also ψ-BVFS as well as union of two ψ-BVFSs is also ψ-BVFS. Further, we introduced the concepts of cut-set of a ψ-BVFS, ψ-BVFSG, ψ-BVF cosets, and ψ-BVNFSG. Furthermore, we proved that intersection of two ψ-BVFSG is also ψ-BVFSG , every BVFSG is ψ-BVFSG and every BVNFSG is also ψ-BVNFSG. In addition, we proved that cut set of $\psi-$BVFSG and $\psi-$BVNFSG are also subgroup and normal subgroup, respectively. After that we generated the results of homomorphism of ψ-BVFSG and ψ-BVNFSG which will helpful in future study. Also, we studied the quotient group and proved results related to it.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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