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A Three-Step Iterative Scheme Based on Green's Function for the Solution of Boundary Value Problems

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Abstract. In this manuscript, we suggest a three-step iterative scheme for finding approximate numerical solutions to boundary value problems (BVPs) in a Banach space setting. The underlying strategy of the scheme is based on embedding Green's function into the three-step M-iterative scheme, which we will call in the paper M-Green's iterative scheme. We assume certain possible mild conditions to prove the convergence and stability results of the suggested scheme. We also prove numerically that our M-Green iterative scheme is more effective than the corresponding Mann-Green and Khan-Green iterative schemes. Our results improve and extend some recent results in the literature of Green's function based iteration schemes.

1. Introduction

Let $V = (V, ||.||)$ be a normed vector space and $A : V \to V$ be a self operator. In this case, A is called a contraction operator on X if for any two elements $v, w \in V$, one has a constant $0 \leq c < 1$ with the following property:

$$||Av - Aw|| \leq c||v - w||. \tag{1.1}$$

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The operator A is called nonexpansive on V if (1.1) is holds for the value $c = 1$. For contraction operators, Banach [1] proved the following well-known result:

Theorem 1.1. *[1] Let V be a Banach space and $A : V \to V$ be a contraction operator. Then the following conditions are true:*

(a) *A admits a unique fixed point, namely, $v^* \in V$.*
(b) *For each $v_0 \in V$, the Picard iteration, $v_{n+1} = Av_n$ $(n = 0, 1, 2, 3, ...)$ converges to v^*.*

In 1965, as many know, Browder [2] was the first to prove a fixed point theorem for nonexpansive operators. However, unlike the Banach result above, the Browder result does not suggest the Picard iteration for nonexpansive operators in general [3–12]. Precisely, there are many nonexpansive operators for which Picard iteration does not work. Another drawback of Picard [13] iterative scheme is that its rate of convergence is slow. In 1953, Mann [14] introduced a new iterative scheme, which is a one-step iteration like the Picard iteration [13] for computing fixed points of nonexpansive operators. In 1975, Ishikawa [15] introduced a two-step iterative scheme that is more general than the Picard [13] and Mann [14] iterative schemes. In 2012, Khan [16] combined the Picard [13] and Mann [14] iteration which is also a two-step iteration. Khan [16] proved that this new iteration is essentially independent of but faster than the Picard [13], Mann [14] and Ishikawa [15] iterative schemes for contraction and as well as for nonexpansive operators. We know that, three-step iterative schemes are essentially better than the one and two-step iterative schemes in many cases. For example, in [17], Glowinski and Le Tallec applied a three-step scheme for computing and sought solutions for various problems and proving that the numerical efficiency of the these schemes is far better than the corresponding one-step and two-steps iterative schemes. Similarly, in [18], Haubruge et al. improved and extended some main outcome of Glowinski and Le Tallec [17] and noticed that the three-step schemes suggests high parallelized schemes when some appropriate assumptions are available. Eventually, we conclude that three-step iterative schemes should be used whenever they are available.

On the other hand, a new three-step iterative scheme was developed by Ullah and Arshad [19] which they called it as M-iterative scheme. This scheme reads as follows:

$$\begin{cases} v_0 \in V, \\ w_n = (1 - a_n)v_n + a_n A v_n, \\ u_n = A w_n, \\ v_{n+1} = A u_n, (n = 0, 1, 2, 3, ...), \end{cases} \tag{1.2}$$

where $a_n \in (0, 1)$.

Although, many authors proved convergence of the Picard [13], Mann [14], Ishikawa [15], Khan [16] and M-iterative scheme of Ullah and Arshad [19] for different classes of nonlinear mappings. On the other side, Khuri and Sayfy [20] introduced a novel setting for Picard and Ishikawa iterative schemes.

Actually these authors modified the so-called schemes and named them as Picard-Green's and Mann-Green's iterative schemes. After that, Khuri and Louhichi [21] modified the Ishikawa iterative scheme using Green's function and obtained the Ishikawa–Green's iterative scheme. They noticed that this new Ishikawa-Green's iterative scheme is better than the Mann-Green's iterative scheme for a broad class of BVPs. Very recently, Ali et al. [22] considered the Khan iterative scheme using Green's function approach and obtained the Khan-Green's iterative scheme. They proved the Khan-Green's iterative scheme is eventually better than the Picard-Green's, Mann-Green's and Ishikawa-Green's iterative schemes. Motivated by above work, we modify the M-iterative scheme using the Green's function approach and prove its convergence to a solution of certain BVP of a broad class. We use different examples to validate our outcome. Graphs and tables in the last section precisely suggest the novel M-Green's iterative approach is high accurate corresponding to the other previous known approaches.

2. Methodology and Green's function

This section is essentially divided into two some subsections. In the first subsection, we provide a brief description of the Green's function associated with a general BVP of second order. In the next subsection, we embed this Green's function into our M-iterative scheme and obtain the desired M-Green's iterative scheme.

2.1. **A short description of the Green's function.**

We start this section with a general BVPs as follows:

$$L_t(v) = v'' = q(\xi, v, v'). \tag{2.1}$$

The notations $L_t(v)$ and $q(\xi, v, v')$ are respectively stand for linear and nonlinear terms with the boundary conditions (BCs) given as follows:

$$\left.\begin{array}{l} B_a[v] = \alpha_1 v(a) + \alpha_2 v'(a) = \gamma, \\ B_b[v] = \beta_1 v(b) + \beta_2 v'(b) = \delta, \end{array}\right\} \tag{2.2}$$

where $a \leq \xi \leq b$ and $\gamma, \delta, \alpha_1, \alpha_2, \beta_1, \beta_2$ are some constants.

There are many papers devoted to the existence of solution for Problem (2.1)–(2.2) (see e.g., [23–25] and others). However, once the existence of solution for a problem is guaranteed then an iterative scheme which can approximate the value of this solution is always desirable. Thus in this paper, our aim is to suggest a new and effective iterative scheme for the so-called problem.

Since our approach is based on the Green's function, so we may denote the Green's function simply by $G_f = G_f(\xi, \eta)$ to the term $L_t(v)$ and it is known that $G_f(\xi, \eta)$ attains the following form:

$$G_f(\xi, \eta) = \begin{cases} a_1 v_1 + a_2 v_2 & \text{if } \; a \leq \xi < \eta \\ b_1 v_1 + b_2 v_2 & \text{if } \; \eta < \xi < b, \end{cases}$$

where a_i, b_i, $(i = 0, 1)$ are some constants and v_i, $(i = 1, 2)$ are two known linearly independent solutions for the $L_t(v) = 0$. It is known that the values of these constants found from the following axioms of G_f.

(a) G_f agrees with homogeneous BCs, that is,

$$B_a[G_f(\xi, \eta)] = B_b[G_f(\xi, \eta)] = 0. \tag{2.3}$$

(b) The function $\mathcal{G}_f$ is essentially continuous at each $\xi = \eta$, that is,

$$a_1 v_1(\eta) + a_2 v_2(\eta) = b_1 v_1(\eta) + b_2 v_2(\eta). \tag{2.4}$$

(c) The function G_f' has a unit jump discontinuity at each $\xi = \eta$, that is,

$$a_1 v_1'(\eta) + a_2 v_2'(\eta) - b_1 v_1'(\eta) - b_2 v_2'(\eta) = -1. \tag{2.5}$$

Now it is worth mentioning that the particular sought solution of the given Problem (2.1)–(2.2) can be set using Green's function as given bellow:

$$v_p = \int_a^b G_f(\xi, \eta) q(\eta, v_p, v_p') d\eta, \tag{2.6}$$

where v_p essentially denotes the particular solution of $v'' = q(\xi, v, v')$.

2.2. **M–Green iterative scheme.** The essential aim of the section is to construct a new iterative scheme that is based on Green's function, and to achieve the aim, we shall take the following problem:

$$L_t(v) + N_t(v) = q(\xi, v), \tag{2.7}$$

and it should be noted that here the notation L_t denotes a linear and the notation N_t stands for linear term. While q may be linear or nonlinear in the variable v.

To achieve our desired objective, we set the following operator:

$$Rv_p = \int_a^b G_f(\xi, \eta) L_t(v_p) d\eta, \tag{2.8}$$

and notice that here notation v_p essentially represents a particular sought solution as before of (2.7) and the operator G_f is the Green's that is relative to the term L_t. Notice that, for the sake of simplicity, we may assume $v_p = v$. So that from (2.8), it follows that

$$\begin{aligned} Rv &= \int_a^b G_f(\xi, \eta)[L_t(v) + N_t(v) - q(\eta, v) - N_t(v) + q(\eta, v)] d\eta \\ &= \int_a^b G_f(\xi, \eta)[L_t(v) + N_t(v) - q(\eta, v)] d\eta + \int_a^b G_f(\xi, \eta)[q(\eta, v) - N_t(v)] d\eta \\ &= v + \int_a^b G_f(\xi, \eta)[L_t(v) + N_t(v) - q(\eta, v)] d\eta. \end{aligned}$$

Accordingly, one can now replace the operator involved in (1.2) with R in order to get the following new modified form of it as follows:

$$\begin{cases} w_n = (1-a_n)v_n + a_n R v_n, \\ u_n = R w_n, \\ v_{n+1} = R u_n, (n = 0, 1, 2, 3, \ldots). \end{cases} \tag{2.9}$$

Accordingly, we get

$$\begin{cases} w_n = (1-a_n)v_n + a_n[v_n + \int_a^b G_f(\xi,\eta)[L_t(v_n) + N_t(v_n) - q(\eta, v_n)]d\eta], \\ u_n = w_n + \int_a^b G_f(\xi,\eta)[L_t(w_n) + N_t(w_n) - q(\eta, w_n)]d\eta, \\ v_{n+1} = u_n + \int_a^b G_f(\xi,\eta)[L_t(u_n) + N_t(u_n) - q(\eta, u_n)]d\eta. \end{cases} \tag{2.10}$$

Finally, it follows that

$$\begin{cases} w_n = v_n + a_n[\int_a^b G_f(\xi,\eta)[L_t(v_n) + N_t(v_n) - q(\eta, v_n)]d\eta], \\ u_n = w_n + \int_a^b G_f(\xi,\eta)[L_t(w_n) + N_t(w_n) - q(\eta, w_n)]d\eta, \\ v_{n+1} = u_n + \int_a^b G_f(\xi,\eta)[L_t(u_n) + N_t(u_n) - q(\eta, u_n)]d\eta. \end{cases} \tag{2.11}$$

3. Convergence theorem

We consider some special assumptions to establish a convergence result of the proposed M–Green's (2.11) towards the sought solution of the problem. We offer an example to support the establish result and to show that the proposed M–Green's iteration scheme produces high accurate numerical results as compared the Mann–Green's and Khan–Green's iteration schemes. To do that, we assume the following broad class of BVP:

$$v''(\xi) = q(\xi, v(\xi), v'(\xi)), \tag{3.1}$$

subject to the boundary conditions:

$$v(0) = \mathcal{C}, \quad v(1) = \mathcal{D}, \tag{3.2}$$

and here it should be keep in mind that the notations $\mathcal{C}$ as well as $\mathcal{D}$ essentially denotes constants. The first target is to find the required Green's function for the equation $v'' = 0$ that obeys the available BCs as suggested in (3.2). Now it follows that $v'' = 0$ admits essentially two different linearly independent sought solutions which we may denote here as $v_1 = 1$ and $v_2 = \xi$. Accordingly, we may observe from (2.3) that Green's function attains essentially a form as follows:

$$G_f(\xi,\eta) = \begin{cases} a_1 + a_2\xi & \text{when } \ a \le \xi < \eta \\ b_1 + b_2\xi & \text{when } \ \eta < \xi \le b. \end{cases}$$

Using the axioms of Green's function, we obtain the values of constant. In this case, Green's function becomes

$$G_f(\xi,\eta) = \begin{cases} \eta(1-\xi) & \text{when } \ a \le \xi < \eta \\ \xi(1-\eta) & \text{when } \ \eta < \xi \le b. \end{cases}$$

To go forward, we shall use (3.1)–(3.2), so that we obtain the M–Green's iterative scheme as given in (2.11) in the following new form:

$$\begin{cases} w_n = v_n + a_n[\int_0^{\xi} \eta(1-\xi)\left(v_n''(\eta) - q(\eta, v_n(\eta), v_n'(\eta))\right) d\eta \\ + \int_1^{\xi} \xi(1-\eta)(v_n''(\eta) - q(\eta, v_n(\eta), v_n'(\eta))) d\eta], \\ u_n = u_n + \int_0^{\xi} \eta(1-\xi)\left(w_n''(\eta) - q(\eta, w_n(\eta), w_n'(\eta))\right) d\eta \\ + \int_1^{\xi} \xi(1-\eta)(w_n''(\eta) - q(\eta, w_n(\eta), w_n'(\eta))) d\eta, \\ v_{n+1} = u_n + \int_0^{\xi} \eta(1-\xi)\left(u_n''(\eta) - q(\eta, u_n(\eta), u_n'(\eta))\right) d\eta \\ + \int_1^{\xi} \xi(1-\eta)(u_n''(\eta) - q(\eta, u_n(\eta), u_n'(\eta))) d\eta. \end{cases} \tag{3.3}$$

It follows that

$$\begin{cases} w_n = v_n + a_n[\int_0^1 G_f(\xi, \eta)\left(v_n''(\eta) - q(\eta, v_n(\eta), v_n'(\eta))\right] d\eta, \\ u_n = w_n + \int_0^1 G_f(\xi, \eta)\left(w_n''(\eta) - q(\eta, w_n(\eta), w_n'(\eta))\right) d\eta, \\ v_{n+1} = u_n + \int_0^1 G_f(\xi, \eta)\left(u_n''(\eta) - q(\eta, u_n(\eta), u_n'(\eta))\right) d\eta. \end{cases} \tag{3.4}$$

Notice that we take the initial point $v_0(\xi)$ that satisfies $v'' = 0$ and the BCs (3.2). Thus, we have $v_0(\xi) = (\mathcal{D} - \mathcal{C})\xi + \mathcal{C}$.

Now we can set the operator $A_{G_f} : C[0,1] \to C[0,1]$ by

$$A_{G_f} v = v + \int_0^1 G_f(\xi, \eta)(v''(\xi) - q(\eta, v(\eta), v''(\eta)))) d\eta. \tag{3.5}$$

Hence, the iteration scheme (3.4) attains the following form:

$$\begin{cases} w_n = (1 - a_n) v_n + a_n A_{G_f} v_n, \\ u_n = A_{G_f} w_n, \\ v_{n+1} = A_{G_f} u_n. \end{cases} \tag{3.6}$$

Now, the iterative scheme (3.6) is our proposed M-Green's iteration. As we know that $V = C[0,1]$ form a Banach space with the supremum norm. Hence we can now prove the following main theoretical result of our paper using our M-Green's iterative scheme (3.6) for solution of the Problem (3.1)–(3.2).

Theorem 3.1. *Set $A_{G_f} : V \to V$ be the operator defined in (3.5) and $\{v_n\}$ be the sequence of M–Green's iterative scheme (3.6). If $\frac{\sqrt{3}}{12} \times \sup_{[0,1]\times\mathbb{R}^3} \left|\frac{\partial q}{\partial v}\right| < 1$ and suppose one of the following condition holds:*

(i) $\sum a_n = \infty$

(ii) $0 < a \le a_n$ *for some* a.

Then $\{v_n\}$ converges the unique fixed point of A_{G_f} and hence to the unique sought solution of the given Problem (3.1) and (3.2).

Proof. Put $\frac{\sqrt{3}}{12} \times \sup_{[0,1]\times\mathbb{R}^3} \left|\frac{dq}{dv}\right| = c$, it follows that A_{G_f} is a c-contraction on V. Hence by Theorem 1.1(a), A_{G_f} has a unique fixed point in $V = C[0,1]$, namely, v^* which is the unique solution for the given problem (3.1) and (3.2).

If the condition (i) is hold, that is, $\sum a_n = \infty$, then we prove that the sequence of M–Green's iterative converges strongly to v^*. To see this, we have

$$\begin{aligned}
||w_n - v^*|| &= ||(1-a_n)v_n + a_n A_{G_f} v_n - v^*|| \\
&\leq (1-a_n)||v_n - v^*|| + a_n||A_{G_f} v_n - v^*|| \\
&= (1-a_n)||v_n - v^*|| + a_n||A_{G_f} v_n - A_{G_f} v^*|| \\
&\leq (1-a_n)||v_n - v^*|| + a_n c||v_n - v^*|| \\
&= [1 - a_n(1-c)]||v_n - v^*||.
\end{aligned}$$

Hence subsequently from the above, we have

$$||w_n - v^*|| \leq [1 - a_n(1-c)]||v_n - v^*||. \tag{3.7}$$

Now using (3.7), one has

$$\begin{aligned}
||v_{n+1} - v^*|| &= ||A_{G_f} u_n - v^*|| = ||A_{G_f} u_n - A_{G_f} v^*|| \\
&\leq c||u_n - v^*|| = c||A_{G_f} w_n - v^*|| \\
&= c||A_{G_f} w_n - A_{G_f} v^*|| \leq c^2||w_n - v^*|| \\
&\leq c^2([1 - a_n(1-c)]||v_n - v^*||).
\end{aligned}$$

Accordingly, we get

$$\begin{aligned}
||w_{n+1} - v^*|| &\leq c^2([1 - a_n(1-c)]||v_n - v^*||) \\
&\leq c^4[1 - a_n(1-c)][1 - a_{n-1}(1-c)]||v_{n-1} - v^*|| \\
&\leq c^6[1 - a_n(1-c)][1 - a_{n-1}(1-c)][1 - a_{n-2}(1-c)]||v_{n-2} - v^*||.
\end{aligned}$$

Inductively, we obtain

$$||v_{n+1} - v^*|| \leq c^{(2n+2)}\Pi_{m=0}^{n}[1 - a_m(1-c)]||v_0 - v^*||. \tag{3.8}$$

We know from the literature that $1 - v \leq e^{-v}$ for all $v \in [0, 1]$. Using this with (3.8), we get

$$||v_{n+1} - v^*|| \leq c^{(2n+2)} e^{-(1-c)\sum_{m=0}^{n} a_m ||v_0 - v^*||}. \tag{3.9}$$

As supposed $\sum a_n = \infty$ and "c" lies in $(0, 1)$, it follows from (3.9) that

$$\lim_{n\to\infty} ||v_{n+1} - v^*|| = 0.$$

Accordingly, $\{v_n\}$ converges to the unique fixed point v^* of A_{G_f} which is the unique solution of the problem (3.1) and (3.2). The case when condition (ii) is hold, that is, $0 < a \leq a_n$, is straightforward and hence omitted. $\square$

4. Stability theorem

The stability of an iterative scheme in the setting of numerical approximations means that a stable iterative scheme should produce if small change in initial approximation does not effect significantly the numerical approximations [26]. Hence a stable iterative scheme does not show excessive sensitivity to small changes in input data or initial approximations. In other words, it should produce solutions that converge towards a consistent outcome as the number of iterations increases, rather than diverging or oscillating wildly. Stability is an important consideration in various computational schemes, such as solving differential equations, optimization problems, or numerical simulations, as it ensures the accuracy and predictability of the results, ultimately leading to the successful and dependable execution of computational tasks. For more details; see [27–31].

This section aims to present a stability theorem for the iterative scheme under consideration. The concept of stability in fixed point iterations can be traced back to the basic work of Urabe [32], which laid the initial foundation. Building upon Urabe's contributions, Harder and Hicks [33] developed a precise mathematical definition for stability. To provide our main result of this section, it is essential to revisit some fundamental concepts, which will be briefly reviewed below.

Definition 4.1. *[33] Suppose we have a mapping A defined on a Banach space V. Consider a sequence of iterates $\{v_n\} \subseteq V$ with the mapping A as given below:*

$$\begin{cases} v_0 \in V, \\ v_{n+1} = \Omega(A, v_n), \end{cases} \tag{4.1}$$

where the the function Ω is associated with mapping A and $\{v_n\}$. If $\{v_n\}$ is strongly convergent to the given fixed point v^* of the mapping A, then $\{v_n\}$ is known as stable if

$$\lim_{n\to\infty} ||\hat{v}_{n+1} - \Omega(A, \hat{v}_n)|| = 0 \text{ implies } \lim_{n\to\infty} \hat{v}_n = v^*,$$

where $\{\hat{v}_n\}$ is arbitrary sequence of elements of the Banach space V.

Next, we give the definition of equivalent sequences in Banach spaces.

Definition 4.2. *[34] Two given sequences, namely, $\{v_n\}$ and $\{\hat{v}_n\}$ are called equivalent to each other if and only if $\lim_{n\to\infty} ||v_n - \hat{v}_n|| = 0$.*

In contrast to the notion of arbitrary sequences, Timis [35] introduced the concept of equivalent sequences to derive a novel mathematical definition of stability. This innovative form of stability is referred to as "w^2-stability" and its formal definition is presented below.

Definition 4.3. *[35] The sequence of iterates $\{v_n\}$ produced by (4.1) convergent to a fixed point v^* of the mapping A is called weak w^2-stable when for any choice of equivalent sequence $\{\hat{v}_n\} \subseteq V$ of the sequence $\{v_n\}$, we have*

$$\lim_{n\to\infty} ||\hat{v}_{n+1} - \Omega(A, \hat{v}_n)|| = 0 \text{ implies } \lim_{n\to\infty} \hat{v}_n = v^*.$$

Using the above weak concept of stability, we now want to show that M-Green's iterative scheme (3.6) is weak w^2 stable.

Theorem 4.1. *Suppose that V, A_{G_f} and $\{v_n\}$ are same as given in Theorem 3.1. Then, the convergence of the sequence of iterates $\{v_n\}$ is weak w^2–stable with respect to A_{G_f}.*

Proof. Assume that we have an equivalent sequence $\{\hat{v}_n\}$ of $\{v_n\}$, that is $\lim_{n\to\infty} ||\hat{v}_n - v_n|| = 0$. Put

$$\epsilon_n = ||\hat{v}_{n+1} - A_{G_f}\hat{u}_n]||,$$

where $\hat{u}_n = A_{G_f}\hat{w}_n$ and $\hat{w}_n = (1-\alpha_n)\hat{v}_n + \alpha_n A_{G_f}\hat{v}_m$.
Let $\lim_{n\to\infty}\epsilon_n = 0$. We first find $||\hat{w}_n - w_n||$. For this,

$$\begin{aligned}
||\hat{w}_n - w_n|| &= ||[(1-\alpha_n)\hat{v}_n + \alpha_n A_{G_f}\hat{v}_n] - [(1-\alpha_n)v_n + \alpha_n A_{G_f}v_n]|| \\
&= ||[(1-\alpha_n)(\hat{v}_n - v_n) + \alpha_n(A_{G_f}\hat{v}_n - A_{G_f}v_n]|| \\
&\leq (1-\alpha_n)||\hat{v}_n - v_n|| + \alpha_n||A_{G_f}\hat{v}_n - A_{G_f}v_n|| \\
&\leq (1-\alpha_n)||\hat{v}_n - v_n|| + \alpha_n c||\hat{v}_n - v_n|| \\
&\leq [1-\alpha_n(1-c)]||\hat{v}_n - v_n||.
\end{aligned}$$

Hence

$$||\hat{w}_n - w_n|| \leq [1-\alpha_n(1-c)]||\hat{v}_n - v_n||. \tag{4.2}$$

Keeping (4.2) in mind, we can proceed as follows:

$$\begin{aligned}
||\hat{v}_{n+1} - v^*|| &\leq ||\hat{v}_{n+1} - v_{n+1}|| + ||v_{n+1} - v^*|| \\
&\leq ||\hat{v}_{n+1} - A_{G_f}\hat{u}_n|| + ||A_{G_f}\hat{u}_n - v_{n+1}|| + ||v_{n+1} - v^*|| \\
&= \epsilon_n + ||F_G\hat{u}_n - v_{n+1}|| + ||v_{n+1} - v^*|| \\
&\leq \epsilon_n + c||\hat{u}_n - u_n|| + ||v_{n+1} - v^*|| \\
&= \epsilon_n + c||A_{G_f}\hat{w}_n - A_{G_f}w_n|| + ||v_{n+1} - v^*|| \\
&\leq \epsilon_n + c^2||\hat{w}_n - w_n|| + ||v_{n+1} - v^*|| \\
&\leq \epsilon_n + c^2[1-\alpha_n(1-c)]||\hat{v}_n - v_n|| + ||v_{n+1} - v^*||.
\end{aligned}$$

Subsequently, we obtained

$$||\hat{v}_{n+1} - v^*|| \leq \epsilon_n + c^2[1-\alpha_n(1-c)]||\hat{v}_n - v_n|| + ||v_{n+1} - v^*||. \tag{4.3}$$

Now, as assumed, $\lim_{n\to\infty}\epsilon_n = 0$ and $\lim_{n\to\infty}||\hat{v}_n - v_n|| = 0$. Moreover, $\lim_{n\to\infty}||v_{n+1} - v^*|| = 0$ due to the convergence of $\{v_n\}$ towards v^*. Subsequently, from (4.3), $\lim_{n\to\infty}||\hat{v}_n - v^*|| = 0$. It follows that $\{v_n\}$ produced by (3.6) is weak w^2-stable with respect to the mapping A_{G_f}. □

5. Numerical experiments

Now we suggest some examples of second order BVPs and connect our M-Green's, Khan-Green's and Mann-Green's iterative schemes with it. In the form of tables and graphs, we provide our final findings. In these tables and graphs, one can see the high accuracy of our new so-called M-Green's iterative scheme.

Example 5.1. *To show the numerical efficiency of our proposed M-Green's iterative scheme, we consider the following BVP:*

$$v''(\xi) + \xi v(\xi) - \xi^3 - 2 = 0, \tag{5.1}$$

with BCs as follows:

$$v(0) = 0, \quad v(1) = 1, \tag{5.2}$$

where $\xi \in [0, 1]$. *It follows that the unique solution of Problem (5.1)–(5.2) is* $v(\xi) = \xi^2$. *It should be noted that starting iterate* $v_0(\xi)$ *in this case follows from* $v''(\xi) = 0$ *and its BCs, as* $v_0(\xi) = \xi$.

Now, using Example 5.1, our proposed M-Green's iterative scheme takes the following form:

$$\begin{cases} w_n = v_n + a_n \int_0^{\xi} \eta(1-\xi)[v_n''(\eta) + \eta v_n(\eta) - \eta^3 - 2]d\eta, \\ +a_n \int_{\xi}^{1} \xi(1-\eta)[v_n''(\eta) + \eta v_n(\eta) - \eta^3 - 2]d\eta, \\ u_n = w_n + \int_0^{\xi} \eta(1-\xi)[w_n''(\eta) + \eta w_n(\eta) - \eta^3 - 2]d\eta, \\ + \int_{\xi}^{1} \xi(1-\eta)[w_n''(\eta) + \eta w_n(\eta) - \eta^3 - 2]d\eta, \\ v_{n+1} = u_n + \int_0^{\xi} \eta(1-\xi)[u_n''(\eta) + \eta u_n(\eta) - \eta^3 - 2]d\eta, \\ + \int_{\xi}^{1} \xi(1-\eta)[u_n''(\eta) + \eta u_n(\eta) - \eta^3 - 2]d\eta. \end{cases} \tag{5.3}$$

The iterative scheme (5.3) is now our proposed M-Green's iterative scheme. The graph of the Green's involved in the scheme is given as Figure 1. Now for $a_n = \frac{1}{2}$, *we choose different cases for* ξ *and we see that in each case, in Table 1 and 2, our proposed M-Green's iterative scheme converges to the sought solution of the problem (5.1)–(5.2). The graphical view in this in Figure 2.*

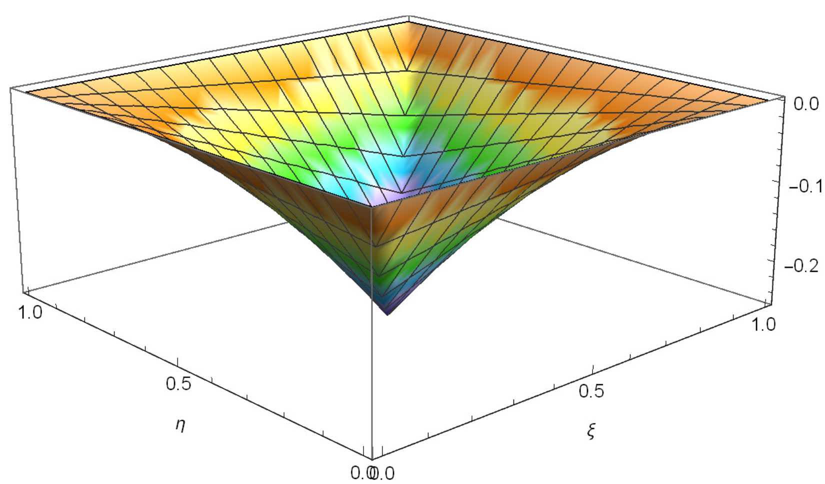

Figure 1. Plot of Green's function for Example 5.1.

Table 1. Numerical values generated by M-Green's iteration for different choices of ξ.

n	$\xi = 0.1$	$\xi = 0.2$	$\xi = 0.3$	$\xi = 0.4$
0	*0.1000000*	*0.2000000*	*0.3000000*	*0.400000*
1	*0.0100886*	*0.0401752*	*0.0902546*	*0.160319*
2	*0.0100001*	*0.0400003*	*0.0900004*	*0.160000*
3	*0.0100000*	*0.0400000*	*0.0900000*	*0.160000*
4	*0.0100000*	*0.0400000*	*0.0900000*	*0.160000*
5	*0.0100000*	*0.0400000*	*0.0900000*	*0.160000*

Table 2. Numerical values generated by M-Green's iteration for different choices of ξ.

n	$\xi = 0.5$	$\xi = 0.6$	$\xi = 0.7$	$\xi = 0.8$	$\xi = 0.9$
0	*0.500000*	*0.600000*	*0.700000*	*0.800000*	*0.900000*
1	*0.250359*	*0.360364*	*0.490329*	*0.640252*	*0.810137*
2	*0.250001*	*0.360001*	*0.490000*	*0.640000*	*0.810000*
3	*0.250000*	*0.360000*	*0.490000*	*0.640000*	*0.810000*
4	*0.250000*	*0.360000*	*0.490000*	*0.640000*	*0.810000*
5	*0.250000*	*0.360000*	*0.490000*	*0.640000*	*0.810000*

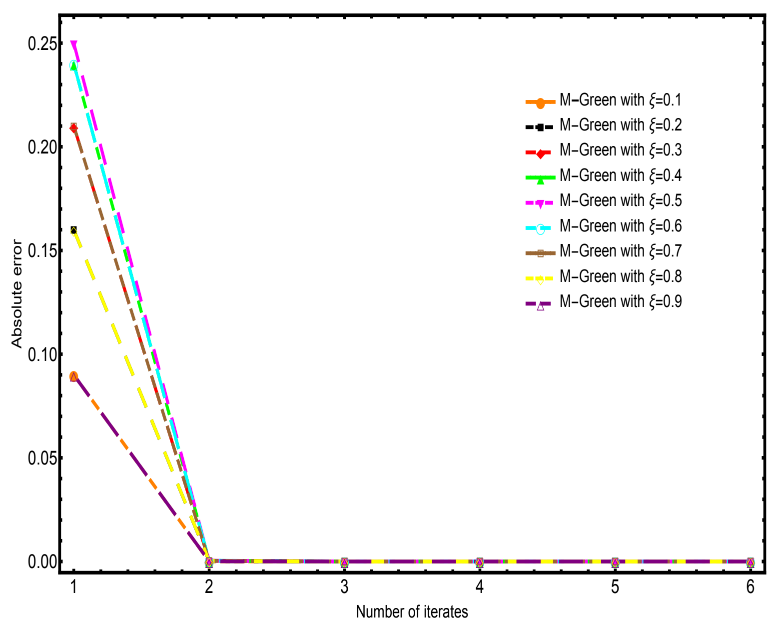

Figure 2. Behaviors of M-Green's iterations for different choices of ξ.

Now, we choose $a_n = 0.9$, we obtain the absolute errors given in the Table 3, for different values of ξ. Clearly our proposed scheme converges fast to the solution. The graphical view is provided in the Figure 3.

Table 3. Absolute errors comparison of different iterative schemes.

ξ	*Exact solution*	**Mann-Green**	**Khan-Green**	**M-Green**
0.1	*0.01*	4.5701×10^{-6}	9.89994×10^{-10}	4.05925×10^{-16}
0.2	*0.04*	8.84515×10^{-6}	1.95814×10^{-9}	8.11851×10^{-16}
0.3	*0.09*	0.0000126001	2.84931×10^{-9}	1.16573×10^{-15}
0.4	*0.16*	0.0000155043	3.57659×10^{-9}	1.47105×10^{-15}
0.5	*0.25*	0.0000171762	4.03244×10^{-9}	1.60982×10^{-15}
0.6	*0.36*	0.000017254	4.10819×10^{-9}	1.66533×10^{-15}
0.7	*0.49*	0.0000154828	3.72156×10^{-9}	1.44329×10^{-15}
0.8	*0.64*	0.0000118068	2.84858×10^{-9}	1.22125×10^{-15}
0.9	*0.81*	6.45478×10^{-6}	1.55255×10^{-9}	6.66134×10^{-16}

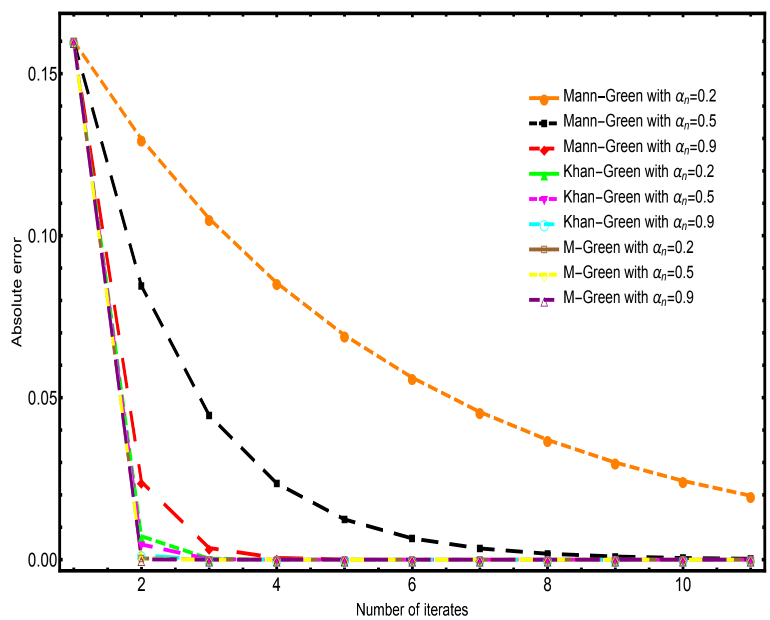

Figure 3. Speed of convergence comparison of different iterative schemes.

Example 5.2. *We now offer a new example to show once again the high accuracy of our scheme on the other iterative schemes of the literature.*

$$v''(\xi) - \frac{3}{2}v(\xi)^2 = 0, \tag{5.4}$$

$$v(0) = 4 \quad \text{and} \quad v(1) = 1. \tag{5.5}$$

In this case, the exact solution of the Problem (5.4)–(5.5) is given by $v(\xi) = \frac{4}{(1+\xi)^2}$. *This problem is solved numerically by Khuri and Louhichi [21] by using Mann-Green's and Ishikawa Green's iterative schemes. We now improve their findings by our M-Green's iterative scheme. By choosing* $v_0(\xi) = 4 - 3\xi$, *and the results are displayed in the next table.*

Now, using Example 5.2, our proposed M-Green's iterative scheme takes the following form:

$$\begin{cases} w_n = v_n + a_n \int_0^{\xi} \eta(1-\xi)[v_n''(\eta) - \frac{3}{2} v_n(\eta)^2] d\eta, \\ +a_n \int_{\xi}^{1} \xi(1-\eta)[v_n''(\eta) - \frac{3}{2} v_n(\eta)^2] d\eta, \\ u_n = w_n + \int_0^{\xi} \eta(1-\xi)[w_n''(\eta) - \frac{3}{2} w_n(\eta)^2] d\eta, \\ + \int_{\xi}^{1} \xi(1-\eta)[w_n''(\eta) - \frac{3}{2} w_n(\xi)^2] d\eta, \\ v_{n+1} = u_n + \int_0^{\xi} \eta(1-\xi)[u_n''(\eta) - \frac{3}{2} u_n(\eta)^2] d\eta, \\ + \int_{\xi}^{1} \xi(1-\eta)[u_n''(\eta) - \frac{3}{2} u_n(\eta)^2] d\eta. \end{cases} \tag{5.6}$$

Now, we choose $a_n = 0.5$*, then again, we see that our scheme (5.6) converges faster to the solution of the Problem (5.4)–(5.5) as compared the other two schemes, see Table 4 and Figure 4.*

Table 4. Absolute errors comparison of different iterative schemes.

ξ	*Exact solution*	**Mann-Green**	**Khan-Green**	**M-Green**
0.1	*3.30579*	*0.002034250*	6.4303×10^{-6}	3.89832×10^{-7}
0.2	*2.77778*	0.001486440	0.0000117717	7.15245×10^{-7}
0.3	*2.36686*	0.000447975	0.0000154153	9.39372×10^{-7}
0.4	*2.04082*	0.000360256	0.0000171661	1.04929×10^{-6}
0.5	*1.77778*	0.000758254	0.0000171076	1.04871×10^{-6}
0.6	*1.56250*	0.000780165	*0.0000154865*	9.51631×10^{-7}
0.7	*1.38408*	*0.000546944*	*0.0000126278*	7.77381×10^{-7}
0.8	*1.23457*	0.000217887	8.87776×10^{-6}	5.47131×10^{-7}
0.9	*1.10803*	0.000027884	4.56883×10^{-6}	2.81581×10^{-7}

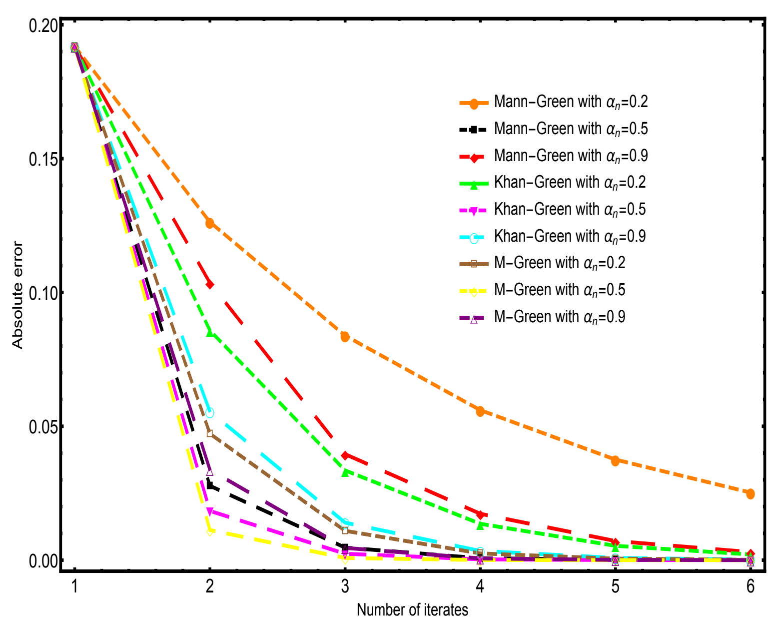

Figure 4. Speed of convergence comparison of different iterative schemes.

6. Conclusion

We provided a new iterative scheme for approximating solutions of BVPs based on Green's function, so-called an M-Green's iterative scheme. We assumed also some possible mild conditions and proved the convergence and stability of this scheme in the setting of Banach space. An example is suggested and proved that its M-Green's iterative scheme suggests high accurate results as compared the Mann-Green's and Khan-Green's iterative schemes. Also this is the first paper which suggested a three-step iterative scheme based on Green's function opposed to the one-step and two-step iterative schemes studied by former authors. Hence, our results improved and extended many previous results such as Ali et al. [22] and Khuri–Louhichi [21].

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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