Int. J. Anal. Appl. (2023), 21:118

International Journal of Analysis and Applications

Exploring the Remarkable Properties of the Double Sadik Transform and Its Applications to Fractional Caputo Partial Differential Equations

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Abstract. The Double Sadik Transform (DST) represents a generalized double integral transform that has emerged as a highly effective analytical technique for solving numerous scientific problems. This study aims to investigate the DST applied to elementary functions and explore its notable properties, including its duality with the Double Laplace Transform and its capability to transform shifting functions, periodic functions, and convolution functions. Furthermore, the DST methodology is employed to resolve prominent linear fractional Caputo partial differential equations with known solutions commonly encountered in diverse mathematical models. The obtained outcomes are expressed in exact closed form, with the most precise results articulated through the Mittag-Leffler function. These results serve to validate the effectiveness and efficiency of the DST approach, establishing it as a valuable tool for addressing scientific problems involving fractional calculus.

1. Introduction

In recent times, the creation of highly efficient and precise tools for handling problems related to fractional calculus has captured the interest of numerous researchers. New analytical and approximate techniques have been discovered, and the incorporation of existing ones has been improved, in order to figure out a variety of mathematical models such as fractional differential and integral equations. Some of the advancements are Adomian Decomposition Method (ADM) [1], Homotopy Perturbation Method (HPM) [2], Fractional Differential Transform Method (FDTM) [3], Homotopy Analysis Method (HAM) [4], Variational Iteration Method (VIM) [5], and references therein.

Received: Aug. 28, 2023.

2020 *Mathematics Subject Classification.* 35R11, 35A22, 26A33.

Key words and phrases. double Sadik transform; double integral transform; Caputo fractional derivatives; exact solution; Mittag-Leffler function.

https://doi.org/10.28924/2291-8639-21-2023-118

ISSN: 2291-8639

Integral transformation is a mathematically powerful tool that has been extensively developed to address problems in the fields of science and engineering. The key advantage of this tool lies not only in its user-friendly nature but also in its remarkable capability to transform complex functions or equations into simpler forms. This frequently results in the derivation of closed-form solutions, thereby highlighting its notable characteristic. Recently, several integral transformations that enjoy widespread popularity include the Laplace transform [6], the Fourier transform [7–9] the Sumudu transform [10], the Elzaki transform [11], the Aboodh transform [12, 13], the Mellin transform [14, 15], and the Sadik transform [16, 17].

Double integral transforms represent an advanced enhancement tool that extends the concept of single integral transforms by subjecting a function to two successive transformations, each in a different variable. These integral transformations have made significant contributions to the solution of mathematical models involving multiple independent variables. Consequently, various double integral transforms have been explored in the existing literature. Several notable examples include the utilization of double Laplace transforms by Anwar et al. [19] to seek solutions for the fractional Caputo heat equation. In 2016, Debnath formulated and applied the double Laplace transform to fractional, integral, and partial differential equations [20]. Hassan and Elzaki [21] employed the double Elzaki transform to solve nonlinear partial differential equations. Eltayeb and Kilicman [22] investigated the relationship between the double Sumudu transform and the double Laplace transform while solving the new wave equation with non-constant coefficients. Sonawane and Kiwne [23] established the properties of the double Kamal transform, including the double Kamal-double Laplace duality and the double Kamal-double Sumudu duality. The double Shehu transform was employed to solve integral equations and partial differential equations, and its main properties and theorems were established [24]. Furthermore, the Laplace-ARA transform, which combines a single Laplace transform with an ARA transform, has been extensively studied, particularly its application to partial differential equations [25]. The recently introduced double Sadik transform has shown promise in solving fractional partial differential equations arising in scientific models [26]. Unfortunately, certain important properties of double integral transforms have not yet received formal acknowledgment or recognition.

This study aims to comprehensively investigate the remarkable properties of the double Sadik transform, including its duality, shifting property, and transformation of elementary functions. Furthermore, it applies the double Sadik transform methodology to solve a linear Caputo fractional partial differential equation. The inspiration for this study arose from an analysis of Debnath's work [20] as well as the accomplished works by Sonawane and Kiwne [23].

The research makes two significant contributions. Firstly, it offers a formal proof of the duality of the double Sadik transformation with the double Laplace transform. This duality plays a pivotal role in formulating the double Sadik transform for fundamental functions. Moreover, crucial characteristics such as convolution transformation, shifting, and translating properties are explored. Secondly, it demonstrates the practical application of the double Sadik transform in solving a range of fractional partial differential equations.

The paper is organized as follows: Section 2 reviews the definition of a single Sadik transformation and presents a table illustrating the Sadik transforms for basic functions. It also includes a brief overview of fractional calculus. Section 3 introduces the definition of the double Sadik transform and outlines its fundamental properties. Additionally, the existence of the Sadik transform is discussed in this section. Section 4 details the application of the double Sadik transform (DST) method to fractional PDEs in the Caputo sense, and several examples are presented to illustrate the validity and effectiveness of the DST method. Finally, the conclusion and discussion are presented in the last section.

2. Review of the Sadik Transform and Fractional Calculus

The Sadik transform is a mathematical operation that was introduced by the Indian mathematician S.L. Shaik in 2018. This generalization of the Laplace transform has been applied in a wide range of fields, including electrical engineering, mechanical engineering, and physics. Here, the definition of the transform is reviewed.

Definition 2.1. *[27] If $f(t)$ is piecewise continuous function on the interval $0 \leq t \leq A$ for any $A > 0$ and $|f(t)| \leq Ke^{Bt}$ when $t \geq M$, for any real constant B and some positive constant K and M. Then Sadik transform of $f(t)$ is defined by*

$$\mathcal{S}[f(t)] = \frac{1}{v^{\beta}} \int_0^{\infty} f(t) e^{-tv^{\alpha}} dt = F(v^{\alpha}, \beta)$$

where v is complex variable, α is any non zero real number, and β is any real number. Here $\mathcal{S}$ is called the Sadik transform operator.

Remark 2.1. *By altering the values of α and β, the Sadik transform changes from one to the other as follows: Laplace transform ($\alpha = 1, \beta = 0$), Aboodh transform ($\alpha = \beta = 1$), Laplace-Carson transform ($\alpha = 1, \beta = -1$), Kamal transform ($\alpha = -1, \beta = 0$), Sumudu transform ($\alpha = -1, \beta = 1$), Elzaki transform ($\alpha = -1, \beta = -1$), Sawi transform ($\alpha = -1, \beta = 2$), Tarig transform ($\alpha = -2, \beta = 1$).*

The Sadik transform is closely related to the Laplace transform, which is a widely used tool in the field of engineering and physics for solving differential equations. Table 1 displays a comparison of Sadik and Laplace transforms for certain kinds of functions. Proof of these Sadik transform properties can be found in [27].

In addition to the aforementioned Sadik transformation of elementary functions, Mittag-Leffler function is a notable mathematical tool that commonly serves a significant role in expressing the closed-form solution of a fractional differential equation.

Table 1. Sadik transform and Laplace transform of some functions

$f(t)$	$\mathcal{S}[f(t)] = F(v^\alpha, \beta)$	$\mathcal{L}[f(t)] = F_L(v)$
1	$\frac{1}{v^{\alpha+\beta}}$	$\frac{1}{v}$
t	$\frac{1}{v^{2\alpha+\beta}}$	$\frac{1}{v^2}$
t^2	$\frac{2!}{v^{3\alpha+\beta}}$	$\frac{2!}{v^3}$
$t^n, n \in \mathbb{N}$	$\frac{n!}{v^{(n+1)\alpha+\beta}}$	$\frac{n!}{v^{n+1}}$
$t^\gamma, \gamma > -1$	$\frac{\Gamma(\gamma+1)}{v^{(\gamma+1)\alpha+\beta}}$	$\frac{\Gamma(\gamma+1)}{v^{\gamma+1}}$
e^{at}	$\frac{1}{v^\beta(v^\alpha - a)}$	$\frac{1}{v-a}$
$\sin at$	$\frac{a}{v^\beta(v^{2\alpha}+a^2)}$	$\frac{a}{v^2+a^2}$
$\cos at$	$\frac{v^\alpha}{v^\beta(v^{2\alpha}+a^2)}$	$\frac{v}{v^2+a^2}$
$\sinh at$	$\frac{a}{v^\beta(v^{2\alpha}-a^2)}$	$\frac{a}{v^2-a^2}$
$\cosh at$	$\frac{v^\alpha}{v^\beta(v^{2\alpha}-a^2)}$	$\frac{v}{v^2-a^2}$

Definition 2.2. *[17] The Mittag-Leffler function is defined by*

$$E_{p,q}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(pk+q)}, \quad t, q \in \mathbb{C}, \ \Re(p) > 0, \ \Re(q) > 0.$$

It is to be noticed that with a value of $q = 1$, the particular case of the Mittag-Leffler function is

$$E_{p,1}(t) = E_p(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(pk+1)}, \quad t, p \in \mathbb{C}, \ \Re(p) > 0.$$

Theorem 2.1. *[17] Let $f(t) = t^{pm+q-1}E_{p,q}(\pm at^p)$. The Sadik transform of f is given by:*

$$\mathcal{S}\left[t^{pm+q-1}E_{p,q}(\pm at^p)\right] = \frac{m!\,v^{\alpha p-(\alpha q+\beta)}}{(v^{\alpha p} \mp a)^{m+1}}$$

where $p, q \in \mathbb{C}, \Re(p) > 0, \Re(q) > 0, \Re(v) > |a|^{\frac{1}{\Re(\alpha p)}}$.

Definition 2.3. *The Riemann-Liouville fractional integral operator of order $\gamma \geq 0$ is defined as*

$$I^\gamma f(t) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}} d\tau, \gamma > 0, \ t > 0, \\ f(t), \ \gamma = 0 \end{cases}$$

Regarding the Riemann-Liouville fractional integral, it can be demonstrated that

1.) I^{γ} is a linear operator
2.) $I^{\gamma_1}I^{\gamma_2}f(t) = I^{\gamma_1+\gamma_2}f(t)$,
3.) $I^{\gamma_1}I^{\gamma_2}f(t) = I^{\gamma_2}I^{\gamma_1}f(t)$.

Definition 2.4. *[18] The Caputo fractional derivative operator D^{γ} of order γ, $(n-1 < \gamma \leq n, n \in \mathbb{N})$ is defined in the following form,*

$$D^{\gamma}f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-\tau)^{-\gamma+n-1} f^{(n)}(\tau) d\tau,$$

$\alpha > 0$, $t > 0$, *where the function $f(t)$ has absolutely continuous derivatives up to order $n-1$.*

One can note that Caputo fractional derivative operator is a linear operation

$$D^{\gamma}(c_1 f(t) + c_2 g(t)) = c_1 D^{\gamma} f(t) + c_2 D^{\gamma} g(t) \tag{2.1}$$

where c_1 and c_2 are constants. Moreover, the following two basic properties can be proved

1.) $D^{\gamma}I^{\gamma}f(t) = f(t)$,
2.) $I^{\gamma}D^{\gamma}f(t) = f(t) - \sum_{k=1}^{n-1} \frac{f^{(k)}(0)}{k!} t^k$.

3. The Double Sadik Transform and its Properties

Definition 3.1. *Let $f(x,t)$ be a function of two variables x and t defined in the positive quadrant of the $xt-$plane. The Sadik transform of $f(x,t)$ with respect to x is defined by*

$$\mathcal{S}_x[f(x,t)] = F(w,t:\alpha,\beta) = \frac{1}{w^{\beta}} \int_0^{\infty} e^{-xw^{\alpha}} f(x,t) dx$$

and the Sadik transform of $f(x,t)$ with respect to t is defined by

$$\mathcal{S}_t[f(x,t)] = F(x,v:\alpha,\beta) = \frac{1}{v^{\beta}} \int_0^{\infty} e^{-tv^{\alpha}} f(x,t) dt.$$

Definition 3.2. *A function $f(x,t)$ is called of exponential order a and $b (a > 0, b > 0)$ on $0 \leq x < \infty, 0 \leq t < \infty$, if there exists a positive constant K such that for all $x > X$ and $t > T$, $|f(x,t)| \leq Ke^{ax+bt}$, we write $f(x,t) = O(e^{ax+bt})$ as $x \to \infty, t \to \infty$.*

Definition 3.3. *[28] Let $f(x,t)$ be a function that can be expressed as a convergent infinite series and $(x,t) \in \mathbb{R}^2$, the double Sadik transform is denoted by $\mathcal{S}_2[f(x,t)] = F(v,w:\alpha,\beta)$ and defined by*

$$\mathcal{S}_2[f(x,t)] = F(w,v:\alpha,\beta) = \frac{1}{v^{\beta}w^{\beta}} \int_0^{\infty}\int_0^{\infty} e^{-(tv^{\alpha}+xw^{\alpha})} f(x,t) dx dt$$

where $x,t > 0$ and v,w are transform variables for t and x respectively, α is any non-zero real number and β is any real number, whenever the double improper integral is convergent. Here $\mathcal{S}_2$ is called the double Sadik transform operator.

Theorem 3.1. *If a function $f(x,t)$ is continuous in every finite intervals $(0,X)$ and $(0,T)$ and of exponential order e^{ax+bt}, then the double Sadik transform of $f(x,t)$ exists for all w,v provided $v^\beta w^\beta \neq 0, \Re(w^\alpha) > a$ and $\Re(v^\alpha) > b$.*

Proof. Consider

$$\begin{aligned}
\left|F(w,v:\alpha,\beta)\right| &= \left|\frac{1}{v^\beta w^\beta}\int_0^\infty\int_0^\infty e^{-(tv^\alpha + xw^\alpha)} f(x,t)dxdt\right| \\
&\leq \frac{1}{|v^\beta w^\beta|}\int_0^\infty\int_0^\infty \left|e^{-(tv^\alpha+xw^\alpha)}\right| \cdot |f(x,t)|dxdt \\
&\leq \frac{K}{|v^\beta w^\beta|}\int_0^\infty e^{-x(\Re(w^\alpha)-a)}dx \cdot \int_0^\infty e^{-t(\Re(v^\alpha)-b)}dt \\
&= \frac{K}{|v^\beta w^\beta|}\left(\frac{1}{(\Re(w^\alpha)-a)(\Re(v^\alpha)-b)}\right) \quad \text{for } \Re(w^\alpha) > a,\ \Re(v^\alpha) > b.
\end{aligned}$$

It follows that $\lim\limits_{w\to\infty, v\to\infty}\left|F(w,v:\alpha,\beta)\right| = 0$. So $\lim\limits_{w\to\infty,v\to\infty} F(w,v:\alpha,\beta) = 0$. □

The preceding definition implies that:

(1) $\mathcal{S}_2$ is a linear operation.
(2) $\mathcal{S}_2[f(x,t)] = \mathcal{S}_t\mathcal{S}_x[f(x,t)] = \mathcal{S}_x\mathcal{S}_t[f(x,t)]$.
(3) If $f(x,t) = \phi(x)\psi(t)$ then

$$\mathcal{S}_2[f(x,t)] = \mathcal{S}_x\mathcal{S}_t[f(x,t)] = \mathcal{S}_x[\phi(x)]\mathcal{S}_t[\psi(t)].$$

Theorem 3.2 (Duality)**.** *If $\mathcal{S}_2[f(x,t)] = F(w,v:\alpha,\beta)$ and $\mathcal{L}_2[f(x,t)] = F_L(w,v)$ then*

$$F(w,v:\alpha,\beta) = \frac{1}{w^\beta v^\beta}F_L(w^\alpha, v^\alpha).$$

Proof. By utilizing definition 3.3, it is simple to demonstrate that

$$\begin{aligned}
\mathcal{S}_2[f(x,t)] &= F(w,v:\alpha,\beta) \\
&= \frac{1}{v^\beta w^\beta}\int_0^\infty\int_0^\infty e^{-(tv^\alpha+xw^\alpha)}f(x,t)dxdt \\
&= \frac{1}{w^\beta v^\beta}F_L(w^\alpha, v^\alpha).
\end{aligned}$$

□

The duality property can be deployed for figuring out the DST of other elementary functions, as shown below. A detailed exposition of the proofs supporting the these results is omitted here due to their evident clarity.

(1) $\mathcal{S}_2[1] = \dfrac{1}{w^{\alpha+\beta}v^{\alpha+\beta}}$
(2) $\mathcal{S}_2[x^{\gamma_1}t^{\gamma_2}] = \dfrac{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}{w^{(\gamma_1+1)\alpha+\beta}v^{(\gamma_2+1)\alpha+\beta}}$
(3) $\mathcal{S}_2[x^n t^m] = \dfrac{n!m!}{w^{(n+1)\alpha+\beta}v^{(m+1)\alpha+\beta}}$

(4) $\mathcal{S}_2[e^{ax+bt}] = \dfrac{1}{w^\beta v^\beta (w^\alpha - a)(v^\alpha - b)}$

(5) $\mathcal{S}_2[\cos(ax+bt)] = \dfrac{w^\alpha v^\alpha - ab}{w^\beta v^\beta (w^{2\alpha} + a^2)(v^{2\alpha} + b^2)}$

(6) $\mathcal{S}_2[\sin(ax+bt)] = \dfrac{av^\alpha + bw^\alpha}{w^\beta v^\beta (w^{2\alpha} + a^2)(v^{2\alpha} + b^2)}$

(7) $\mathcal{S}_2[\cosh(ax+bt)] = \dfrac{w^\alpha v^\alpha + ab}{w^\beta v^\beta (w^{2\alpha} - a^2)(v^{2\alpha} - b^2)}$

(8) $\mathcal{S}_2[\sinh(ax+bt)] = \dfrac{av^\alpha + bw^\alpha}{w^\beta v^\beta (w^{2\alpha} - a^2)(v^{2\alpha} - b^2)}$

(9) $\mathcal{S}_2[f(x \pm t)] = \dfrac{1}{w^\alpha \mp v^\alpha}\left[\frac{1}{v^\beta}\mathcal{S}_x[f(x)] \mp \frac{1}{w^\beta}\mathcal{S}_t[f(t)]\right]$

(10) If $\mathcal{S}_x[f(x)] = F(w : \alpha, \beta)$ and $\mathcal{S}_t[g(t)] = G(v : \alpha, \beta)$ then

$$\mathcal{S}_2[f(x)] = \frac{1}{v^{\alpha+\beta}} F(w : \alpha, \beta) \text{ and } \mathcal{S}_2[g(t)] = \frac{1}{w^{\alpha+\beta}} G(v : \alpha, \beta).$$

Theorem 3.3. *If $\mathcal{S}_x[f(x)] = F(w : \alpha, \beta)$ and $\mathcal{S}_t[g(t)] = G(v : \alpha, \beta)$ then for any constants $a, b > 0$*

$$\mathcal{S}_2[f(ax)g(bt)] = \frac{1}{(ab)^{1+\frac{\beta}{\alpha}}} F\Big(\frac{w}{a^{\frac{1}{\alpha}}} : \alpha, \beta\Big) \cdot G\Big(\frac{v}{b^{\frac{1}{\alpha}}} : \alpha, \beta\Big).$$

Proof. Since

$$\mathcal{L}_2[f(ax)g(bt)] = \frac{1}{ab}\Big[F_L\Big(\frac{w}{a}\Big)G_L\Big(\frac{v}{b}\Big)\Big] = H_L(w, v).$$

Hence,

$$\begin{aligned}
\mathcal{S}_2[f(ax)g(bt)] &= H(w, v : \alpha, \beta) \\
&= \frac{1}{w^\beta v^\beta} H_L(w^\alpha, v^\alpha) \\
&= \frac{1}{w^\beta v^\beta} \cdot \frac{1}{ab}\Big[F_L\Big(\frac{w^\alpha}{a}\Big)G_L\Big(\frac{v^\alpha}{b}\Big)\Big] \\
&= \frac{1}{(\frac{w}{a^{\frac{1}{\alpha}}})^\beta (\frac{v}{b^{\frac{1}{\alpha}}})^\beta} \cdot \frac{1}{(ab)^{1+\frac{\beta}{\alpha}}}\Big[F_L\Big(\frac{w}{a^{\frac{1}{\alpha}}}\Big)^\alpha G_L\Big(\frac{v}{b^{\frac{1}{\alpha}}}\Big)^\alpha\Big] \\
&= \frac{1}{(ab)^{1+\frac{\beta}{\alpha}}} \cdot \Big[\frac{1}{(\frac{w}{a^{\frac{1}{\alpha}}})^\beta} F_L\Big(\frac{w}{a^{\frac{1}{\alpha}}}\Big)^\alpha \cdot \frac{1}{(\frac{v}{b^{\frac{1}{\alpha}}})^\beta} G_L\Big(\frac{v}{b^{\frac{1}{\alpha}}}\Big)^\alpha\Big] \\
&= \frac{1}{(ab)^{1+\frac{\beta}{\alpha}}} F\Big(\frac{w}{a^{\frac{1}{\alpha}}} : \alpha, \beta\Big) \cdot G\Big(\frac{v}{b^{\frac{1}{\alpha}}} : \alpha, \beta\Big).
\end{aligned}$$

□

Theorem 3.4. *If $\mathcal{S}_2[f(x, t)] = F(w, v : \alpha, \beta)$ then*

$$\mathcal{S}_2[f(x-a, t-b)H(x-a, t-b)] = e^{-aw^\alpha - bv^\alpha} F(w, v : \alpha, \beta)$$

here $H(x-a, t-b)$ is a Heaviside step function.

Proof. By definition 3.3, one can show that

$$\begin{aligned}
&\mathcal{S}_2[f(x-a,t-b)H(x-a,t-b)]\\
&=\frac{1}{w^\beta v^\beta}\int_0^\infty\int_0^\infty e^{-xw^\alpha-tv^\alpha}f(x-a,t-b)H(x-a,t-b)dxdt\\
&=\frac{1}{w^\beta v^\beta}\int_b^\infty\int_a^\infty e^{-xw^\alpha}e^{-tv^\alpha}f(x-a,t-b)dxdt.
\end{aligned}$$

Let $u=x-a, s=t-b$ then

$$\begin{aligned}
\mathcal{S}_2[f(x-a,t-b)H(x-a,t-b)]&=\frac{1}{w^\beta v^\beta}\int_0^\infty\int_0^\infty e^{-(u+a)w^\alpha}e^{-(s+b)v^\alpha}f(u,s)duds\\
&=\frac{e^{-aw^\alpha-bv^\alpha}}{w^\beta v^\beta}\int_0^\infty\int_0^\infty e^{-uw^\alpha-sv^\alpha}f(u,s)duds\\
&=e^{-aw^\alpha-bv^\alpha}F(w,v:\alpha,\beta).
\end{aligned}$$

□

Theorem 3.5. *If $f(x,t)$ is a periodic function i.e. $f(x,t)=f(x+T_1,t+T_2)$ then*

$$\mathcal{S}_2[f(x,t)]=(1-e^{-T_1w^\alpha-T_2v^\alpha})^{-1}\frac{1}{w^\beta v^\beta}\int_0^{T_2}\int_0^{T_1}e^{-w^\alpha x-v^\alpha t}f(x,t)dxdt.$$

Proof. According to definition (3.3), it can be shown that

$$\begin{aligned}
\mathcal{S}_2[f(x,t)]&=\frac{1}{w^\beta v^\beta}\int_0^\infty\int_0^\infty e^{-xw^\alpha-tv^\alpha}f(x,t)dxdt\\
&=\frac{1}{w^\beta v^\beta}\Big[\int_0^{T_1}\int_0^{T_2}e^{-xw^\alpha-tv^\alpha}f(x,t)dxdt+\int_{T_1}^\infty\int_{T_2}^\infty e^{-xw^\alpha-tv^\alpha}f(x,t)dxdt\Big].
\end{aligned}$$

Let $x=\xi+T_1, t=\eta+T_2$ then the periodic property yields

$$\begin{aligned}
\mathcal{S}_2[f(x,t)]&=\frac{1}{w^\beta v^\beta}\Big[\int_0^{T_1}\int_0^{T_2}e^{-xw^\alpha-yv^\alpha}f(x,t)dxdt\\
&+\int_0^\infty\int_0^\infty e^{-(\xi+T_1)w^\alpha}e^{-(\eta+T_2)v^\alpha}f(\xi+T_1,\eta+T_2)d\xi d\eta\Big]\\
&=\frac{1}{w^\beta v^\beta}\Big[\int_0^{T_1}\int_0^{T_2}e^{-\xi w^\alpha-\eta v^\alpha}f(\xi,\eta)d\xi d\eta\Big]+e^{-T_1w^\alpha-T_2v^\alpha}\mathcal{S}_2[f(x,t)].
\end{aligned}$$

Hence,

$$\mathcal{S}_2[f(x,t)]=(1-e^{-T_1w^\alpha-T_2v^\alpha})^{-1}\frac{1}{w^\beta v^\beta}\int_0^{T_2}\int_0^{T_1}e^{-w^\alpha x-v^\alpha t}f(x,t)dxdt.$$

□

Definition 3.4. *The convolution of $f(x,t)$ and $g(x,t)$ is defined by*

$$(f**g)(x,t)=\int_0^t\int_0^x f(x-\xi,t-\eta)g(\xi,\eta)d\xi d\eta$$

Note that $(f**g)(x,t)=(g**f)(x,t)$

Theorem 3.6. *If $\mathcal{S}_2[f(x,t)] = F(w,v:\alpha,\beta)$ and $\mathcal{S}_2[g(x,t)] = G(w,v:\alpha,\beta)$ then*

$$\mathcal{S}_2[(f**g)(x,t)] = w^{\beta}v^{\beta}F(w,v:\alpha,\beta)G(w,v:\alpha,\beta)$$

Proof. Since

$$\mathcal{L}_2[f(x,t)**g(x,t)] = F_L(w,v)G_L(w,v) = H_L(w,v).$$

Hence

$$\begin{aligned}\mathcal{S}_2[f(x,t)**g(x,t)] &= H(w,v:\alpha,\beta)\\ &= \frac{1}{w^{\beta}v^{\beta}}H_L(w^{\alpha},v^{\alpha})\\ &= \frac{1}{w^{\beta}v^{\beta}}\cdot\Big[F_L(w^{\alpha},v^{\alpha})G_L(w^{\alpha},v^{\alpha})\Big]\\ &= w^{\beta}v^{\beta}\Big[\frac{1}{w^{\beta}v^{\beta}}F_L(w^{\alpha},v^{\alpha})\cdot\frac{1}{w^{\beta}v^{\beta}}G_L(w^{\alpha},v^{\alpha})\Big]\\ &= w^{\beta}v^{\beta}F(w,v:\alpha,\beta)G(w,v:\alpha,\beta).\end{aligned}$$

□

Before implementing the above properties, the Sadik transform of the fractional derivative is stated. The proof of the theorem can be found in [26]

Theorem 3.7. *[26] Let $\mathcal{S}_2[f(x,t)] = F(w,v:\alpha,\beta)$ then the double Sadik transform for the partial fractional Caputo derivatives are*

$$\begin{aligned}\mathcal{S}_2[\frac{\partial^{\gamma_1}f(x,t)}{\partial x^{\gamma_1}}] &= w^{\gamma_1\alpha}F(w,v:\alpha,\beta) - \sum_{k=0}^{m-1}w^{(\gamma_1-1-k)\alpha-\beta}\mathcal{S}_t\big[\frac{\partial^k f(0,t)}{\partial x^k}\big],\\ & \quad m-1<\gamma_1\leq m, m\in\mathbb{N},\\ \mathcal{S}_2[\frac{\partial^{\gamma_2}f(x,t)}{\partial t^{\gamma_2}}] &= v^{\gamma_2\alpha}F(w,v:\alpha,\beta) - \sum_{k=0}^{n-1}v^{(\gamma_2-1-k)\alpha-\beta}\mathcal{S}_x\big[\frac{\partial^k f(x,0)}{\partial t^k}\big],\\ & \quad n-1<\gamma_2\leq n, n\in\mathbb{N}.\end{aligned}$$

It is worth noting that in cases where both γ_1 and γ_2 take on integer values, the above theorem aligns with the findings presented by Singh subsequently:

Theorem 3.8. *[28] Let $\mathcal{S}_2[f(x,t)] = F(w,v:\alpha,\beta)$ then the double Sadik transform for the partial derivatives of an arbitrary integer order are*

$$\begin{aligned}\mathcal{S}_2[\frac{\partial^{m}f(x,t)}{\partial x^{m}}] &= w^{m\alpha}F(w,v:\alpha,\beta) - \sum_{k=0}^{m-1}w^{(m-1-k)\alpha-\beta}\mathcal{S}_t\big[\frac{\partial^k f(0,t)}{\partial x^k}\big],\\ \mathcal{S}_2[\frac{\partial^{n}f(x,t)}{\partial t^{n}}] &= v^{n\alpha}F(w,v:\alpha,\beta) - \sum_{k=0}^{n-1}v^{(n-1-k)\alpha-\beta}\mathcal{S}_x\big[\frac{\partial^k f(x,0)}{\partial t^k}\big].\end{aligned}$$

The proofs of these theorems are provided in the referenced source, which we have omitted in this context.

4. Implementation of Double Sadik Transform Method to Fractional Partial Differential Equation in Scientific Models

In this section, the practical implications of the double Sadik transform for resolving essential fractional partial differential equations are discussed. Let us consider the linear fractional partial differential equations in the form

$$\sum_{k=1}^{N} A_k \frac{\partial^{\gamma_k} u(x,t)}{\partial t^{\gamma_k}} + \sum_{k=1}^{n} a_k \frac{\partial^{k} u(x,t)}{\partial t^{k}} = \sum_{j=1}^{M} B_j \frac{\partial^{\eta_j} u(x,t)}{\partial x^{\eta_j}} + \sum_{j=0}^{m} b_j \frac{\partial^{j} u(x,t)}{\partial x^{j}} + g(x,t) \tag{4.1}$$

$N_k - 1 < \gamma_k \leq N_k$, $M_j - 1 < \eta_j \leq M_j$ with respect to the initial conditions

$$\frac{\partial^k u(x,0)}{\partial t^k} = f_k(x), \quad k = 0,1,\dots,n-1, x \in \mathbb{R}^+ \tag{4.2}$$

and boundary conditions

$$\frac{\partial^j u(0,t)}{\partial x^j} = h_j(t), \quad j = 0,1,\dots,m-1, t \in \mathbb{R}^+. \tag{4.3}$$

Here A_k, B_k, a_k, b_k are constants and $g(x,t)$ is given function.
Let denote $\mathcal{S}_2[u(x,t)] = U(w, v : \alpha, \beta)$ and the single Sadik transform of initial and boundary conditions are

$$\mathcal{S}_x[f_k(x)] = F_k(w:\alpha,\beta),\ \mathcal{S}_t[h_j(t)] = H_j(v:\alpha,\beta),\ k = 0,\dots n-1, j = 0,\dots,m-1. \tag{4.4}$$

For the sake of convenience, α and β are omitted. From now, the transformed function $U(w, v : \alpha, \beta)$ is written by $U(w, v)$.

By utilizing the double Sadik transform on both sides of equation (4.1) and incorporating the prescribed conditions specified by (4.4), an algebraic equation of

$$\begin{aligned}
&\sum_{k=1}^{N} A_k \Big[v^{\gamma_k \alpha} U(w,v) - \sum_{i=0}^{N_k-1} v^{(\gamma_k-1-i)\alpha-\beta} F_i(w) \Big] + \sum_{k=1}^{n} a_k \Big[v^{k\alpha} U(w,v) - \sum_{i=0}^{k-1} v^{(k-1-i)\alpha-\beta} F_i(w) \Big] \\
&= \sum_{j=1}^{M} B_j \Big[w^{\eta_j \alpha} U(w,v) - \sum_{i=0}^{M_j-1} w^{(\eta_j-1-i)\alpha-\beta} H_i(v) \Big] + \sum_{j=1}^{m} b_j \Big[w^{j\alpha} U(w,v) - \sum_{i=0}^{j-1} w^{(j-1-i)\alpha-\beta} H_i(v) \Big] \\
&b_0 U(w,v) + G(w,v),
\end{aligned}$$

is derived. Subsequent manipulation of this expression yields

$$\begin{aligned}
&\Big[\sum_{k=1}^{N} A_k v^{\gamma_k \alpha} + \sum_{k=1}^{n} a_k v^{k\alpha} - \sum_{j=1}^{M} B_j w^{\eta_j \alpha} - \sum_{j=1}^{m} b_j w^{j\alpha} - b_0 \Big] U(w,v) \\
&= \sum_{k=1}^{N} A_k \Big[\sum_{i=0}^{N_k-1} v^{(\gamma_k-1-i)\alpha-\beta} F_i(w) \Big] + \sum_{k=1}^{n} a_k \Big[\sum_{i=0}^{k-1} v^{(k-1-i)\alpha-\beta} F_i(w) \Big] \\
&- \sum_{j=1}^{M} B_j \Big[\sum_{i=0}^{M_j-1} w^{(\eta_j-1-i)\alpha-\beta} H_i(v) \Big] - \sum_{j=1}^{m} b_j \Big[\sum_{i=0}^{j-1} w^{(j-1-i)\alpha-\beta} H_i(v) \Big] + G(w,v),
\end{aligned}$$

which can be explicitly solved for $U(w,v)$ as

$$U(w,v)=\frac{1}{\Delta}\Bigg[\sum_{k=1}^{N}A_k\Big[\sum_{i=0}^{N_k-1}v^{(\gamma_k-1-i)\alpha-\beta}F_i(w)\Big]+\sum_{k=1}^{n}a_k\Big[\sum_{i=0}^{k-1}v^{(k-1-i)\alpha-\beta}F_i(w)\Big]$$
$$-\sum_{j=1}^{M}B_j\Big[\sum_{i=0}^{M_j-1}w^{(\eta_j-1-i)\alpha-\beta}H_i(v)\Big]-\sum_{j=1}^{m}b_j\Big[\sum_{i=0}^{j-1}w^{(j-1-i)\alpha-\beta}H_i(v)\Big]+G(w,v)\Bigg] \tag{4.5}$$

where $\Delta=\sum_{k=1}^{N}A_kv^{\gamma_k\alpha}+\sum_{k=1}^{n}a_kv^{k\alpha}-\sum_{j=1}^{M}B_jw^{\eta_j\alpha}-\sum_{j=1}^{m}b_jw^{j\alpha}-b_0$. Therefore, the exact solution $u(x,t)$ to the problem (4.1)-(4.3) is obtained by performing the inverse double Sadik transform of $U(w,v)$.

4.1. **Applications.** This section comprises multiple illustrated examples that demonstrate the application of the DST to solve significant linear fractional Caputo partial differential equations. Moreover, a range of noteworthy homogeneous and inhomogeneous problems have been investigated to confirm the capabilities of this technique.

Example 4.1 (**Fractional Newell-Whitehead-Segel equation**). *Consider* $N=1, A_1=1, a_k=0, B_k=0, m=2, b_0=-3, b_1=0, b_2=1$, *the fractional Newell-Whitehead-Segel equation is given*

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma}=\frac{\partial^2 u(x,t)}{\partial x^2}-3u(x,t),\quad 0<\gamma\le 1.$$

The initial and boundary conditions are

$$u(x,0)=e^{2x},\ u(0,t)=E_\gamma(t^\gamma),\ u_x(0,t)=2E_\gamma(t^\gamma),$$

and the exact solution of this problem is $u(x,t)=e^{2x}E_\gamma(t^\gamma)$.

Here $f_0(x)=e^{2x}, h_0(t)=E_\gamma(t^\gamma)$ *and* $h_1(t)=2E_\gamma(t^\gamma)$. *The single Sadik transform of these functions are*

$$F_0(w)=\frac{1}{w^\beta(w^\alpha-2)},\ H_0(v)=\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-1}\ \text{ and }\ H_1(v)=2\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-1}.$$

After taking the double Sadik transform and utilizing the given conditions, the resulting equation is

$$v^{\gamma\alpha}U(w,v)-\frac{v^{(\gamma-1)\alpha-\beta}}{w^\beta(w^\alpha-2)}=w^{2\alpha}U(w,v)-w^{\alpha-\beta}\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-1}-2w^{-\beta}\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-1}-3U(w,v),$$

which can be effortlessly rearranged to

$$(v^{\gamma\alpha}-w^{2\alpha}+3)U(w,v)=\frac{v^{(\gamma-1)\alpha-\beta}}{w^\beta}\Big[\frac{1}{w^\alpha-2}-\frac{w^\alpha}{v^{\gamma\alpha}-1}-\frac{2}{v^{\gamma\alpha}-1}\Big].$$

Upon simplification, it is found that

$$U(w,v)=\frac{1}{w^\beta(w^\alpha-2)}\cdot\frac{v^{(\gamma-1)\alpha-\beta}}{v^{\gamma\alpha}-1}.$$

The exact solution is obtained by applying the inverse double Sadik transform, which gives

$$u(x,t)=e^{2x}E_\gamma(t^\gamma).$$

Example 4.2 (**Fractional diffusion equation**). *[19] If $N = 1, A_1 = 1, a_k = 0, B_k = 0, m = 2, b_0 = b_1 = 0, b_2 = \frac{1}{\pi^2}$, the time-fractional diffusion equation is derived*

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \frac{1}{\pi^2}\frac{\partial^2 u(x,t)}{\partial x^2}, \quad x, t > 0,\ 0 < \gamma \leq 1.$$

The initial and boundary conditions are

$$u(x,0) = \sin \pi x,\ u(0,t) = 0,\ u_x(0,t) = \pi E_\gamma(-t^\gamma),$$

and the exact solution of this problem is $u(x,t) = \sin \pi x E_\gamma(-t^\gamma)$.

Note that $f_0(x) = \sin \pi x, h_0(t) = 0, h_1(t) = \pi E_\gamma(-t^\gamma)$ and the single Sadik transform of these functions are

$$F_0(w) = \frac{\pi}{w^\beta(w^{2\alpha} + \pi^2)},\ H_0(v) = 0 \ \text{and}\ H_1(v) = \pi\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}+1}.$$

Applying the double Sadik transform and utilizing the corresponding conditions yields the equation

$$v^{\gamma\alpha}U(w,v) - \frac{\pi v^{(\gamma-1)\alpha-\beta}}{w^\beta(w^{2\alpha}+\pi^2)} = \frac{1}{\pi^2}w^{2\alpha}U(w,v) - w^{-\beta}\frac{v^{\gamma\alpha-\alpha-\beta}}{\pi(v^{\gamma\alpha}+1)}.$$

This equation is then rearranged to obtain

$$(v^{\gamma\alpha} - \frac{w^{2\alpha}}{\pi^2})U(w,v) = \frac{v^{\gamma\alpha-\alpha-\beta}}{w^\beta}(\frac{\pi}{w^{2\alpha}+\pi^2} - \frac{1}{\pi(v^{\gamma\alpha}+1)}).$$

Upon simplification, it becomes evident that

$$U(w,v) = \frac{\pi}{w^\beta(w^{2\alpha}+\pi^2)} \cdot \frac{v^{(\gamma-1)\alpha-\beta}}{v^{\gamma\alpha}+1}.$$

Thus, by utilizing the inverse double Sadik transform, the exact solution can be obtained as

$$u(x,t) = \sin \pi x \cdot E_\gamma(-t^\gamma).$$

It is noteworthy that if $\gamma = 1$, the obtained solution reduces to the classical diffusion equation solution for $u(x,t) = e^{-t}\sin \pi x$.

Example 4.3 (**Fractional wave equation**). *[29] Consider $A_k = 0, a_1 = 1, B_1 = -1, b_j = 0$, the linear inhomogeneous space-fractional wave equation is derived*

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^\eta u(x,t)}{\partial x^\eta} = \sin x + t\cos x, \quad x, t > 0,\ 0 < \eta \leq 1.$$

The initial and boundary conditions are

$$u(x,0) = x^{\eta+1}E_{2,\eta+2}(-x^2) - x^{2\eta}E_{2,2\eta+1}(-x^2),\ u(0,t) = 0,$$

and the exact solution of this problem is $u(x,t) = x^{\eta+1}E_{2,\eta+2}(-x^2) + tx^\eta E_{2,\eta+1}(-x^2) - x^{2\eta}E_{2,2\eta+1}(-x^2)$.

Here $f_0(x) = x^{\eta+1}E_{2,\eta+2}(-x^2) - x^{2\eta}E_{2,2\eta+1}(-x^2)$*,* $h_0(t) = 0$ *and the single Sadik transform of these functions are*

$$F_0(w) = \frac{w^{-\eta\alpha-\beta}}{w^{2\alpha}+1} - \frac{w^{\alpha-2\eta\alpha-\beta}}{w^{2\alpha}+1}, \quad \text{and } H_0(v) = 0.$$

By applying the double Sadik transform and utilizing the given conditions, the equation

$$v^{\alpha}U(w,v) - v^{-\beta}\Big[\frac{w^{-\eta\alpha-\beta}}{w^{2\alpha}+1} - \frac{w^{\alpha-2\eta\alpha-\beta}}{w^{2\alpha}+1}\Big] + w^{\eta\alpha}U(w,v) = \frac{1}{v^{\alpha+\beta}}\cdot\frac{1}{w^{\beta}(w^{2\alpha}+1)} + \frac{1}{v^{2\alpha+\beta}}\frac{w^{\alpha}}{w^{\beta}(w^{2\alpha}+1)}$$

is obtained. Rearranging the equation yields

$$(v^{\alpha}+w^{\eta\alpha})U(w,v) = \frac{1}{v^{\beta}w^{\beta}(w^{2\alpha}+1)}\big[w^{-\eta\alpha} + \frac{1}{v^{\alpha}}\big] + \frac{w^{\alpha}}{v^{\alpha+\beta}w^{\beta}(w^{2\alpha}+1)}\big[\frac{1}{v^{\alpha}} - \frac{v^{\alpha}}{w^{2\eta\alpha}}\big],$$

which simplifies to

$$U(w,v) = \frac{1}{v^{\alpha+\beta}}\cdot\frac{w^{2\alpha-(\alpha(\eta+2)+\beta)}}{w^{2\alpha}+1} + \frac{1}{v^{2\alpha+\beta}}\cdot\frac{w^{2\alpha-(\alpha(\eta+1)+\beta)}}{w^{2\alpha}+1} - \frac{1}{v^{\alpha+\beta}}\cdot\frac{w^{2\alpha-(\alpha(2\eta+1)+\beta)}}{w^{2\alpha}+1}.$$

As a result, the exact solution,

$$u(x,t) = x^{\eta+1}E_{2,\eta+2}(-x^2) + tx^{\eta}E_{2,\eta+1}(-x^2) - x^{2\eta}E_{2,2\eta+1}(-x^2)$$

can be reached by using the inverse double Sadik transform. Note that if $\eta = 1$ *this solution is reduced to that of the classical wave equation* $u(x,t) = t\sin x$*.*

Example 4.4 (**Fractional Klein-Gordon equation**)**.** *If* $A_1 = 1, a_k = 0, B_j = 0, b_0 = 1, b_1 = 0, b_2 = 1$ *then the linear homogeneous time-fractional Klein-Gordon equation is obtained [29]*

$$\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}} - \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) = 0, \quad x,t \ge 0,\ 1 < \gamma \le 2.$$

The initial and boundary conditions are

$$u(x,0) = \sin x + 1,\ u_t(x,0) = 0,\ u(0,t) = E_{\gamma}(t^{\gamma}),\ u_x(0,t) = 1$$

and the exact solution of this problem is $u(x,t) = \sin x + E_{\gamma}(t^{\gamma})$*.*

Note that $f_0(x) = \sin x + 1$*,* $f_1(x) = 0$*,* $h_0(t) = E_{\gamma}(t^{\gamma})$*,* $h_1(t) = 1$ *and the single Sadik transform of these functions are*

$$F_0(w) = \frac{1}{w^{\beta}(w^{2\alpha}+1)} + \frac{1}{w^{\alpha+\beta}},\ F_1(w) = 0,\ H_0(v) = \frac{v^{(\gamma-1)\alpha-\beta}}{v^{\gamma\alpha}-1} \text{ and } H_1(v) = \frac{1}{v^{\alpha+\beta}}.$$

After applying the double Sadik transform and utilizing the given conditions, the resulting equation is

$$v^{\gamma\alpha}U(w,v) - v^{(\gamma-1)\alpha-\beta}\Big[\frac{1}{w^{\beta}(w^{2\alpha}+1)} + \frac{1}{w^{\alpha+\beta}}\Big] - w^{2\alpha}U(w,v)$$
$$+w^{\alpha-\beta}\cdot\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-1} + w^{-\beta}\cdot\frac{1}{v^{\alpha+\beta}} - U(w,v) = 0,$$

which can be rearranged to obtain

$$(v^{\gamma\alpha} - w^{2\alpha} - 1)U(w,v) = \big[\frac{v^{\gamma\alpha-\alpha-\beta}}{w^{\beta}(w^{2\alpha}+1)} - \frac{1}{w^{\beta}v^{\alpha+\beta}}\big] + v^{\gamma\alpha-\alpha-\beta}\big[\frac{1}{w^{\alpha+\beta}} - \frac{w^{\alpha-\beta}}{v^{\gamma\alpha}-1}\big].$$

Upon simplifying, it is found that

$$U(w,v)=\frac{1}{v^{\alpha+\beta}}\cdot\frac{1}{w^{\beta}(w^{2\alpha}+1)}+\frac{1}{w^{\alpha+\beta}}\cdot\frac{v^{\gamma\alpha-(\alpha+\beta)}}{v^{\gamma\alpha}-1}.$$

Subsequently, the inverse double Sadik transform is employed to determine the exact solution,

$$u(x,t)=\sin x+E_{\gamma}(t^{\gamma}).$$

Example 4.5 (**Fractional inhomogeneous Klein-Gordon equation**). *Consider* $N=1, M=1, A_1=1, B_1=1, a_k=0, b_0=-1, b_j=0$, *the linear inhomogeneous space-time fractional Klein-Gordon equation is provided [29]*

$$\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}}-\frac{\partial^{\eta}u(x,t)}{\partial x^{\eta}}+u(x,t)=6x^3\frac{t^{3-\gamma}}{\Gamma(4-\gamma)}+(x^3-6\frac{x^{3-\eta}}{\Gamma(4-\eta)})t^3,\quad x,t\geq 0,\ 1<\gamma,\eta\leq 2.$$

The initial and boundary the conditions are

$$u(x,0)=u_t(x,0)=u(0,t)=u_x(0,t)=0$$

and the exact solution of this problem is $u(x,t)=x^3t^3$.

In this context, $f_0(x)=f_1(x)=h_0(t)=h_1(t)=0$ *and their single Sadik transforms are*

$$F_0(w)=F_1(w)=H_0(v)=H_1(v)=0.$$

By applying the double Sadik transform and utilizing the relevant conditions, it is found that

$$(v^{\gamma\alpha}-w^{\eta\alpha}+1)U(w,v)=\frac{6}{w^{4\alpha+\beta}}\cdot\frac{3!}{v^{(4-\gamma)\alpha+\beta}}+\frac{3!}{w^{4\alpha+\beta}}\cdot\frac{3!}{v^{4\alpha+\beta}}-6\frac{3!}{v^{4\alpha+\beta}}\cdot\frac{1}{w^{(4-\eta)\alpha+\beta}},$$

which can be further simplified to

$$U(w,v)=\frac{3!}{w^{4\alpha+\beta}}\cdot\frac{3!}{v^{4\alpha+\beta}}.$$

Consequently, the exact solution,

$$u(x,t)=x^3t^3,$$

can be derived by taking the inverse double Sadik transform.

Example 4.6 (**Fractional inhomogeneous Burger's equation**). *For* $N=1, A_1=1, a_k=0, B_k=0, b_0=-1, b_1=0, b_2=1$, *the linear inhomogeneous time-fractional Burger's equation is obtained*

$$\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}}-\frac{\partial^{2}u(x,t)}{\partial x^{2}}+\frac{\partial u(x,t)}{\partial x}=\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)}+2x-2,\quad x,t\geq 0,\ 0<\gamma\leq 1.$$

The initial and boundary conditions are

$$u(x,0)=x^2,\ u(0,t)=t^2,\ u_x(0,t)=0$$

and the exact solution of this problem is $u(x,t)=x^2+t^2$.

It should be noted that the functions $f_0(x) = x^2$, $h_0(t) = t^2$, and $h_1(t) = 0$ have respective single Sadik transforms

$$F_0(w) = \frac{2!}{w^{3\alpha+\beta}},\ H_0(v) = \frac{2!}{v^{3\alpha+\beta}},\ \textit{and}\ H_1(v) = 0.$$

Upon applying the double Sadik transform and utilizing the conditions, the resulting equation

$$\begin{aligned} &v^{\gamma\alpha}U(w,v) - v^{(\gamma-1)\alpha-\beta}\cdot\frac{2!}{w^{3\alpha+\beta}} - w^{2\alpha}U(w,v) + w^{\alpha-\beta}\cdot\frac{2!}{v^{3\alpha+\beta}} + w^{\alpha}U(w,v) - w^{-\beta}\cdot\frac{2!}{v^{3\alpha+\beta}} \\ &= \frac{2}{v^{(3-\gamma)\alpha+\beta}}\cdot\frac{1}{w^{\alpha+\beta}} + 2\frac{1}{w^{2\alpha+\beta}}\cdot\frac{1}{v^{\alpha+\beta}} - \frac{2}{v^{\alpha+\beta}}\cdot\frac{1}{w^{\alpha+\beta}} \end{aligned}$$

can be expressed as

$$(v^{\gamma\alpha} - w^{2\alpha} + w^{\alpha})U(w,v) = \frac{2}{w^{3\alpha+\beta}}\frac{(v^{\gamma\alpha} + w^{\alpha} - w^{2\alpha})}{v^{\alpha+\beta}} + \frac{2}{v^{3\alpha+\beta}}\left(\frac{v^{\gamma\alpha}}{w^{\alpha+\beta}} + \frac{1-w^{\alpha}}{w^{\beta}}\right).$$

Further simplification yields

$$U(w,v) = \frac{1}{w^{\alpha+\beta}}\cdot\frac{2!}{v^{3\alpha+\beta}} + \frac{1}{v^{\alpha+\beta}}\cdot\frac{2!}{w^{3\alpha+\beta}}.$$

Thus, the inverse double Sadik transform can be employed to obtain the exact solution

$$u(x,t) = t^2 + x^2.$$

This solution is consistent with the findings reported in reference [2].

Example 4.7 (**Fractional homogeneous Burger's equation**). *If $A_1 = 1, a_k = 0, B_k = 0, b_0 = 0, b_1 = -1, b_2 = 1$, the linear homogeneous time-fractional Burger's equation is found*

$$\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}} - \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} = 0,\ 0 < \gamma \le 1.$$

The initial and boundary conditions are

$$u(x,0) = e^{-x},\ u(0,t) = E_{\gamma}(2t^{\gamma}),\ u_x(0,t) = -E_{\gamma}(2t^{\gamma})$$

and the exact solution of this problem is $u(x,t) = e^{-x}E_{\gamma}(2t^{\gamma})$.

Note that $f_0(x) = e^{-x}$, $h_0(t) = E_{\gamma}(2t^{\gamma})$, $h_1(t) = -E_{\gamma}(2t^{\gamma})$ and the single Sadik transform of these functions are

$$F_0(w) = \frac{1}{w^{\beta}(w^{\alpha}+1)},\ H_0(v) = \frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-2},\ H_1(v) = -H_0(v) = -\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-2}.$$

After taking the double Sadik transform and using the given conditions, the resulting equation is

$$\begin{aligned} v^{\gamma\alpha}U(w,v) - v^{(\gamma-1)\alpha-\beta}\cdot\frac{1}{w^{\beta}(w^{\alpha}+1)} - w^{2\alpha}U(w,v) + w^{\alpha-\beta}\cdot\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-2} \\ +w^{-\beta}\cdot\frac{-v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-2} + w^{\alpha}U(w,v) - w^{-\beta}\cdot\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-2} = 0 \end{aligned}$$

which is effortlessly reduced to

$$(v^{\gamma\alpha} - w^{2\alpha} + w^{\alpha})U(w,v) = \frac{1}{v^{\alpha+\beta}w^{\beta}}\left(\frac{v^{\gamma\alpha}}{w^{\alpha}+1} - \frac{w^{\alpha}v^{\alpha}}{v^{\gamma\alpha}-2} + \frac{2v^{\alpha}}{v^{\gamma\alpha}-2}\right).$$

Simplifying the equation, it is found that

$$U(w,v)=\frac{1}{w^{\beta}(w^{\alpha}+1)}\cdot\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-2}.$$

Hence, the exact solution is obtained by applying the inverse double Sadik transform, which gives

$$u(x,t)=e^{-x}E_{\gamma}(2t^{\gamma}).$$

Example 4.8 (**Fractional Fokker-Planck equation**). *Consider* $A_1=1, a_k=0, B_j=0, b_0=0, b_1=-1, b_2=1$, *the linear homogeneous time-fractional Fokker-Planck equation is given [29]*

$$\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}}-\frac{\partial^{2}u(x,t)}{\partial x^{2}}+\frac{\partial u(x,t)}{\partial x}=0,\ \ x,t\geq 0,\ \ 0<\gamma\leq 1.$$

The initial and boundary conditions

$$u(x,0)=x,\ u(0,t)=\frac{t^{\gamma}}{\Gamma(\gamma+1)},\ u_x(0,t)=1$$

and the exact solution of this problem is $u(x,t)=x+\frac{t^{\gamma}}{\Gamma(\gamma+1)}$.

Note that $f_0(x)=x,\ h_0(t)=\frac{t^{\gamma}}{\Gamma(\gamma+1)},\ h_1(t)=1$ *and thier single Sadik transform are*

$$F_0(w)=\frac{1}{w^{2\alpha+\beta}},\ H_0(v)=\frac{1}{v^{(\gamma+1)\alpha+\beta}},\ H_1(v)=\frac{1}{v^{\alpha+\beta}}.$$

Applying the double Sadik transform and utilizing the corresponding conditions, leads to

$$\begin{aligned}v^{\gamma\alpha}U(w,v)-v^{(\gamma-1)\alpha-\beta}\cdot\frac{1}{w^{2\alpha+\beta}}&=w^{2\alpha}U(w,v)-w^{\alpha-\beta}\cdot\frac{1}{v^{\gamma\alpha+\alpha+\beta}}\\&-w^{-\beta}\cdot\frac{1}{v^{\alpha+\beta}}+w^{\alpha}U(w,v)-w^{-\beta}\cdot\frac{1}{v^{\gamma\alpha+\alpha+\beta}}.\end{aligned}$$

This equation is then arranged to obtain

$$\begin{aligned}&(v^{\gamma\alpha}-w^{2\alpha}-w^{\alpha})U(w,v)=(\frac{v^{(\gamma-1)\alpha-\beta}}{w^{2\alpha+\beta}}-\frac{w^{-\beta}}{v^{\alpha+\beta}})-\frac{w^{-\beta}}{v^{\gamma\alpha+\alpha+\beta}}-\frac{w^{\alpha-\beta}}{v^{\gamma\alpha+\alpha+\beta}}\\&=(\frac{v^{(\gamma-1)\alpha-\beta}}{w^{2\alpha+\beta}}-\frac{w^{2\alpha}w^{-\beta}}{w^{2\alpha}v^{\alpha+\beta}}-\frac{w^{\alpha}}{w^{2\alpha+\beta}v^{\alpha+\beta}})+(\frac{v^{\gamma\alpha}}{w^{\alpha+\beta}v^{\gamma\alpha+\alpha+\beta}}-\frac{w^{2\alpha}}{w^{2\alpha+\beta}v^{\gamma\alpha+\alpha+\beta}}-\frac{w^{2\alpha}}{w^{\alpha+\beta}v^{\gamma\alpha+\alpha+\beta}}).\end{aligned}$$

Upon simplification, it becomes evident that

$$U(w,v)=\frac{1}{v^{\alpha+\beta}}\cdot\frac{1}{(w^{2\alpha+\beta})}+\frac{1}{w^{\alpha+\beta}}\cdot\frac{1}{v^{\gamma\alpha+\alpha+\beta}}.$$

Hence, by utilizing the inverse double Sadik transform, the exact solution

$$u(x,t)=x+\frac{t^{\gamma}}{\Gamma(\gamma+1)}$$

can be obtained.

Example 4.9 (**Homogeneous Fractional KdV equation**). *For* $N=1, A_1=1, a_k=0, B_j=0, b_0=0, b_1=-2, b_2=0, b_3=-1$, *the linear homogeneous time-fractional KdV equation is found*

$$\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}}+2\frac{\partial u(x,t)}{\partial x}+\frac{\partial^{3}u(x,t)}{\partial x^{3}}=0,\ \ x,t\geq 0,\ \ 0<\gamma\leq 1.$$

The initial and boundary conditions are

$$u(x,0)=\sin x,\ u(0,t)=-t^{\gamma}E_{2\gamma,\gamma+1}(-t^{2\gamma}),\ u_x(0,t)=E_{2\gamma,1}(-t^{2\gamma}),\ u_{xx}(0,t)=t^{\gamma}E_{2\gamma,\gamma+1}(-t^{2\gamma})$$

and the exact solution of this problem is $u(x,t)=\sin x\cdot E_{2\gamma,1}(-t^{2\gamma})-\cos x\cdot t^{\gamma}E_{2\gamma,\gamma+1}(-t^{2\gamma})$.

Note that $f_0(x)=\sin x,\ h_0(t)=-t^{\gamma}E_{2\gamma,\gamma+1}(-t^{2\gamma}),\ h_1(t)=E_{2\gamma,1}(-t^{2\gamma}),h_2(t)=t^{\gamma}E_{2\gamma,\gamma+1}(-t^{2\gamma})$ *and the single Sadik transform of these functions are*

$$F_0(w)=\frac{1}{w^{\beta}(w^{2\alpha}+1)},\ H_0(v)=-\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{2\gamma\alpha}+1},\ H_1(v)=\frac{v^{2\gamma\alpha-\alpha-\beta}}{v^{2\gamma\alpha}+1},\ H_2(v)=\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{2\gamma\alpha}+1}.$$

By applying the double Sadik transform and using the given conditions, the equation

$$\begin{aligned}&v^{\gamma\alpha}U(w,v)-v^{(\gamma-1)\alpha-\beta}\cdot\frac{1}{w^{\beta}(w^{2\alpha}+1)}+2w^{\alpha}U(w,v)+2w^{-\beta}\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{2\gamma\alpha}+1}\\&+w^{3\alpha}U(w,v)+w^{2\alpha-\beta}\cdot\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{2\gamma\alpha}+1}-w^{\alpha-\beta}\cdot\frac{v^{2\gamma\alpha-\alpha-\beta}}{v^{2\gamma\alpha}+1}-w^{-\beta}\cdot\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{2\gamma\alpha}+1}=0\end{aligned}$$

is obtained. Rearranging the equation yields

$$(v^{\gamma\alpha}+2w^{\alpha}+w^{3\alpha})U(w,v)=\frac{v^{(\gamma-1)\alpha-\beta}}{w^{\beta}(w^{2\alpha}+1)}+\frac{v^{\gamma\alpha-\alpha-\beta}}{w^{\beta}(v^{2\gamma\alpha}+1)}\Big(w^{\alpha}v^{\gamma\alpha}-w^{2\alpha}-1\Big),$$

which simplifies to

$$U(w,v)=\frac{1}{w^{\beta}(w^{2\alpha}+1)}\cdot\frac{v^{2\gamma\alpha-\alpha-\beta}}{v^{2\gamma\alpha}+1}-\frac{w^{\alpha}}{w^{\beta}(w^{2\alpha}+1)}\cdot\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{2\gamma\alpha}+1}.$$

Hence, the exact solution

$$u(x,t)=\sin x\cdot E_{2\gamma,1}(-t^{2\gamma})-\cos x\cdot t^{\gamma}E_{2\gamma,\gamma+1}(-t^{2\gamma})$$

can be reached by using the inverse double Sadik transform. The solution is analogous to the solution obtained in reference [3].

Example 4.10 (**Fractional KdV equation**). *If* $N=1,A_1=1,a_k=0,B_j=0,b_0=0,b_1=-1,b_2=0,b_3=-1$ *then the linear inhomogeneous time-fractional KdV equation is obtained [29]*

$$\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}}+\frac{\partial u(x,t)}{\partial x}+\frac{\partial^{3}u(x,t)}{\partial x^{3}}=2\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\cdot\cos x,\ \ x,t\geq 0,\ \ 0<\gamma\leq 1.$$

The initial and boundary conditions are

$$u(x,0)=0,\ u(0,t)=t^2,\ u_x(0,t)=0,\ u_{xx}(0,t)=-t^2$$

and the exact solution of this problem is $u(x,t)=t^2\cos x$.

Note that $f_0(x)=0,\ h_0(t)=t^2,\ h_1(t)=0,h_2(t)=-t^2$ *and the single Sadik transform of these functions are*

$$F_0(w)=H_1(v)=0,\ H_0(v)=\frac{2!}{v^{3\alpha+\beta}},H_2(v)=-\frac{2!}{v^{3\alpha+\beta}}.$$

After applying the double Sadik transform and utilizing the given conditions, the resulting equation is

$$v^{\gamma\alpha}U(w,v)+w^{\alpha}U(w,v)-w^{-\beta}\frac{2!}{v^{3\alpha+\beta}}+w^{3\alpha}U(w,v)-w^{2\alpha-\beta}\frac{2!}{v^{3\alpha+\beta}}+w^{-\beta}\frac{2!}{v^{3\alpha+\beta}}$$
$$=\frac{2}{v^{(3-\gamma)\alpha+\beta}}\cdot\frac{w^{\alpha}}{w^{\beta}(w^{2\alpha}+1)},$$

which can be rearranged to obtain

$$(v^{\gamma\alpha}+w^{\alpha}+w^{3\alpha})U(w,v)=\frac{2!}{w^{\beta}v^{3\alpha+\beta}}\big(w^{2\alpha}+\frac{w^{\alpha}v^{\gamma\alpha}}{w^{2\alpha}+1}\big).$$

Simplifying the equation, it is found that

$$U(w,v)=\frac{2}{v^{3\alpha+\beta}}\cdot\frac{w^{\alpha}}{w^{\beta}(w^{2\alpha}+1)}.$$

Subsequently, the inverse double Sadik transform can be relied on to determine the exact solution,

$$u(x,t)=t^{2}\cos x.$$

Example 4.11 (Fractional KdV-Burger's equation). *[29] Consider $N=1, A_1=1, B_j=0, a_k=0, b_0=0, b_1=-1, b_2=1, b_3=-1$, the linear inhomogeneous time-fractional KdV-Burger's equation is found*

$$\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}}+\frac{\partial u(x,t)}{\partial x}-\frac{\partial^{2}u(x,t)}{\partial x^{2}}+\frac{\partial^{3}u(x,t)}{\partial x^{3}}=e^{-x},\ \ x,t\geq 0,\ \ 0<\gamma\leq 1.$$

The initial and boundary conditions are

$$u(x,0)=e^{-x},\ u(0,t)=u_{xx}(0,t)=\frac{1}{3}[4E_{\gamma}(3t^{\gamma})-1],\ u_{x}(0,t)=-\frac{1}{3}[4E_{\gamma}(3t^{\gamma})-1]$$

and the exact solution of this problem is $u(x,t)=\frac{1}{3}e^{-x}[4E_{\gamma}(3t^{\gamma})-1]$.

Note that $f_0(x)=e^{-x}$, $h_0(t)=h_2(t)=\frac{1}{3}[4E_{\gamma}(3t^{\gamma})-1]$, $h_1(t)=-\frac{1}{3}[4E_{\gamma}(3t^{\gamma})-1]$ and the single Sadik transform of these functions are

$$F_0(w)=\frac{1}{w^{\beta}(w^{\alpha}+1)},\ H_0(v)=H_2(v)=\frac{1}{3}\big[4\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-3}-\frac{1}{v^{\alpha+\beta}}\big],\ H_1(v)=-H_0(v).$$

By applying the double Sadik transform and using the relavant conditions, it is found that the resulting equation

$$v^{\gamma\alpha}U(w,v)-v^{\gamma\alpha-\alpha-\beta}\frac{1}{w^{\beta}(w^{\alpha}+1)}+w^{\alpha}U(w,v)-w^{-\beta}\cdot\frac{1}{3}\big[4\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-3}-\frac{1}{v^{\alpha+\beta}}\big]$$
$$-w^{2\alpha}U(w,v)+w^{\alpha-\beta}\cdot\frac{1}{3}\big[4\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-3}-\frac{1}{v^{\alpha+\beta}}\big]-w^{-\beta}\cdot\frac{1}{3}\big[4\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-3}-\frac{1}{v^{\alpha+\beta}}\big]$$
$$+w^{3\alpha}U(w,v)-w^{2\alpha-\beta}\cdot\frac{1}{3}\big[4\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-3}-\frac{1}{v^{\alpha+\beta}}\big]+w^{\alpha-\beta}\cdot\frac{1}{3}\big[4\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-3}-\frac{1}{v^{\alpha+\beta}}\big]$$
$$-w^{-\beta}\cdot\frac{1}{3}\big[4\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-3}-\frac{1}{v^{\alpha+\beta}}\big]=\frac{1}{v^{\alpha+\beta}}\cdot\frac{1}{w^{\beta}(w^{\alpha}+1)},$$

can be expressed as

$$(v^{\gamma\alpha} + w^{\alpha} - w^{2\alpha} + w^{3\alpha})U(w, v) = \frac{v^{\gamma\alpha-\alpha-\beta}}{w^{\beta}(w^{\alpha}+1)} + \frac{1}{v^{\alpha+\beta}} \cdot \frac{1}{w^{\beta}(w^{\alpha}+1)} + \frac{1}{3}\Big[4\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-3} - \frac{1}{v^{\alpha+\beta}}\Big] \cdot \frac{(3-2w^{\alpha}+w^{2\alpha})}{w^{\beta}}.$$

Further simplification yields

$$U(w, v) = \frac{1}{w^{\beta}(w^{\alpha}+1)} \cdot \frac{v^{\gamma\alpha}+1}{v^{\alpha+\beta}(v^{\gamma\alpha}-3)} = \frac{1}{w^{\beta}(w^{\alpha}+1)} \cdot \frac{1}{3}\Big[4\frac{v^{\gamma\alpha-\alpha-\beta}}{v^{\gamma\alpha}-3} - \frac{1}{v^{\alpha+\beta}}\Big].$$

Hence, the inverse double Sadik transform can be deployed to obtain the exact solution

$$u(x, t) = \frac{1}{3}e^{-x}[4E_{\gamma}(3t^{\gamma}) - 1].$$

5. Discussion and Conclusions

The double Sadik transform is an efficient analytical tool for solving fractional models in science and engineering. We have successfully demonstrated the significant properties of this type of double integral transform and applied it to different types of linear Caputo fractional partial differential equations, including the fractional diffusion equation, the fractional wave equation, the fractional Newell-Whitehead-Segel equation, the fractional KdV equation, the fractional Klein-Gordon equation, the fractional Fokker-Planck equation, and the fractional Burger equation. Furthermore, this technique is applicable to both homogeneous and inhomogeneous problems. The outcomes achieved are presented in a closed form, which corresponds to conventional methods involving the double Laplace transform [20], [29], double Elzaki transform [21], double Kamal transform [23], and so on. The proposed improvements can be regarded as a generalization of other double integral transform methods. For these reasons, this method is a viable and practical way to tackle fractional scientific problems. Nevertheless, the efficacy of this approach is limited to a specific class of linear fractional partial differential equations. To address nonlinear problems, an development has been required. In future research, we intend to enhance this method with the goal of making it more effective in solving nonlinear fractional scientific problems.

Acknowledgments: This research project was financially supported by Thailand Science Research and Innovation (TSRI). The author is grateful to the reviewers for their insightful comments and suggestions.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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