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Normal Surfaces along a Curve on a Surface in Euclidean 3-Space

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Abstract. Curves on surfaces and their frames play an important role in differential geometry and in many branches of science such as mechanics and physics. So, we are interested in studying one of these surfaces along a curve lying on a surface. In this paper, we define a surface normal to a surface along a curve lying on a surface in Euclidean 3-space $\mathbb{E}^3$. Then, we analyze the necessary and sufficient conditions for that surface to be a ruled surface. Finally, we illustrate the convenience and efficiency of this approach with some representative examples.

1. Introduction

The problem of finding surfaces with a given common curve as a special curve play an important role in geometric design. The first paper related with this type of the problem proposed by Wang et.al., [1]. They parameterized the surface by using the Serret–Frenet frame of the given curve and gave the necessary and sufficient condition to satisfy the geodesic requirement. The basic idea is to regard the wanted surface as an extension from the given characteristic curve, and represent it as a linear combination of the marching-scale functions: $u(s,t)$, $v(s,t)$, $w(s,t)$ and the three vector functions $\mathbf{t}(s)$, $\mathbf{n}(s)$, $\mathbf{b}(s)$, which are the unit tangent, principal normal and binormal vectors of the curve, respectively. With the given geodesic curve and isoparametric constraints, they derived the necessary and sufficient conditions for the correct parametric representation of the surface pencil. The extension to ruled and developable surfaces is also outlined. Kasap et al. [2] generalized the marching-scale functions of Wang and gave a sufficient condition for a given curve to be a geodesic on a surface. With the inspiration of the work of Wang, Li et.al. [3], they changed the characteristic curve from geodesic to a line of curvature and defined the surface pencil with a common line of

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curvature. Bayram et.al. [4] tackled the problem of constructing surfaces passing through a given asymptotic curve. Important contributions to surface passing through a given curve have been studied in [5–8].

However, the relevant work on surfaces through characteristic curve on a surface depending on the Darboux frame is rare. So, this led us to offer an approach for designing a surface possessing a given curve on a surface, we call it a normal surface along the curve. Then, we analyze the necessary and sufficient condition for that surface to be a normal ruled surface. Moreover, some examples are illustrated to explain the applications of the theoretical results.

2. Preliminaries

In this section, we list some notions, formulas and conclusions for space curves, and ruled surfaces in Euclidean 3-space $\mathbb{E}^3$ (see for instance, [9, 10]).

Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve; by $\kappa(s)$ and $\tau(s)$ we denote the natural curvature and torsion of $\alpha = \alpha(s)$, respectively. We assume $\alpha^{''}(s) \neq 0$ for all $s \in [0, L]$, since this would give us a straight line. In this paper, $\alpha^{'}(s)$ denotes the derivative of α with respect to arc length parameter s. For each point of $\alpha(s)$, the set $\{t(s),\ n(s),\ b(s)\}$ is called the Serret–Frenet frame along $\alpha(s)$, where $t(s) = \alpha^{'}(s)$ is the unit tangent, $n(s) = \alpha^{''}(s)/\left\|\alpha^{''}(s)\right\|$ is the unit principal normal, and $b(s) = t(s) \times n(s)$ is the unit binormal vector. The arc-length derivative of the Serret–Frenet frame is governed by the relations:

$$\begin{pmatrix} t^{'}(s) \\ n^{'}(s) \\ b^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}, \tag{2.1}$$

Let M be a regular surface, and $\alpha : I \subseteq \mathbb{R} \to M$ be a unit speed curve on M. If we denote the Darboux frame along the curve $\alpha = \alpha(s)$ by $\{e_1(s),\ e_2(s),\ e_3(s)\}$; $t = e_1(s)$ be the unit tangent vector, $e_3 = e_3(s)$ is the surface unit normal restricted to α, and $e_2 = e_3 \times e_1$ be the unit tangent to the surface M. Then, the rotation matrix between Serret–Frenet frame and Darboux frame is

$$\begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.2}$$

Hence, we have the derivative formulae of the Darboux frame as follows

$$\begin{pmatrix} e_1^{'} \\ e_2^{'} \\ e_3^{'} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{2.3}$$

where

$$\left.\begin{aligned} \kappa_n &= \kappa \sin\vartheta =< e_1^{'}, e_3>, \\ \kappa_g &= \kappa \cos\vartheta = \det\left(\alpha^{'}, \alpha^{''}, e_2\right), \\ \tau_g &= \tau + \vartheta^{'} = \det\left(\alpha^{'}, e_2, e_2^{'}\right), \end{aligned}\right\} \tag{2.4}$$

we call $\kappa_g = \kappa_g(s)$ a geodesic curvature, $\kappa_n = \kappa_n(s)$ a normal curvature, and $\tau_g = \tau - \vartheta^{'}$ a geodesic torsion of $\alpha(s)$. In terms of these quantities, the geodesics, asymptotic lines, and line of curvatures on a smooth surface may be characterized as loci on a surface which $\kappa_g = 0$, $\kappa_n = 0$, and $\tau_g = 0$, respectively. Further, we have

$$\left.\begin{aligned} \kappa(s) &= \sqrt{\kappa_g^2 + \kappa_n^2}, \\ \tau_g(s) &= \vartheta^{'} + \tau. \end{aligned}\right\} \tag{2.5}$$

3. Normal surface family

In this section, we consider a surface normal to the surface M along a regular curve $\alpha = \alpha(s)$, such that the surface tangent plane is coincident with the subspace $Sp\{e_1, e_3\}$, that is expressing the surface along $\alpha(s)$ as follows:

$$M_n : P(s, t) = \alpha(s) + u(s, t)e_1(s) + v(s, t)e_3(s); \quad 0 \leq t \leq T, \quad 0 \leq s \leq L, \tag{3.1}$$

where $u(s, t)$, and $v(s, t)$ are C^1 functions. If the parameter t is seen as the time, the functions $u(s, t)$, and $v(s, t)$ can then be viewed as directed marching distances of a point unit in the time t in the direction e_1 and e_3, respectively, and the position vector $\alpha(s)$ is seen as the initial location of this point on M. It is easily checked that the two tangent vectors of M_n are given by

$$\left.\begin{aligned} P_s(s, t) &= (1 + u_s - v\kappa_n)e_1 + (u\kappa_g - v\tau_g)e_2 + (v_s + u\kappa_n)e_3, \\ P_t(s, t) &= u_t e_1 + v_t e_3. \end{aligned}\right\} \tag{3.2}$$

The lowercase subscript letters s, and t denote partial derivatives corresponding to the indicated variable, e.g., $P_s = \frac{\partial P}{\partial s}$, $P_t = \frac{\partial P}{\partial t}$. Thus, the normal vector of M_n is given by

$$N(s, t) := P_s \times P_t = \eta_1(s, t)e_1 + \eta_2(s, t)e_2 + \eta_3(s, t)e_3, \tag{3.3}$$

where

$$\left.\begin{aligned} \eta_1(s, t) &= v_t(u\kappa_g - v\tau_g), \\ \eta_2(s, t) &= u_t(v_s + u\kappa_n) - v_t(1 + u_s - u\kappa_g - v\kappa_n), \\ \eta_3(s, t) &= -u_t(u\kappa_g - v\tau_g). \end{aligned}\right\} \tag{3.4}$$

Our goal is to find the necessary and sufficient conditions for which the surface M_n is normal to the surface M along $\alpha(s)$. First, since $\alpha(s)$ is an isoparametric curve on the surface M_n, there exists a parameter $t_0 \in [0, T]$ such that $P(s, t_0) = \alpha(s)$, $0 \leq t_0 \leq T$, $0 \leq s \leq L$, that is,

$$\left.\begin{aligned} u(s, t_0) &= v(s, t_0) = 0, \\ u_s(s, t_0) &= v_s(s, t_0) = 0. \end{aligned}\right\} \tag{3.5}$$

Secondly, when $t = t_0$, i.e., along the curve $\alpha(s)$ of M, the surface normal is

$$N(s, t_0) = -v_t(s, t_0)e_2, \tag{3.6}$$

this is the reason why we call M_n the normal surface of M along the curve $\alpha(s)$. Therefore, we have the following theorem.

Theorem 3.1. *The surface M_n is a normal surface along the curve $\alpha(s)$ of M if and only if*

$$\left.\begin{aligned} &u(s, t_0) = v(s, t_0) = 0, \\ &u_s(s, t_0) = v_s(s, t_0) = 0, \\ &v_t(s, t_0) \neq 0, \quad 0 \leq t_0 \leq T, \quad 0 \leq s \leq L. \end{aligned}\right\} \tag{3.7}$$

Proof. First, since $\alpha(s)$ is an isoparametric curve on the surface M_n, there exists a parameter $t_0 \in [0, T]$ such that $P(s, t_0) = \alpha(s)$, $0 \leq t_0 \leq T$, $0 \leq s \leq L$, that is

$$\left.\begin{aligned} &u(s, t_0) = v(s, t_0) = 0, \\ &u_s(s, t_0) = v_s(s, t_0) = 0. \end{aligned}\right\} \tag{3.8}$$

Secondly, when $t = t_0$, i.e., along the curve $\alpha(s)$ of M, the surface normal is

$$N(s, t_0) = -v_t(s, t_0)e_2. \tag{3.9}$$

Thus, the proof is completed. □

We will call the set of surfaces defined by Eqs. (3.1) and (3.7) an isoparametric normal surface family, since the common curve is an isoparametric curve on these surfaces. Any surface M_n defined by Eq. (3.1) and satisfying Eqs. (3.7) is a member of this family. For the purposes of simplification and better analysis, next we study the case when the marching-scale functions $u(s, t)$, and $v(s, t)$ can be decomposed into two factors:

$$\begin{aligned} u(s, t) &= l(s)U(t), \\ v(s, t) &= n(s)V(t). \end{aligned} \tag{3.10}$$

Here $l(s)$, $n(s), U(t)$ and $V(t)$ are C^1 functions and not identically zero. Thus, from Theorem 3.1, we can get the following corollary.

Corollary 3.1. The necessary and sufficient condition for M_n being a normal along the curve $\alpha(s)$ of M is:

$$\left.\begin{aligned} &U(t_0) = V(t_0) = 0, \ \ l(s) = const. \neq 0, \ \ n(s) = const. \neq 0, \\ &\tfrac{dV(t_0)}{dt} = const. \neq 0, \ \ 0 \leq t_0 \leq T, \quad 0 \leq s \leq L. \end{aligned}\right\} \tag{3.11}$$

Note that, to obtain a normal surface family, we can first design the marching-scale functions in Eq. (3.11), and then apply them to Eq. (3.1) to derive the final parametrization. For convenience in practice, the marching-scale functions can be further constrained to be in more restricted forms and still possess enough degrees of freedom to define a large class of normal surface family along the curve

$\alpha(s)$ of M. Specifically, let us suppose that $u(s,t)$, and $v(s,t)$ can be chosen in two different forms:
(1) If we choose

$$\begin{cases} u(s,t) = \sum_{k=1}^{p} a_{1k} l(s)^k U(t)^k, \\ v(s,t) = \sum_{k=1}^{p} a_{2k} m(s)^k V(t)^k. \end{cases} \tag{3.12}$$

Thus, we can simply express the sufficient condition for which M_n being a normal along the curve $\alpha(s)$ of M as follows:

$$\begin{cases} U(t_0) = V(t_0) = 0, \\ a_{21} \neq 0,\ m(s) \neq 0, \text{ and } \frac{dV(t_0)}{dt} \neq 0, \end{cases} \tag{3.13}$$

where $l(s)$, $m(s)$, $U(t)$, and $V(t)$ are C^1 functions, $a_{ij} \in \mathbb{R}$ ($i = 1,2$; $j = 1,2,...,p$) and $l(s)$, and $n(s)$ are not identically zero.
(2) If we choose

$$\begin{cases} u(s,t) = f(\sum_{k=1}^{p} a_{1k} l(s)^k U(t)^k), \\ v(s,t) = g(\sum_{k=1}^{p} a_{2k} n(s)^k V(t)^k), \end{cases} \tag{3.14}$$

then we can rewrite the condition (3.11) as:

$$\begin{cases} U(t_0) = V(t_0) = v(t_0) = f(0) = g(0) = 0, \\ a_{21} \neq 0,\ \frac{dV(t_0)}{dt} = const \neq 0,\ n(s) \neq 0,\ g'(0) \neq 0, \end{cases} \tag{3.15}$$

where $l(s)$, $n(s)$, $U(t)$, $V(t)$, f, and g are C^1 functions.

Example 3.1. We consider a surface of revolution parameterized by

$$M : X(s,t) = (s, e^s\ \sin t, e^s\ \cos t)\,.$$

The curve

$$\alpha(s) = \left(s, e^s \sin s^2, e^s \cos s^2\right),\ s \in \mathbb{R},$$

is a regular curve on the surface M. In this case the subspace $Sp\{e_1, e_3\}$ of $\alpha(s)$ is

$$e_1(s) = \frac{\dot{\alpha}(s)}{\|\dot{\alpha}(s)\|} = \left(\frac{1}{\sqrt{1+e^{2s}(1+4s^2)}}, \frac{e^s\left(2s\cos s^2 + \sin s^2\right)}{\sqrt{1+e^{2s}(1+4s^2)}}, \frac{e^s\left(\cos s^2 - 2s\sin s^2\right)}{\sqrt{1+e^{2s}(1+4s^2)}}\right),$$

and

$$\begin{aligned} e_3(s) &= \frac{X_s \times X_t}{\|X_s \times X_t\|} \\ &= \left(-\frac{e^{2s}}{\sqrt{e^{2s}+e^{4s}}}, \frac{e^s \sin s^2}{\sqrt{e^{2s}+e^{4s}}}, \frac{e^s \cos s^2}{\sqrt{e^{2s}+e^{4s}}}\right). \end{aligned}$$

Using Eq. (3.1), the normal surface family can be calculated as follows:

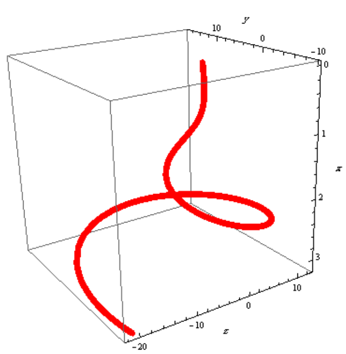
$$M_n \quad : \quad P(s,t) = \begin{pmatrix} s, \\ e^s \sin(s^2), \\ e^s \cos(s^2) \end{pmatrix}$$
$$+u(s,t)\begin{pmatrix} \frac{1}{\sqrt{1+e^{2s}(1+4s^2)}}, \\ \frac{e^s\left(2s\cos(s^2)+\sin(s^2)\right)}{\sqrt{1+e^{2s}(1+4s^2)}}, \\ \frac{e^s\left(\cos(s^2)-2s\sin(s^2)\right)}{\sqrt{1+e^{2s}(1+4s^2)}} \end{pmatrix} + v(s,t)\begin{pmatrix} -\frac{e^{2s}}{\sqrt{e^{2s}+e^{4s}}}, \\ \frac{e^s\sin(s^2)}{\sqrt{e^{2s}+e^{4s}}}, \\ \frac{e^s\cos(s^2)}{\sqrt{e^{2s}+e^{4s}}} \end{pmatrix}.$$

It is very clear that the functions $u(s,t)$, and $v(s,t)$ can control the shape of the surface, and if these functions are given, then we immediately obtain a normal surface in this family. So, we consider the following cases.

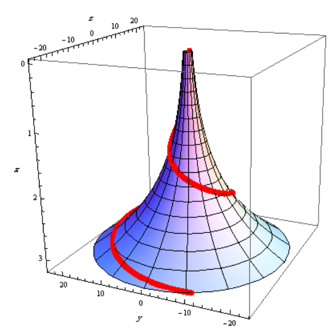
Case(3.1). We choose $u(s,t) = s\sin t$, and $v(s,t) = t\cos s$, and $t \in [0,T]$. Obviously, Eqs. (3.11) are satisfied, and the normal surface is given by

$$M_n : P(s,t) = \left\{ \begin{array}{c} s - \frac{te^{2s}\cos s}{\sqrt{e^{2s}+e^{4s}}} + \frac{s\sin t}{\sqrt{1+e^{2s}(1+4s^2)}}, \\ e^s\left(\sin(s^2) + \frac{t\cos s\sin(s^2)}{\sqrt{e^{2s}+e^{4s}}} + \frac{s\left(2s\cos(s^2)+\sin(s^2)\right)\sin t}{\sqrt{1+e^{2s}(1+4s^2)}}\right), \\ e^s\left(\cos(s^2) + \frac{t\cos s\cos(s^2)}{\sqrt{e^{2s}+e^{4s}}} + \frac{s\left(\cos(s^2)-2s\sin(s^2)\right)\sin t}{\sqrt{1+e^{2s}(1+4s^2)}}\right) \end{array} \right\}.$$

The surfaces M, M_n, and $M \cup M_n$ along to the curve α are shown in Figs. (1a, 1b), and Figs. (2a, 2b).

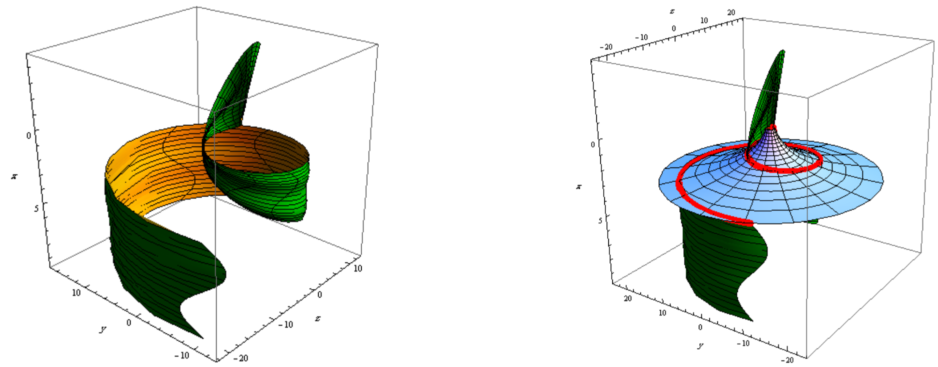


(a)



(b)

Figure 1. (*a*) The curve α. (*b*) The curve on the surface M.



(a) (b)

Figure 2. (a) The normal surface M_n. (b) The surface M_n normal to M along the curve α.

Case(3.2). If we choose $u(s,t) = (1+\sin(t)) + \sum_{k=2}^{4} a_{1k}(1+\sin(t))^k$, $v(s,t) = \cos(t) + \sum_{k=2}^{4} a_{2k}\cos^k(t)$, $t_0 = 0$, $t_0 = 3\pi/2$, a_{1k}, $a_{2k} \in \mathbb{R}$, and $t \in [0, 2\pi]$, then Eqs. (3.11) are satisfied.

Hence, the normal surface can be represented as follows:

$$M_n : P(s,t) = \{P_1(s,t), P_2(s,t), P_3(s,t)\},$$

where

$$P_1(s,t) = \left(\begin{array}{c} s - \frac{e^{2s}\left(1+\cos t+(1+\cos t)^2+2(1+\cos t)^3+3(1+\cos t)^4\right)}{\sqrt{e^{2s}+e^{4s}}} \\ + \frac{1+\sin t+(1+\sin t)^2+2(1+\sin t)^3+3(1+\sin t)^4}{\sqrt{1+e^{2s}(1+4s^2)}} \end{array} \right),$$

$$P_2(s,t) = e^s \left(\begin{array}{c} \sin\left(s^2\right) + \frac{\left(7+21\cos t+25\cos^2(t)+14\cos^3(t)+3\cos^4(t)\right)\sin\left(s^2\right)}{\sqrt{e^{2s}+e^{4s}}} + \\ \frac{\left(2s\cos\left(s^2\right)+\sin\left(s^2\right)\right)\left(7+21\sin t+25\sin^2(t)+14\sin^3(t)+3\sin^4(t)\right)}{\sqrt{1+e^{2s}(1+4s^2)}} \end{array} \right),$$

$$P_3(s,t) = e^s \left(\begin{array}{c} \cos\left(s^2\right) + \frac{\cos\left(s^2\right)\left(7+21\cos t+25\cos^2(t)+14\cos^3(t)+3\cos^4(t)\right)}{\sqrt{e^{2s}+e^{4s}}} + \\ \frac{\left(\cos\left(s^2\right)-2s\sin\left(s^2\right)\right)\left(7+21\sin t+25\sin^2(t)+14\sin^3(t)+3\sin^4(t)\right)}{\sqrt{1+e^{2s}(1+4s^2)}} \end{array} \right).$$

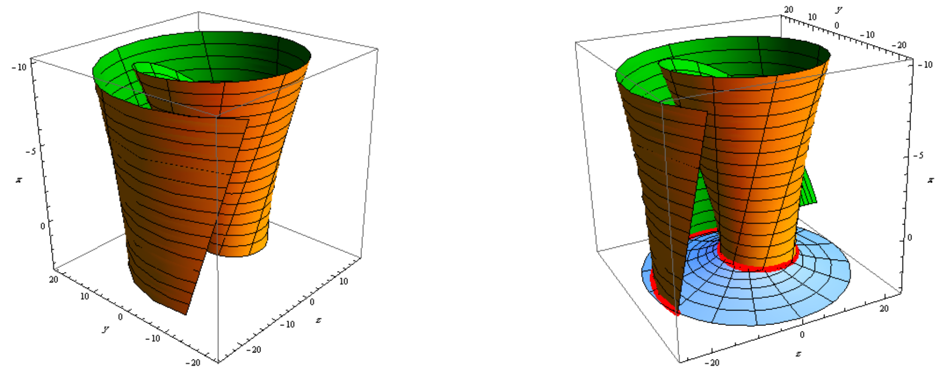
In this case, the surfaces M_n, and $M \cup M_n$ along the curve α are shown in Figs. (3a, 3b).

Example 3.2. Let M be a surface given by

$$M : X(s,t) = \left(\cos s - \frac{t}{\sqrt{2}}\cos s, \sin s - \frac{t}{\sqrt{2}}\sin s, \frac{s}{\sqrt{2}}\right),$$

and the curve is expressed as:

$$\alpha(s) = \left(\cos s - \frac{\cos s}{\sqrt{2}}, \text{sins} - \frac{\sin s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right).$$



(a) (b)

Figure 3. (a) The normal surface M_n. (b) The surface M_n normal to M along the curve α.

In this case, the subspace $Sp\{e_1, e_3\}$ of $\alpha(s)$ is given by

$$\begin{aligned} e_1(s) &= \left(-\frac{1}{2}\sqrt{2-\sqrt{2}}\text{sins}, \frac{1}{2}\sqrt{2-\sqrt{2}}\text{coss}, \frac{1}{\sqrt{4-2\sqrt{2}}}\right), \\ e_3(s) &= \left(\frac{\text{sins}}{\sqrt{4-2\sqrt{2}}}, -\frac{\text{coss}}{\sqrt{4-2\sqrt{2}}}, \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}}\right). \end{aligned}$$

Also, using Eq. (3.1), the normal surface family can be represented as:

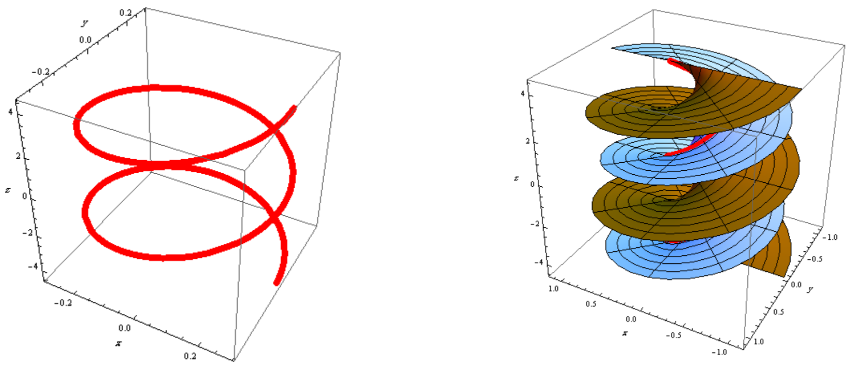
$$M_n \ : \ P(s,t) = \left(\text{coss} - \frac{\text{coss}}{\sqrt{2}}, \text{sins} - \frac{\text{sins}}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$
$$+u(s,t)\begin{pmatrix} -\frac{1}{2}\sqrt{2-\sqrt{2}}\text{sins}, \\ \frac{1}{2}\sqrt{2-\sqrt{2}}\text{coss}, \\ \frac{1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix} + v(s,t)\begin{pmatrix} \frac{\text{sins}}{\sqrt{4-2\sqrt{2}}}, \\ -\frac{\text{coss}}{\sqrt{4-2\sqrt{2}}}, \\ \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}.$$

It is very clear that the functions $u(s,t)$, and $v(s,t)$ can control the shape of the surface, and if these functions are given, then we immediately obtain a normal surface in this family. So, we consider the following cases.

Case(3.3). We choose $u(s,t) = e^s \sin t$, and $v(s,t) = \sin t \cos s$, and $t \in [0,T]$. Obviously, Eqs. (3.11) are satisfied, and the normal surface is given by

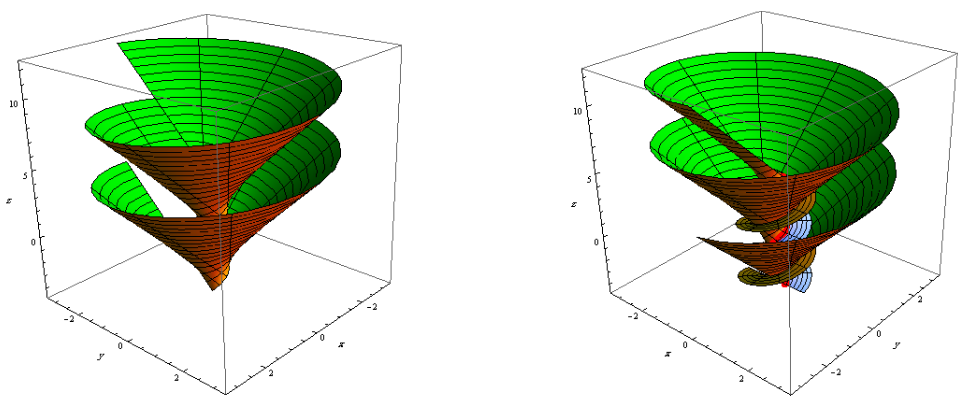
$$M_n : P(s,t) = \left\{ \begin{array}{c} -\frac{1}{2}\sqrt{2-\sqrt{2}}e^s \sin s \sin t + \cos s\left(1 - \frac{1}{\sqrt{2}} + \frac{\sin s \sin t}{\sqrt{4-2\sqrt{2}}}\right), \\ \left(1-\frac{1}{\sqrt{2}}\right)\sin s - \frac{\cos s\left(\left(-2+\sqrt{2}\right)e^s + \sqrt{2}\cos s\right)\sin t}{2\sqrt{2-\sqrt{2}}}, \\ \frac{\sqrt{2-\sqrt{2}}s + \left(e^s + \left(-1+\sqrt{2}\right)\cos s\right)\sin t}{\sqrt{4-2\sqrt{2}}} \end{array} \right\}.$$

The surfaces M, M_n, and $M \cup M_n$ along the curve α are shown in Figs. (4a, 4b), and Figs. (5a, 5b).



(a) (b)

Figure 4. (a) The base curve α. (b) The surface M.



(a) (b)

Figure 5. (a) The normal surface M_n. (b) The surface M_n normal to M along the curve α.

Case(3.4). If we choose $u(s,t) = (1+\sin(t)) + \sum\limits_{k=2}^{4} a_{1k}(1+\sin(t))^k$, $v(s,t) = \cos(t) + \sum\limits_{k=2}^{4} a_{2k}\cos^k(t)$, $t_0 = 0$, $t_0 = 3\pi/2$, a_{1k}, $a_{2k} \in \mathbb{R}$, and $t \in [0, 2\pi]$, then Eqs. (3.11) are satisfied. Hence, the normal surface can be represented as follows:

$$M_n : P(s,t) = \{P_1(s,t), P_2(s,t), P_3(s,t)\},$$

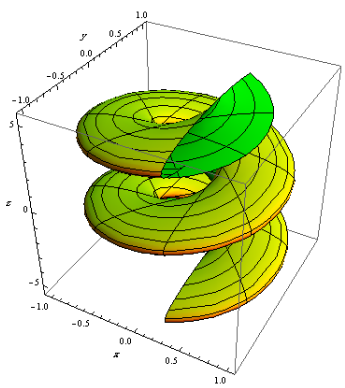
where

$$P_1(s,t) = \left(\begin{array}{c} \cos s - \frac{\cos s}{\sqrt{2}} + \frac{\left(1+\cos t+(1+\cos t)^2+2(1+\cos t)^3+3(1+\cos t)^4\right)\sin s}{\sqrt{4-2\sqrt{2}}} - \\ \frac{1}{2}\sqrt{2-\sqrt{2}}\sin s \left(\begin{array}{c} 1+\sin t+(1+\sin t)^2 \\ +2(1+\sin t)^3+3(1+\sin t)^4 \end{array} \right) \end{array} \right),$$

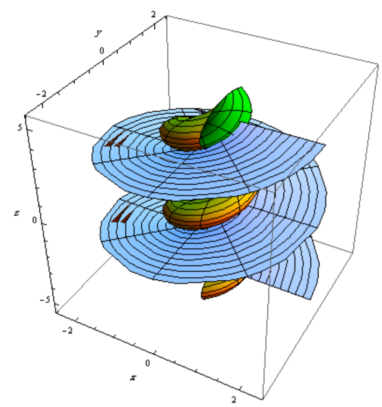
$$P_2(s,t)=\begin{pmatrix} -\frac{\cos s\left(1+\cos t+(1+\cos t)^2+2(1+\cos t)^3+3(1+\cos t)^4\right)}{\sqrt{4-2\sqrt{2}}}+\sin s-\frac{\sin s}{\sqrt{2}}+ \\ \frac{1}{2}\sqrt{2-\sqrt{2}}\cos s\begin{pmatrix} 1+\sin t+(1+\sin t)^2 \\ +2(1+\sin t)^3+3(1+\sin t)^4 \end{pmatrix} \end{pmatrix},$$

$$P_3(s,t)=\begin{pmatrix} \frac{u}{\sqrt{2}}+\frac{\left(-1+\sqrt{2}\right)\left(1+\cos t+(1+\cos t)^2+2(1+\cos t)^3+3(1+\cos t)^4\right)}{\sqrt{4-2\sqrt{2}}}+ \\ \frac{1+\sin t+(1+\sin t)^2+2(1+\sin t)^3+3(1+\sin t)^4}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}.$$

In this case, the surfaces M_n, and $M\cup M_n$ along the curve α are shown in Figs. (6a, 6b).



(a)



(b)

Figure 6. (a) The normal surface M_n. (b) The surface M_n normal to M along the curve α.

3.1. **Normal ruled surfaces.** A ruled surface is a surface generated by a straight line moving along a curve. The various positions of the generating lines are called the rulings or generators of the surface. Suppose that M_n is a ruled surface along the curve $\alpha(s)$ of M, then there exists t_0 such that $P(s,t_0)=\alpha(s)$. This follows that the ruled surface can be expressed as

$$M_n:\ P(s,t)=P(s,t_0)+(t-t_0)e(s),\ 0\leq s\leq L,\ \text{with } t,\ t_0\in[0,T],$$

where $e(s)$ denotes the direction of the rulings. According to the Eq. (3.1), we have

$$(t-t_0)e(s)=u(s,t)e_1(s)+v(s,t)e_3(s),\ 0\leq s\leq L,\ \text{with } t,\ t_0\in[0,T], \tag{3.16}$$

which is a system of two equations with two unknown functions $u(s,t)$, and $v(s,t)$. For simplicity, we omit variable s. The solutions of the above system can be deduced as follows:

$$\begin{aligned} u(s,t)&=(t-t_0)<e,e_1>=(t-t_0)\det(e,e_2,e_3),\\ v(s,t)&=(t-t_0)<e,e_3>=(t-t_0)\det(e,e_1,e_2). \end{aligned} \tag{3.17}$$

The above equations are just the necessary and sufficient conditions for which M_n is a ruled surface with a directrix $\alpha(s)$ on M.

Now, we need to check if M_n is the normal ruled surface with a directrix $\alpha(s)$ of M by using the conditions given in Theorem 3.1. It is evident that in this case, these conditions become

$$\det(e, e_1, e_2) = < e, e_3 > \neq 0. \tag{3.18}$$

It follows that at any point on the curve $\alpha(s)$; the ruling direction $e(s)$ must be in the plane $Sp\{e_1, e_3\}$. This leads to

$$e(s) = \beta e_1 + \gamma e_3,\ \gamma(s) \neq 0,\ 0 \leq s \leq L, \tag{3.19}$$

for some real functions $\beta(s)$, and $\gamma(s)$. Substituting it into the expressions in Eq. (3.16), we get

$$u(s,t) = \beta(s)t,\ v(s,t) = \gamma(s)t;\ \gamma(s) \neq 0,\ 0 \leq s \leq L.$$

Hence, the isoparametric ruled surface family with a directrix $\alpha(s)$ on M can be expressed as:

$$\left.\begin{array}{l} M_n : P(s,t) = \alpha(s) + te(s), \\ e(s) = \beta e_1 + \gamma e_3,\ 0 \leq s \leq L,\ 0 \leq t \leq T, \end{array}\right\} \tag{3.20}$$

where the functions $\beta(s)$ and $\gamma(s) \neq 0$, can control the shape of the ruled surface family. However, the normal vector to the ruled surface M_n is

$$N(s,t) = t\left(\beta\kappa_g - \gamma\tau_g\right)\left(\gamma e_1 + \beta e_3\right) + \left[-\gamma + t\left(\beta^2 + \gamma^2\right)\kappa_n + t\left(\beta\gamma' - \gamma\beta'\right)\right] e_2,$$

and thus when $t_0 = 0$, i.e., along the curve $\alpha(s)$, the surface normal is

$$N(s,0) = -\gamma e_2.$$

So, the normal vector of M_n at $P(s,t_0) = \alpha(s)$ is orthogonal to the normal vector of M at $\alpha(s)$. Thus, M_n is the normal ruled surface of M along $\alpha(s)$.

Theorem 3.2. *The necessary and sufficient condition for M_n being a normal ruled surface along $\alpha(s)$ of M is that there exist a parameter $t_0 \in [0,T]$, and the functions $\beta(s)$, and $\gamma(s) \neq 0$, so that M_n can be represented by Eq. (3.19).*

Proof. Since the functions $\beta(s)$ and $\gamma(s) \neq 0$, can control the shape of the ruled surface family. Then, the normal vector to the ruled surface M_n is

$$\begin{aligned} N(s,t) &= t\left(\beta\kappa_g - \gamma\tau_g\right)\left(\gamma e_1 + \beta e_3\right) \\ &\quad + \left[-\gamma + t\left(\beta^2 + \gamma^2\right)\kappa_n + t\left(\beta\gamma' - \gamma\beta'\right)\right] e_2, \end{aligned}$$

and thus when $t_0 = 0$, i.e., along the curve $\alpha(s)$, the surface normal is

$$N(s,0) = -\gamma e_2.$$

So, the normal vector of M_n at $P(s,t_0) = \alpha(s)$ is orthogonal to the normal vector of M at $\alpha(s)$. Thus, M_n is a normal ruled surface of M along $\alpha(s)$. Hence, this completes the proof. $\square$

By Theorem 3.2, we not only prove the existence of the normal surface, but also give the concrete expression of the surface. Every member of the isoparametric normal ruled surface family along $\alpha(s)$ of M is decided by two family of parameters $\beta(s)$ and $\gamma(s) \neq 0$, i.e., by the direction vector function $e(s)$.

Example 3.3. We consider a surface parameterized by

$$M : X(s,t) = \left(1+\cos s - \frac{\sqrt{2}t\sin s}{\sqrt{3+\cos s}}, \frac{\sqrt{2}t\cos s}{\sqrt{3+\cos s}} + \sin s, \frac{\sqrt{2}t\cos s}{\sqrt{3+\cos s}} + 2\sin\frac{s}{2}\right).$$

This surface is a ruled surface such that the base curve is

$$\alpha(s) = (1+\cos s, \sin s, 2\sin\frac{s}{2}), \ \ s \in \mathbb{R}.$$

Therefore, $\alpha(s)$ is a regular curve on the surface M. In this case the subspace $Sp\{e_1, e_3\}$ of $\alpha(s)$ is

$$e_1(s) = \frac{\dot{\alpha}(s)}{\|\dot{\alpha}(s)\|} = \left(-\frac{\sqrt{2}\sin s}{\sqrt{3+\cos s}}, \frac{\sqrt{2}\cos s}{\sqrt{3+\cos s}}, \frac{\sqrt{2}\cos\frac{s}{2}}{\sqrt{3+\cos s}}\right),$$

and

$$\begin{aligned} e_3(s) &= \frac{X_s \times X_t}{\|X_s \times X_t\|} \\ &= \left(\frac{-3\sin\frac{s}{2} - \sin\frac{3s}{2}}{\sqrt{13+3\cos s}}, \frac{2\sqrt{2}\cos^3\left(\frac{s}{2}\right)}{\sqrt{13+3\cos s}}, \frac{-2\sqrt{2}}{\sqrt{13+3\cos s}}\right). \end{aligned}$$

Thus, using Eq. (3.19), the normal surface family can be represented as:

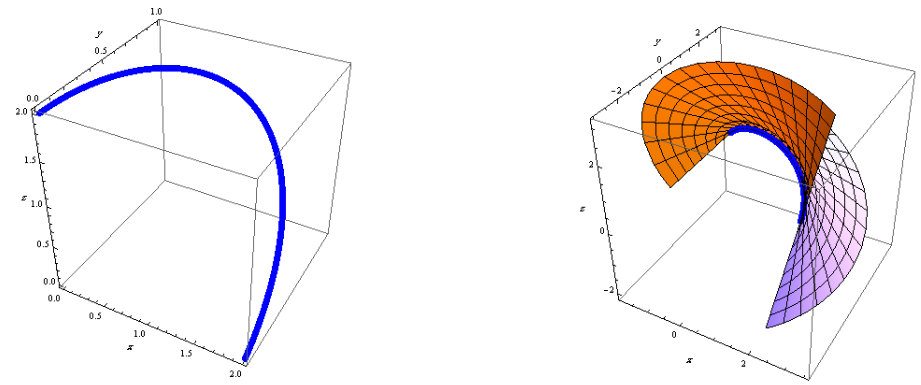
$$\begin{aligned} M_n \ \ : \ \ & P(s,t) = \left\{1+\cos s, \sin s, 2\sin\frac{s}{2}\right\} + \\ & t\left(\beta(s)\left\{\begin{array}{c} -\frac{\sqrt{2}\sin s}{\sqrt{3+\cos s}}, \\ \frac{\sqrt{2}\cos s}{\sqrt{3+\cos s}}, \\ \frac{\sqrt{2}\cos\frac{s}{2}}{\sqrt{3+\cos s}} \end{array}\right\} + \gamma(s)\left\{\begin{array}{c} \frac{-3\sin\frac{s}{2} - \sin\frac{3s}{2}}{\sqrt{13+3\cos s}}, \\ \frac{2\sqrt{2}\cos^3\left(\frac{s}{2}\right)}{\sqrt{13+3\cos s}}, \\ \frac{-2\sqrt{2}}{\sqrt{13+3\cos s}} \end{array}\right\}\right). \end{aligned}$$

The functions $\beta(s)$ and $\gamma(s)$ can control the shape of the surface and it is very clear that if these functions are given, then we immediately obtain the normal surface in the family. In the following, we consider two cases.

Case(3.5). We choose $\beta(s) = \sin s$, and $\gamma(s) = s$. Obviously, Eqs. (3.15-3.19) are satisfied, and the normal surface in this family is given by:

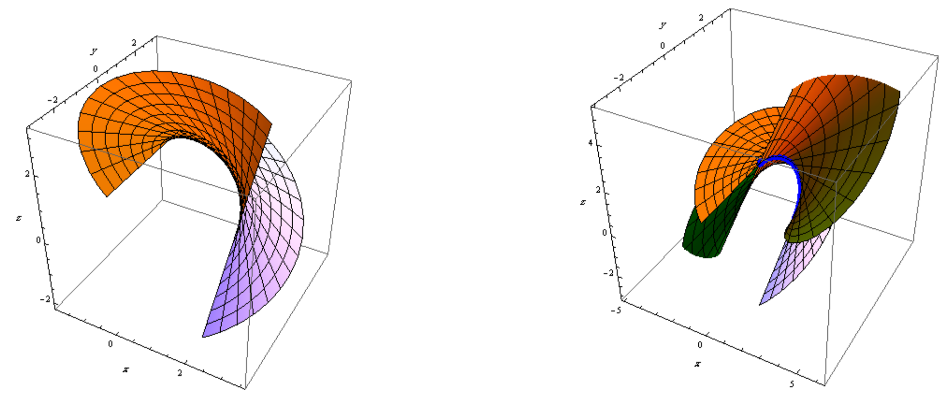
$$M_n : P(s,t) = \left\{\begin{array}{c} 1+\cos s + t\left(-\frac{\sqrt{2}\sin^2 s}{\sqrt{3+\cos s}} - \frac{s\left(3\sin\left(\frac{s}{2}\right)+\sin\left(\frac{3s}{2}\right)\right)}{\sqrt{26+6\cos s}}\right), \\ \frac{2\sqrt{2}s\ \ t\ \ \cos^3\left(\frac{s}{2}\right)}{\sqrt{13+3\cos s}} + \sin s + \frac{\sqrt{2}t\cos s\sin s}{\sqrt{3+\cos s}}, \\ 2\sin\left(\frac{s}{2}\right) + \sqrt{2}t\left(-\frac{2u}{\sqrt{13+3\cos s}} + \frac{\cos\left(\frac{s}{2}\right)\sin s}{\sqrt{3+\cos s}}\right) \end{array}\right\}.$$

The surfaces M, M_n, and $M \cup M_n$ along the curve α, are shown in Figs. (7a, 7b), and Figs. (8a, 8b).



(a) (b)

Figure 7. (a) The base curve α. (b) The ruled surface M.



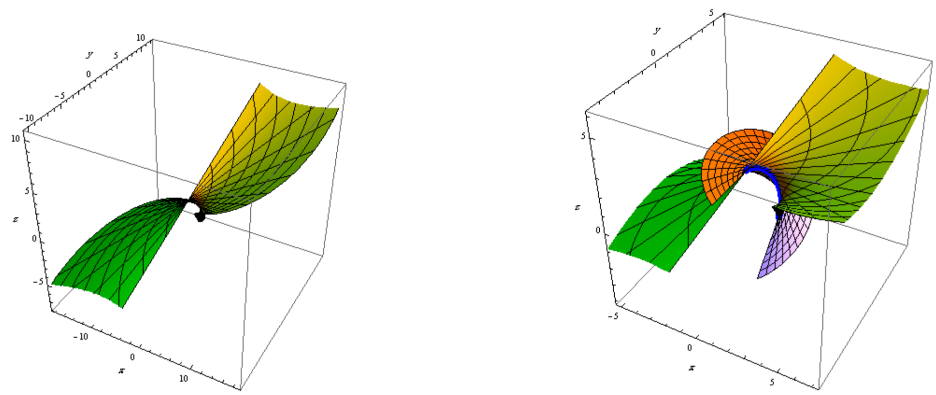
(a) (b)

Figure 8. (a) The normal surface M_n. (b) The surface M_n normal to M along the curve α.

Case(3.6). If we choose $\beta(s) = s^2$ and $\gamma(s) = 2s$, then the normal surface in this family is given by:

$$M_n : P(s,t) = \left\{ \begin{array}{c} 1+\cos s+\sqrt{2}st\left(-\frac{s\sin s}{\sqrt{3+\cos s}}-\frac{3\sin\left(\frac{s}{2}\right)+\sin\left(\frac{3s}{2}\right)}{\sqrt{13+3\cos s}}\right), \\ \frac{\sqrt{2}s^2t\cos s}{\sqrt{3+\cos s}}+\frac{4\sqrt{2}st\ \cos^3\left(\frac{s}{2}\right)}{\sqrt{13+3\cos s}}+\sin s, \\ \frac{\sqrt{2}s^2t\ \cos\left(\frac{s}{2}\right)}{\sqrt{3+\cos s}}-\frac{4\sqrt{2}st}{\sqrt{13+3\cos s}}+2\sin\left(\frac{s}{2}\right) \end{array} \right\}.$$

The surfaces M, M_n, and $M \cup M_n$ along the curve α, are shown in Figs. (9a, 9b).



(a) (b)

Figure 9. (*a*) The normal surface M_n. (*b*) The surface M_n normal to M along the curve α.

4. Conclusion

In the three-dimensional Euclidean space E^3, a surface normal to a surface along a curve on the surface has been defined. Also, the necessary and sufficient conditions for that surface to be a ruled surface have been investigated. Moreover, we have illustrated the convenience and efficiency of this approach by some representative examples. In future works, we plan to study the normal surfaces in Lorentz-Minkowski space for different queries and further improve the results in this paper, combined with the techniques and results in the related papers [11–16].

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References

[1] G.J. Wang, K. Tang, C.L. Tai, Parametric Representation of a Surface Pencil With a Common Spatial Geodesic, Computer-Aided Design. 36 (2004), 447–459. https://doi.org/10.1016/s0010-4485(03)00117-9.

[2] E. Kasap, F.T. Akyildiz, K. Orbay, A Generalization of Surfaces Family With Common Spatial Geodesic, Appl. Math. Comput. 201 (2008), 781–789. https://doi.org/10.1016/j.amc.2008.01.016.

[3] C.Y. Li, R.H. Wang, C.G. Zhu, Parametric Representation of a Surface Pencil With a Common Line of Curvature, Computer-Aided Design. 43 (2011), 1110–1117. https://doi.org/10.1016/j.cad.2011.05.001.

[4] E. Bayram, F. Güler, E. Kasap, Parametric Representation of a Surface Pencil With a Common Asymptotic Curve, Computer-Aided Design. 44 (2012), 637–643. https://doi.org/10.1016/j.cad.2012.02.007.

[5] C.Y. Li, R.H. Wang, C.G. Zhu, A Generalization of Surface Family With Common Line of Curvature, Appl. Math. Comput. 219 (2013), 9500–9507. https://doi.org/10.1016/j.amc.2013.03.077.

[6] C.Y. Li, R.H. Wang, C.G. Zhu, An Approach for Designing a Developable Surface Through a Given Line of Curvature, Computer-Aided Design. 45 (2013), 621–627. https://doi.org/10.1016/j.cad.2012.11.001.

[7] E. Bayram, E. Ergun, E. Kasap, Surface family with a common natural asymptotic lift, J. Sci. Arts. 2 (2015), 117–124.

[8] E. Bayram, M. Bilici, Surface family with a common involute asymptotic curve, Int. J. Geom. Methods Mod. Phys. 13 (2016), 1650062. https://doi.org/10.1142/s0219887816500626.

[9] M.P. Do Carmo, Differential Geometry of Curves and Surfaces, Prentice Hall, Englewood Cliffs, 1976.

[10] H. Pottman, J. Wallner, Computational Line Geometry, Springer-Verlag, Berlin, Heidelberg, 2001.

[11] M.K. Saad, R.A. Abdel-Baky, F. Alharbi, A. Aloufi, Characterizations of Some Special Curves in Lorentz-Minkowski Space, Math. Stat. 8 (2020), 299–305. https://doi.org/10.13189/ms.2020.080308.

[12] H.S. Abdel-Aziz, H. Serry, M.K. Saad, Evolution Equations of Pseudo Spherical Images for Timelike Curves in Minkowski 3-Space, Math. Stat. 10 (2022), 884–893. https://doi.org/10.13189/ms.2022.100420.

[13] Y. Li, Ali.H. Alkhaldi, A. Ali, R.A. Abdel-Baky, M.K. Saad, Investigation of Ruled Surfaces and Their Singularities According to Blaschke Frame in Euclidean 3-Space, AIMS Math. 8 (2023), 13875–13888. https://doi.org/10.3934/math.2023709.

[14] M.K. Saad, H.S. Abdel-Aziz, H.A. Ali, On Magnetic Curves According to Killing Vector Fields in Euclidean 3-Space, Int. J. Anal. Appl. 20 (2022), 18. https://doi.org/10.28924/2291-8639-20-2022-18.

[15] R.A. Abdel-Baky, M.K. Saad, Osculating Surfaces Along a Curve on a Surface in Euclidean 3-Space, J. Math. Comput. Sci. 12 (2022), 84. https://doi.org/10.28919/jmcs/7109.

[16] M.K. Saad, R.A. Abdel-Baky, On Ruled Surfaces According to Quasi-Frame in Euclidean 3-Space, Aust. J. Math. Anal. Appl. 17 (2020), 11.