Int. J. Anal. Appl. (2023), 21:128

International Journal of Analysis and Applications

Cubic Pythagorean Hesitant Fuzzy Linear Spaces and Its Relevance in Multi Criteria Decision Making

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Abstract. Pythagorean fuzzy sets and interval valued Pythagorean fuzzy sets have an important role in decision making techniques. Pythagorean hesitant fuzzy sets are time and again used in dealing with uncertain and vague data. The motive of this paper is to introduce the notion cubic Pythagorean hesitant fuzzy linear spaces. We also present the notion of $P(R)$-intersection, $P(R)$-union of cubic Pythagorean hesitant fuzzy linear spaces with examples. Secondly, a series of operators like cubic Pythagorean hesitant fuzzy weighted averaging aggregation operators, cubic Pythagorean hesitant fuzzy order weighted averaging aggregation operators and cubic Pythagorean hesitant fuzzy hybrid order weighted averaging aggregation operators are developed. Then, these aggregation operators are further extended to cubic Pythagorean hesitant fuzzy prioritized weighted averaging aggregation operators by assigning priorities to the criteria. A real life MCDM problem has been illustrated and the effectiveness of the results are compared with those solved using cubic picture hesitant fuzzy prioritized weighted averaging aggregation operators.

1. Introduction

In consideration with the imprecision data in decision making L.A.Zadeh [16] introduced the idea of fuzzy sets that includes membership function that attributes to each element a membership value

Received: Sep. 10, 2023.

2020 *Mathematics Subject Classification.* Primary 46B20; Secondary 46B99, 46A45, 46B45.

Key words and phrases. cubic Pythagorean hesitant fuzzy linear spaces; cubic Pythagorean hesitant fuzzy weighted; order weighted and hybrid averaging aggregation operators.

https://doi.org/10.28924/2291-8639-21-2023-128

ISSN: 2291-8639

in the closed interval 0 and 1. For instance, if the membership value is 0 then the decision makers are not interested in including that particular criteria in decision making problems and if it is 1 then the decision makers are totally in agreement to handle a real life MCDM problem with that criteria. Preferably, fuzzy input can be of two forms, one is quantifiable and the other is qualitative. The quantifiable fuzzy input can further be dealt by FS, IVFS, IFS [2], cubic sets [6], PFS [4], PyFS [15], hesitant fuzzy sets [13] and so on. But fuzzy sets involves only membership function. Attansov [2] presented intuitionistic fuzzy sets that consists both membership function as well as non membership function with the condition that sum of the two degrees do not go beyond 1. Now at times the sum exceeds the restriction, but sum of their squares is ≤ 1. And thus led to the initiation of new concept Pythagorean fuzzy sets by Yager [15]. In this way Pythagorean fuzzy sets can process the fuzzy input with greater potent than intuitionistic fuzzy sets. For instance, if a decision maker allocates favourable membership value of 0.8 and non membership value of 0.9. Undeniably, intuitionistic fuzzy sets is unable to process this input but can be approached successfully approached by Pythagorean fuzzy sets. Through the concept of hybrid cubic structure proposed by Y.B.Jun [6] Cubic Pythagorean fuzzy sets are introduced.

In recent times fuzzy linear spaces have been gaining importance and their extensions include Interval valued fuzzy linear spaces, cubic linear spaces [13], cubic Γ $n-$normed linear spaces [7], $N-$cubic sets applied to linear spaces [8], cubic picture fuzzy linear spaces [9], cubic Pythagorean fuzzy linear spaces [10]. The purpose of this paper is to introduce the notion of cubic Pythagorean hesitant fuzzy linear spaces, combining interval-valued Pythagorean hesitant fuzzy linear space and Pythagorean hesitant fuzzy linear space, and to discuss properties with examples.

MCDM is a determining branch of the decision-making theory which has been widely used in solving real life problems using various aggregation operators developed by researchers. To handle the MCDM problems more effectively aggregation operators namely weighted picture fuzzy aggregation operators [3], Hesitant Pythagorean fuzzy weighted averaging (HPFWA) [5], Hesitant Pythagorean fuzzy weighted geometric (HPFWG), Hesitant Pythagorean fuzzy order weighted averaging (HPFOWA, Pythagorean Fuzzy Dombi Aggregation Operators [1], Hesitant Pythagorean fuzzy order weighted geometric averaging (HPFOWG) were introduced based on Pythagorean fuzzy environments. Later on, khan et al. developed prioritized aggregation operators for MCDM problems on Pythagorean fuzzy set by setting up priorities for the criteria.

In this paper we introduce the notion of cubic Pythagorean hesitant fuzzy linear spaces, combining interval-valued Pythagorean hesitant fuzzy linear space and Pythagorean hesitant fuzzy linear space, and to discuss prudent properties with examples. Taking into account the above mentioned theory, this paper proposes cubic Pythagorean hesitant fuzzy prioritized weighted averaging aggregation operators, cubic Pythagorean hesitant fuzzy prioritized order weighted averaging aggregation operators and cubic Pythagorean hesitant fuzzy prioritized hybrid weighted averaging aggregation operators. In addition, to this using the above referred operators a real-life MCDM problem has been evaluated and the results are compared with those based on operators in picture fuzzy environment.

2. Preliminaries

Definition 2.1. *A hesitant fuzzy set on a non empty set H is defined as a function $\hat{m}(h)$ while applied to H which returns a finite subset in $[0,1]$ that is*

$$m_f = \{< h, \hat{m}(h) > \mid\ h \in H\}$$

where $\hat{m}(h)$ is a set of some different values in $[0,1]$ which denotes the possible membership degrees of $h \in H$ to the set m_f.

Definition 2.2. *An interval valued hesitant fuzzy set $\widetilde{m_f}$ on non empty set H is defined as*

$$\widetilde{m_f} = \{< h, \widetilde{\hat{m}}(h) > \big|\ h \in H\}$$

where $\widetilde{\hat{m}}(h)$ is a set of some different interval values in $[0,1]$ which denotes the possible membership degrees of $h \in H$ to the set $\widetilde{m_f}$.

Definition 2.3. *A cubic hesitant fuzzy set $C_{m_f} = \{< h, \widetilde{m_f}(h), m_f(h) > \mid\ h \in H\}$ where $\widetilde{m_f}$ is an interval valued hesitant fuzzy set and m_f is an hesitant fuzzy set and is simply denoted by $C_{m_f} =< \widetilde{m_f}, m_f >$.*

Definition 2.4. *A Pythagorean fuzzy Set P_y on a non empty set H can be defined as*

$$P_y = \{(h, \eta_{p_y}(h), \theta_{p_y}(h)) | h \in H\}$$

where $\eta_{p_y} : H \to [0,1]$ expresses the degree of membership of $h \in H$ and $\theta_{p_y} : H \to [0,1]$ expresses the degree of non membership of $h \in H$ satisfying the condition that $0 \le \eta_{p_y}(h) \le 1$, $0 \le \theta_{p_y}(h) \le 1$ and $0 \le \eta^2_{p_y}(h) + \theta^2_{p_y}(h) \le 1$ for all $h \in H$.
Now the degree of indeterminacy of h to P is given as

$$D_{py}(h) = \sqrt{1 - \eta^2_{p_y}(h) - \theta^2_{p_y}(h)}$$

$D_{p_y}(h)$ satisfies the condition that $0 \le D_{p_y}(h)$ for every $h \in H$.

Definition 2.5. *An interval valued Pythagorean fuzzy set on H is described as*

$$\widetilde{P_y} = \{(h, \widetilde{\eta_{p_y}}(h), \widetilde{\theta_{p_y}}(h) | h \in H)\}$$

where $\widetilde{\eta_{p_y}}(h) = [\eta^l_{p_y}(h), \eta^u_{p_y}(h)] \subset [0,1]$ and $\widetilde{\theta_{p_y}}(h) = [\theta^l_{p_y}(h), \theta^u_{p_y}(h)] \subset [0,1]$ with $\eta^l_{p_y}(h) = inf\ \eta_{p_y}(h)$ and $\eta^u_{p_y}(h) = sup\ \eta_{p_y}(h)$ like wise $\theta^l_{p_y}(h) = inf\ \theta_{p_y}(h)$ and $\theta^u_{p_y}(h) = sup\ \theta_{p_y}(h)$.

Definition 2.6. *A cubic Pythagorean fuzzy set of H is a structure mentioned as*

$$C_{P_y} = \{h, \widetilde{P_y}(h), P_y(h) | h \in H\},$$

in which $\widetilde{P_y}$ is an interval valued Pythagorean fuzzy set in H and P_y is a Pythagorean fuzzy set in H.

Definition 2.7. *A Pythagorean hesitant fuzzy set P_{y_h} on a non empty set H is described as*

$$P_{y_h} = \{(h, \hat{\eta}_{p_y}(h), \hat{\theta}_{p_y}(h)) \,\big|\, h \in H\}$$

where $\hat{\eta}_{p_y}(h), \hat{\theta}_{p_y}$ represents different figures of membership and non membership degrees in $[0,1]$.

Definition 2.8. *An interval valued Pythagorean hesitant fuzzy set on a non empty set H is described as*

$$\widetilde{P_{y_h}} = \{(h, \tilde{\hat{\eta}}_{p_y}(h), \tilde{\hat{\theta}}_{p_y}(h) \,\big|\, h \in H\}$$

where $\tilde{\hat{\eta}}_{p_y}(h), \tilde{\hat{\theta}}_{p_y}(h)$ gives different interval values of membership and non membership degrees in $[0,1]$.

Definition 2.9. *A cubic Pythagorean hesitant fuzzy set on a non empty set H is described as*

$$C_{P_{y_h}} = \{h, \widetilde{P_{y_h}}, P_{y_h} \,\big|\, h \in H\}$$

where $\widetilde{P_{y_h}}$ is an interval valued Pythagorean hesitant fuzzy set in H and P_{y_h} is a Pythagorean hesitant fuzzy set in H. Simply described as cubic Pythagorean hesitant fuzzy set as $C_{P_{y_h}} = (\widetilde{P_{y_h}}, P_{y_h})$.

Definition 2.10. *Let $C_{P_{y_{h1}}} = \{\widetilde{P_{y_{h_1}}}, P_{y_{h_1}}\}$ and $C_{P_{y_{h2}}} = \{\widetilde{P_{y_{h_2}}}, P_{y_{h_2}}\}$ be two cubic Pythagorean hesitant fuzzy sets then*

(i) **P-Union**

$$C_{P_{y_{h1}}} \cup_p C_{P_{y_{h2}}} = \Big\{\big(l, \gamma \in \widetilde{P_{y_{h_1}}} \cup \widetilde{P_{y_{h_2}}}(l), \kappa \in P_{y_{h_1}} \cup P_{y_{h_2}} \,\Big|\, \gamma = [\gamma^-, \gamma^+] \geq \max\{[\tilde{\hat{\eta}}_i^-(l), \tilde{\hat{\eta}}_i^+(l)]^-, [\tilde{\hat{\theta}}_i^-(l), \tilde{\hat{\theta}}_i^+(l)]^-\}, \kappa \geq \max\{\hat{\eta}^-(l), \hat{\theta}^-(l)\}\big)\Big\}.$$

(ii) **P-Intersection**

$$C_{P_{y_{h1}}} \cap_p C_{P_{y_{h2}}} = \Big\{\big(l, \gamma \in \widetilde{P_{y_{h_1}}} \cup \widetilde{P_{y_{h_2}}}(l), \kappa \in P_{y_{h_1}} \cup P_{y_{h_2}} \,\Big|\, \gamma = [\gamma^-, \gamma^+] \leq \min\{[\tilde{\hat{\eta}}_i^-(l), \tilde{\hat{\eta}}_i^+(l)]^+, [\tilde{\hat{\theta}}_i^-(l), \tilde{\hat{\theta}}_i^+(l)]^+\}, \kappa \leq \min\{\hat{\eta}^+(l), \hat{\theta}^+(l)\}\big)\Big\}.$$

(iii) **R-Union**

$$C_{P_{y_{h1}}} \cup_R C_{P_{y_{h2}}} = \Big\{\big(l, \gamma \in \widetilde{P_{y_{h_1}}} \cup \widetilde{P_{y_{h_2}}}(l), \kappa \in P_{y_{h_1}} \cup P_{y_{h_2}} \,\Big|\, \gamma = [\gamma^-, \gamma^+] \geq \max\{[\tilde{\hat{\eta}}_i^-(l), \tilde{\hat{\eta}}_i^+(l)]^-, [\tilde{\hat{\theta}}_i^-(l), \tilde{\hat{\theta}}_i^+(l)]^-\}, \kappa \leq \min\{\hat{\eta}^+(l), \hat{\theta}^+(l)\}\big)\Big\}.$$

(iii) **R-Intersection**

$$C_{P_{y_{h1}}} \cap_R C_{P_{y_{h2}}} = \Big\{\big(l, \gamma \in \widetilde{P_{y_{h_1}}} \cup \widetilde{P_{y_{h_2}}}(l), \kappa \in P_{y_{h_1}} \cup P_{y_{h_2}} \,\Big|\, \gamma = [\gamma^-, \gamma^+] \leq \min\{[\tilde{\hat{\eta}}_i^-(l), \tilde{\hat{\eta}}_i^+(l)]^+, [\tilde{\hat{\theta}}_i^-(l), \tilde{\hat{\theta}}_i^+(l)]^+\}, \kappa \geq \max\{\hat{\eta}^-(l), \hat{\theta}^-(l)\}\big)\Big\}.$$

3. Results

Definition 3.1. *For a non empty linear space $\pounds$ over a field F a hesitant fuzzy linear space is defined as a pair $\pounds^{h_f}$=$(\pounds, \hat{h})$ and $\hat{h} : \pounds \to \phi([0,1])$ where $\phi([0,1])$ denote set all finite subsets of* $[0,1]$ *and satisfy the following condition*

$$\pounds^{h_f}(\alpha l_1 * \beta l_2) \geq \pounds^{h_f}(l_1) \cap \pounds^{h_f}(l_2).$$

for $l_1, l_2 \in \pounds$ *and* $\alpha, \beta \in F$.

Definition 3.2. *For a non empty linear space* $\pounds$ *over a field* F *an interval valued hesitant fuzzy linear space is defined as a pair* $\widetilde{\pounds^{h_f}}=(\pounds, \tilde{\tilde{h}})$ *and* $\tilde{\tilde{h}} : \pounds \to D(\phi([0,1]))$ *where* $D(\phi([0,1]))$ *denote collection of all finite sub intervals of* $[0,1]$ *and satisfy following condition*

$$\widetilde{\pounds^{h_f}}(\alpha l_1 * \beta l_2) \geq \min\{\widetilde{\pounds^{h_f}}(l_1), \widetilde{\pounds^{h_f}}(l_2)\}$$

for $l_1, l_2 \in \pounds$ *and* $\alpha, \beta \in F$.

Definition 3.3. *For a non empty linear space* $\pounds$ *over a field* F *a Cubic hesitant fuzzy set* $C_{h_f}=\{< m, \widetilde{h_f}(m), h_f(m) > | m \in M\}$ *is said to be a Cubic hesitant fuzzy linear space of* $\pounds$ *if the following conditions are satisfied*

$$\widetilde{h_f}(\alpha l_1 * \beta l_2) \geq \min\{\widetilde{h_f}(l_1), \widetilde{h_f}(l_2)\}$$

$$h_f(\alpha l_1 * \beta l_2) \leq \max\{h_f(l_1), h_f(l_2)\}$$

Definition 3.4. *For a non empty linear space* $\pounds$ *over a field* F *a Pythagorean hesitant fuzzy linear space is defined as a pair* $\pounds^{P_{y_h}} = (\pounds, \hat{\eta}, \hat{\theta})$ *where* $\hat{\mu} : \pounds \to \phi([0,1])$ *is the degree of positive membership,* $\hat{\eta} : \pounds \to \phi([0,1])$ *is the degree of membership and* $\hat{\theta} : \pounds \to \phi([0,1])$ *is the degree of non membership and satisfy the following condition*

$$\pounds^{P_{y_h}}(\alpha l_1 * \beta l_2) \geq \pounds^{P_{y_h}}(l_1) \cap \pounds^{P_{y_h}}(l_2)$$

for any $l_1, l_2 \in \pounds$ *and* $\alpha, \beta \in F$.

Definition 3.5. *For a non empty linear space* $\pounds$ *over a field* F *an interval valued Pythagorean hesitant fuzzy linear space is defined as pair* $\widetilde{\pounds^{P_{y_h}}} = (\pounds, \tilde{\hat{\eta}}, \tilde{\hat{\theta}})$ *where* $\tilde{\hat{\eta}} : \pounds \to \phi(D[0,1])$ *and* $\tilde{\hat{\vartheta}} : \pounds \to \phi(D[0,1])$ *are the degree of membership and non membership degrees respectively and satisfy the following condition*

$$\widetilde{\pounds^{P_{y_h}}}(\alpha l_1 * \beta l_2) \geq min\{\widetilde{\pounds^{P_{y_h}}}(l_1), \widetilde{\pounds^{P_{y_h}}}(l_2)\}$$

for any $l_1, l_2 \in \pounds$ *and* $\alpha, \beta \in F$.

Definition 3.6. *For a non empty linear space* $\pounds$ *over a field* F *a Cubic Pythagorean hesitant fuzzy set* $C_{P_{y_h}} = \{m, \widetilde{P_{y_h}}(m), P_{y_h}(m) | m \in M\}$ *is said to be a Cubic Pythagorean hesitant fuzzy linear space of* $\pounds$ *if the following conditions are satisfied*

$$\widetilde{P_{y_h}}(\alpha l_1 * \beta l_2) \geq \min\left(\widetilde{P_{y_h}}(l_1), \widetilde{P_{y_h}}(l_2)\right)$$

$$P_{y_h}(\alpha l_1 * \beta l_2) \leq \max\left(P_{y_h}(l_1), P_{y_h}(l_2)\right)$$

Example 3.1. *The example provided below explains the above definition*

Table 1. Values of interval valued Pythagorean hesitant fuzzy sets and Picture hesitant Fuzzy sets

	$\widetilde{P_{y_h}}$	P_{y_h}
l_1	$\left\{\begin{array}{l}([0.6,\ 0.8],\ [0.5,\ 0.6]),\\ ([0.3,\ 0.5],\ [0.5,\ 0.8])\end{array}\right.$	$\left\{\begin{array}{l}(0.09,\ 0.21),\\ (0.32,\ 0.43)\end{array}\right.$
l_1	$\left\{\begin{array}{l}([0.5,\ 0.7],\ [0.4,\ 0.5]),\\ ([0.1,\ 0.2],\ [0.3,\ 0.62])\end{array}\right.$	$\left\{\begin{array}{l}(0.5,\ 0.6),\\ (0.19,\ 0.31)\end{array}\right.$
l_3	$\left\{\begin{array}{l}([0.6,\ 0.8],\ [0.6,\ 0.8]),\\ ([0.6,\ 0.8],\ [0.53,\ 0.85])\end{array}\right.$	$\left\{\begin{array}{l}(0.5,\ 0.7),\\ (0.62,\ 0.75)\end{array}\right.$

From the above table we notice that $\widetilde{P_{y_h}}$ satisfies the condition required for it to be an inter valued Pythagorean hesitant fuzzy linear space and likewise P_{y_h} is a Pythagorean hesitant fuzzy linear space of £ over field $GF(2)$ with the binary operation $l_2 * l_3 = l_1$ and $\beta = 1, \alpha = 0$. And hence the above example in fact satisfied the conditions required for the Cubic Pythagorean Hesitant Fuzzy set to be a Cubic Pythagorean Hesitant Fuzzy linear space.

3.1. **Internal and external Cubic Pythagorean hesitant fuzzy linear spaces.**

3.2. **Definition.** *A cubic Pythagorean hesitant fuzzy set $C_{P_{y_h}} = \{m, \widetilde{P_{y_h}}(m), P_{y_h}(m) | m \in M\}$ is said to be an internal cubic Pythagorean hesitant fuzzy linear space(shortly, ICPyHFLS) if*

$$(\widetilde{P_{y_h}})^-(\alpha l_1 * \beta l_2) \leq P_{y_h}(\alpha l_1 * \beta l_2) \leq (\widetilde{P_{y_h}})^+(\alpha l_1 * \beta l_2)$$

for all $l_1, l_2 \in$ and $\alpha, \beta \in F$.

3.3. **Example.** *Let us consider the values of interval valued Pythagorean hesitant fuzzy set and Pythagorean hesitant fuzzy set as in table 1. Now for $\beta = \gamma = 1$ and $l_2 * l_3 = l_1$ in 3.2 we have*

$$(\widetilde{P_{y_h}})^-(l_2 * l_3) \leq P_{y_h}(l_2 * l_3) \leq (\widetilde{P_{y_h}})^+(l_2 * l_3)$$
$$(\widetilde{P_{y_h}})^-(l_1) \leq P_{y_h}(l_1) \leq (\widetilde{P_{y_h}})^+(l_1)$$

$\Rightarrow 0.32 \in [0.3, 0.5], 0.43 \in [0.4, 0.8], 0.19 \in [0.1, 0.2], 0.31 \in [0.3, 0.62], 0.62 \in [0.6, 0.8], 0.75 \in [0.53, 0.85]$. *Hence $C_{P_{y_h}} = \{m, \widetilde{P_{y_h}}(m), P_{y_h}(m) | m \in M\}$ is an internal cubic picture hesitant fuzzy linear space.*

3.4. **Definition.** *A cubic Pythagorean hesitant fuzzy set $C_{P_{y_h}} = \{m, \widetilde{P_{y_h}}(m), P_{y_h}(m) | m \in M\}$ in a linear space over a field F is said to be an external cubic hesitant Pythagorean fuzzy linear space(shortly, ECHPyFLS) if*

$$P_{y_h}(\alpha l_1 * \beta l_2) \notin \left(\left(\widetilde{P_{y_h}}\right)^-(\alpha l_1 * \beta l_2), \left(\widetilde{P_{y_h}}\right)^+(\alpha l_1 * \beta l_2) \right)$$

for all $l_1, l_2 \in$ and $\alpha, \beta \in F$.

3.5. **Example.** *Let us consider the values of interval valued hesitant Pythagorean fuzzy set and hesitant Pythagorean fuzzy set as in table 1. Now for $\alpha = \beta = 1$ and $l_2 * l_3 = l_1$ in 3.4 we have*

$$P_{y_h}(l_2 * l_3) \notin \left((\widetilde{P_{y_h}})^-(l_2 * l_3), (\widetilde{P_{y_h}})^+(l_2 * l_3) \right)$$
$$P_{y_h}(l_1) \notin \left((\widetilde{P_{y_h}})^-(l_1), (\widetilde{P_{y_h}})^+(l_1) \right)$$

$\Rightarrow 0.09 \notin [06, 0.8]$ *and* $0.21 \notin [0.5, 0.6]$. *Hence* $C_{P_{y_h}} = \{\widetilde{P_{y_h}}, P_{y_h}\}$ *is an external cubic hesitant Pythagorean fuzzy linear space.*

3.6. **Proposition.** *Let* $C_{P_{y_{h1}}} = \{\widetilde{P_{y_{h_1}}}, P_{y_{h_1}}\}$ *and* $C_{P_{y_{h2}}} = \{\widetilde{P_{y_{h_2}}}, P_{y_{h_2}}\}$ *be two ICHPyFLS. Then their $P-$union and $P-$intersection is again an ICHPyFLS.*

3.7. **Proof.** *Since* $C_{P_{y_{h1}}} = \{\widetilde{P_{y_{h_1}}}, P_{y_{h_1}}\}$ *and* $C_{P_{y_{h2}}} = \{\widetilde{P_{y_{h_2}}}, P_{y_{h_2}}\}$ *are ICHPyFLS in £, we have*

$$(\widetilde{P_{y_{h1}}})^-(\alpha l_1 * \beta l_2) \leq P_{y_{h1}}(\alpha l_1 * \beta l_2) \leq (\widetilde{P_{y_{h1}}})^+(\alpha l_1 * \beta l_2)$$
$$(\widetilde{P_{y_{h_2}}})^-(\alpha l_1 * \beta l_2) \leq P_{y_{h_2}}(\alpha l_1 * \beta l_2) \leq (\widetilde{P_{y_{h_2}}})^+(\alpha l_1 * \beta l_2)$$

for all $l_1, l_2 \in$ *and* $\alpha, \beta \in F$.
Thus we obtain that

$$(\widetilde{P_{y_{h1}}} \cup \widetilde{P_{y_{h2}}})^-(\alpha l_1 * \beta l_2) \leq P_{y_{h1}} \cup P_{y_{h2}}(\alpha l_1 * \beta l_2) \leq (\widetilde{P_{y_{h1}}} \cup \widetilde{P_{y_{h2}}})^+(\alpha l_1 * \beta l_2)$$
$$(\widetilde{P_{y_{h1}}} \cap \widetilde{P_{y_{h1}}})^-(\alpha l_1 * \beta l_2) \leq P_{y_{h1}} \cap P_{y_{h1}}(\alpha l_1 * \beta l_2) \leq (\widetilde{P_{y_{h1}}} \cap \widetilde{P_{y_{h1}}})^+(\alpha l_1 * \beta l_2)$$

Hence $P-$union and $P-$intersection of $C_{P_{y_{h1}}} = \{\widetilde{P_{y_{h1}}}, P_{y_{h1}}\}$ *and* $C_{Ph_{f2}} = \{\widetilde{P_{y_{h1}}}, P_{y_{h1}}\}$ *is again an ICHPyFLS.*

4. Score function and cubic Pythagorean hesitant fuzzy weighted averaging aggregation operators

Let$C_{P_{y_h}} = \{m, \widetilde{P_{y_h}}(m), P_{y_h}(m) | m \in M\}$ *be a cubic Pythagorean hesitant fuzzy linear space of £ over field F. Then*

$$Sc(C_{P_{y_h}}) = \frac{1}{2n}\left[\frac{1}{2n}\sum_{i=1}^{n}\left((\hat{\eta}_i{}^-)^2 + (\hat{\eta}_i{}^+)^2 - (\hat{\theta}_i^-)^2 - (\hat{\theta}_i^+)^2\right) + (\hat{\eta}_i)^2 - (\hat{\theta}_i)^2\right]$$

Definition 4.1. *Let* $C_{P_{y_{h_k}}} = (\widetilde{P_{y_{h_k}}}, P_{y_{h_k}})(k = 1, 2, \ldots n)$ *be a collection of cubic Pythagorean hesitant fuzzy elements of linear space £ over field F, then their CPyHFWA is described as follows*

$$CPyHFWA(C_{P_{y_{h_1}}}, C_{P_{y_{h_2}}}, \ldots, C_{P_{y_{h_n}}}) = \sum_{k=1}^{n} \tau_k A_k$$

where τ_k $(k = 1, 2, \ldots, n)$ *be the weight vector of* $C_{P_{y_{h_k}}} = (\widetilde{P_{y_{h_k}}}, P_{y_{h_k}})$ *with* $\tau_k \geq 0$ *and* $\sum_{k=1}^{n} \tau_k = 1$.

Definition 4.2. *Let $C_{P_{y_{h_k}}} = (\widetilde{P_{y_{h_k}}}, P_{y_{h_k}})(k = 1, 2, \dots n)$ be a collection of cubic Pythagorean hesitant fuzzy elements of linear space £ over field F, the aggregation of CPyHFWA is described as follows*
$CPyHFWA(C_{P_{y_{h_1}}}, C_{P_{y_{h_2}}}, \dots, C_{P_{y_{h_n}}})=$

$$\left\{\begin{array}{l}\left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta_k}^{-}\right)^2\right)^{\tau_k}}, \sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta_k}^{+}\right)^2\right)^{\tau_k}}\right], \\ \left[\prod_{i=1}^{n}\left(\hat{\theta_k}^{-}\right)^{\tau_k}, \prod_{i=1}^{n}\left(\hat{\theta_k}^{+}\right)^{\tau_k}\right], \\ \left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta_k}\right)^2\right)^{\tau_k}}, \prod_{i=1}^{n}\left(\hat{\theta_k}\right)^{\tau_k}\right]\end{array}\right.$$

Definition 4.3. *$C_{P_{y_{h_{\omega(k)}}}} = (\widetilde{P_{y_{h_k}}}, P_{y_{h_k}})(k = 1, 2, \dots n)$ be a collection of cubic Pythagorean hesitant fuzzy elements of linear space £ over field F, then by using definition 4.1 the aggregation of CPyH-FOWA is described as follows*
$CPyHFOWA(C_{P_{y_{h_1}}}, C_{P_{y_{h_2}}}, \dots, C_{P_{y_{h_n}}})=$

$$\left\{\begin{array}{l}\left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta}_{\omega(k)}^{-}\right)^2\right)^{\tau_k}}, \sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta}_{\omega(k)}^{+}\right)^2\right)^{\tau_k}}\right], \\ \left[\prod_{i=1}^{n}\left(\hat{\theta}_{\omega(k)}^{-}\right)^{\tau_k}, \prod_{i=1}^{n}\left(\hat{\theta}_{\omega(k)}^{+}\right)^{\tau_k}\right], \\ \left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta}_{\omega(k)}\right)^2\right)^{\tau_k}}, \prod_{i=1}^{n}\left(\hat{\theta}_{\omega(k)}\right)^{\tau_k}\right]\end{array}\right.$$

where $C_{P_{y_{h_{\omega(k)}}}}$ is the k^{th} largest value consequently by total order $C_{P_{y_{h_{\omega(1)}}}} \geq C_{P_{y_{h_{\omega(2)}}}} \geq \dots \geq C_{P_{y_{h_{\omega(n)}}}}$.

Definition 4.4. *Let $C'_{P_{y_{h_{\omega k)}}}} = (\widetilde{P_{y_{h_k}}}, P_{y_{h_k}})(k = 1, 2, \dots n)$ be a collection of cubic Pythagorean hesitant fuzzy elements of linear space £ over field F, then by using definition 4.1 the aggregation of CPyHFHWA is described as follows*
$CPyHFHWA(C_{P_{y_{h_{1)}}}}, C_{P_{y_{h_{2)}}}}, \dots, C_{P_{y_{h_{n)}}}})=$

$$\left\{\begin{array}{l}\left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta}'^{-}_{\omega(k)}\right)^2\right)^{\tau_k}}, \sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta}'^{+}_{\omega(k)}\right)^2\right)^{\tau_k}}\right], \\ \left[\prod_{i=1}^{n}\left(\hat{\theta}'^{-}_{\omega(k)}\right)^{\tau_k}, \prod_{i=1}^{n}\left(\hat{\theta}'^{+}_{\omega(k)}\right)^{\tau_k}\right], \\ \left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta}'_{\omega(k)}\right)^2\right)^{\tau_k}}, \prod_{i=1}^{n}\left(\hat{\theta}'_{\omega(k)}\right)^{\tau_k}\right]\end{array}\right.$$

where $C'_{P_{y_{h_{\omega k)}}}}$ is the k^{th} largest value consequently by total order $C'_{P_{y_{h_{\omega 1)}}}} \geq C'_{P_{y_{h_{\omega 2)}}}} \geq \dots \geq C'_{P_{y_{h_{\omega n)}}}}$ and $C'_{P_{y_{h_{\omega k)}}}} = n\tau_k C_{P_{y_{h_{\omega k)}}}}$ where τ_k are the weight vectors such that $\tau_k \geq 0$.

5. Cubic Pythagorean hesitant fuzzy prioritized weighted averaging aggregation operators

Definition 5.1. *Let* $C_{P_{y_{h_k}}} = (\widetilde{P_{y_{h_k}}}, P_{y_{h_k}})(k = 1, 2, \ldots n)$ *be a collection of cubic Pythagorean hesitant fuzzy elements of linear space £ over field F, then their CPyHFPWA is described as follows*

$$CPyHFPWA(C_{P_{y_{h_1}}}, C_{P_{y_{h_2}}}, \ldots, C_{P_{y_{h_2}}}) = \sum_{k=1}^{n} p_k A_k$$

where $p_k = \frac{S_k}{\sum\limits_{j=1}^{n} S_j}$ $(k = 1, 2, \ldots, n)$ *and* $S_j = \prod\limits_{K=1}^{j-1} Sc(C_{Ph_f})(j = 2, \ldots, n), S_1 = 1.$

Definition 5.2. *Let* $C_{P_{y_{h_k}}} = (\widetilde{P_{y_{h_k}}}, P_{y_{h_k}})(k = 1, 2, \ldots n)$ *be a collection of cubic Pythagorean hesitant fuzzy elements of linear space £ over field F, then by using definition 5.1 the aggregation of CPyHF-PWA is described as follows*

$CPyHFPWA(C_{P_{y_{h_1}}}, C_{P_{y_{h_2}}}, \ldots, C_{P_{y_{h_n}}})=$

$$\left\{\begin{array}{l} \left[\sqrt{1-\prod_{i=1}^{n}\left(1-(\hat{\eta}_k{}^{-})^2\right)^{p_k}}, \sqrt{1-\prod_{i=1}^{n}\left(1-(\hat{\eta}_k{}^{+})^2\right)^{p_k}}\right], \\ \left[\prod_{i=1}^{n}(\hat{\theta}_k{}^{-})^{p_k}, \prod_{i=1}^{n}(\hat{\theta}_k{}^{+})^{p_k}\right], \\ \left[\sqrt{1-\prod_{i=1}^{n}\left(1-(\hat{\eta}_k)^2\right)^{p_k}}, \prod_{i=1}^{n}(\hat{\theta}_k)^{p_k}\right] \end{array}\right.$$

$C_{P_{y_{h_{\omega(k)}}}} = (\widetilde{P_{y_{h_k}}}, P_{y_{h_k}})(k = 1, 2, \ldots n)$ *be a collection of cubic Pythagorean hesitant fuzzy elements of linear space £ over field F, then by using definition 5.1 the aggregation of CPyHFPOWA is described as follows*

$CPyHFPOWA(C_{P_{y_{h_1}}}, C_{P_{y_{h_2}}}, \ldots, C_{P_{y_{h_n}}})=$

$$\left\{\begin{array}{l} \left[\sqrt{1-\prod_{i=1}^{n}\left(1-(\hat{\eta}^{-}_{\omega(k)})^2\right)^{p_k}}, \sqrt{1-\prod_{i=1}^{n}\left(1-(\hat{\eta}^{+}_{\omega(k)})^2\right)^{p_k}}\right], \\ \left[\prod_{i=1}^{n}(\hat{\theta}^{-}_{\omega(k)})^{p_k}, \prod_{i=1}^{n}(\hat{\theta}^{+}_{\omega(k)})^{p_k}\right], \\ \left[\sqrt{1-\prod_{i=1}^{n}\left(1-(\hat{\eta}_{\omega(k)})^2\right)^{p_k}}, \prod_{i=1}^{n}(\hat{\theta}_{\omega(k)})^{p_k}\right] \end{array}\right.$$

where $C_{P_{y_{h_{\omega(k)}}}}$ *is the* k^{th} *largest value consequently by total order* $C_{P_{y_{h_{\omega(1)}}}} \geq C_{P_{y_{h_{\omega(2)}}}} \geq \ldots \geq C_{P_{y_{h_{\omega(n)}}}}$.

Definition 5.3. *Let* $C'_{P_{y_{h_{\omega k)}}}} = (\widetilde{P_{y_{h_k}}}, P_{y_{h_k}})(k = 1, 2, \ldots n)$ *be a collection of cubic Pythagorean hesitant fuzzy elements of linear space £ over field F, then by using definition 5.1 the aggregation of CPyHFPHWA is described as follows*

$CPyHFPHWA(C_{P_{y_{h_{1)}}}}, C_{P_{y_{h_{2)}}}}, \ldots, C_{P_{y_{h_{n)}}}})=$

$$\begin{cases} \left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta}'^{-}_{\omega(k)}\right)^2\right)^{p_k}}, \sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta}'^{+}_{\omega(k)}\right)^2\right)^{p_k}}\right], \\ \left[\prod_{i=1}^{n}\left(\hat{\theta}'^{-}_{\omega(k)}\right)^{p_k}, \prod_{i=1}^{n}\left(\hat{\theta}'^{+}_{\omega(k)}\right)^{p_k}\right], \\ \left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\hat{\eta}'_{\omega(k)}\right)^2\right)^{p_k}}, \prod_{i=1}^{n}\left(\hat{\theta}'_{\omega(k)}\right)^{p_k}\right] \end{cases}$$

where $C'_{P_{y_{h_{\omega k)}}}}$ *is the* k^{th} *largest value consequently by total order* $C'_{P_{y_{h_{\omega 1)}}}} \geq C'_{P_{y_{h_{\omega 2)}}}} \geq \ldots \geq C'_{P_{y_{h_{\omega n)}}}}$ *and* $C'_{P_{y_{h_{\omega k)}}}} = n\tau_k C_{P_{y_{h_{\omega k)}}}}$ *where* τ_k *are the weight vectors such that* $p_k \geq 0$.

6. MCDM method utilizing cubic pythagorean hesitant fuzzy aggregation operators

In this section we utilize the above mentioned operators to deal with the different MCDM problems under CPHF environment. Consider a group of investors looking to invest their money in the best financial platforms F_1, F_2 *and* F_3 *respectively. The required criteria are as follows:*

(1) Low risk factor(C_1*).*

(2) High returns(C_2*).*

(3) Duration of investment(C_3*).*

(4) Ease of settlement(C_4*).*

and the average weights of criteria from the investors are $\tau = (0.2, 0.1, 0.3, 0.4)$. *Now we list the ratings of three alternatives* F_1, F_2 *and* F_3 *under four criteria* C_1, C_2, C_3 *and* C_4 *in the table as below.*

	C_1	C_1
F_1	$([0.1, 0.3], [0.22, 0.34])$	$(0.08, 0.21)$
F_2	$([0.1, 0.4], [0.2, 0.31])$	$(0.05, 0.15)$
F_3	$([0.05, 0.2], [0.2, 0.3])$	$(0.01, 0.32)$

	C_2	C_2
F_1	$([0.1, 0.2], [0.2, 0.4])$	$(0.12, 0.25)$
F_2	$([0.05, 0.15], [0.2, 0.51])$	$(0.08, 0.32)$
F_3	$([0.1, 0.5], [0.2, 0.35])$	$(0.24, 0.33)$

	C_3	C_3
F_1	$([0.20, 0.30], [0.29, 0.39])$	$(0.21, 0.35)$
F_2	$([0.14, 0.15], [0.18, 0.23])$	$(0.18, 0.12)$
F_3	$([0.41, 0.43], [0.27, 0.33])$	$(0.26, 0.43)$

	C_4	C_4
F_1	$([0.39, 0.41], [0.22, 0.29])$	$(0.30, 0.31)$
F_2	$([0.18, 0.19], [0.12, 0.15])$	$(0.24, 0.12)$
F_3	$([0.27, 0.33], [0.25, 0.35])$	$(0.34, 0.36)$

step 1

Table 2. Case 1: Operate the aggregation operator utilizing CPyHFWA operator

I_1	$([0.29, 0.34], [0.23, 0.33])$	$(0.26, 0.28)$
I_2	$([0.16, 0.23], [0.15, 0.22])$	$(0.20, 0.14)$
I_3	$([0.3, 0.37], [0.24, 0.33])$	$(0.28, 0.36)$

Table 3. Case 2: Operate the aggregation operator utilizing CPyHFOWA operator

I_1	$([0.38, 0.32], [0.23, 0.33])$	$(0.28, 0.36)$
I_2	$([0.28, 0.34], [0.23, 0.33])$	$(0.27, 0.28)$
I_3	$([0.18, 0.24], [0.15, 0.22])$	$(0.21, 0.14)$

Table 4. Case 3: Operate the aggregation operator utilizing CPyHFHOWA operator

I_1	$([0.4, 0.41], [0.25, 0.36])$	$(0.37, 0.3)$
I_2	$([0.33, 0.4], [0.25, 0.36])$	$(0.35, 0.3)$
I_3	$([0.23, 0.3], [0.16, 0.24])$	$(0.26, 0.11)$

Table 5. Now we find out the score function as defined above

Case 1	*Case 2*	*Case 3*
$Sc(F_1) = 0.0001$	$Sc(F_1) = 0.015$	$Sc(F_1) = -0.005$
$Sc(F_2) = 0.006$	$Sc(F_2) = 0.013$	$Sc(F_2) = 0.003$
$Sc(F_3) = -0.00591$	$Sc(F_3) = 0.017$	$Sc(F_3) = 0.008$

step 3: Now rank all the alternatives

Table 6. Now rank all the alternatives

	Score	*Ranking*
CPyHFWA	$Sc(F_2) > Sc(F_1) > Sc(F_3)$	$I_2 > I_1 > I_3$
CPyHFOWA	$Sc(F_3) > Sc(F_1) > Sc(F_2)$	$I_3 > I_1 > I_2$
CPyHFHOWA	$Sc(F_3) > Sc(F_2) > Sc(F_1)$	$I_3 > I_2 > I_1$

Now the above mentioned required criteria are prioritized as $C_1 > C_2 > C_3 > C_4$.

Step 1

Table 7. Case 1: Operate the aggregation operator utilizing CPyHFPWA operator

I_1	$([0.5, 0.64], [0.0023, 0.014])$	$(0.46, 0.0056)$
I_2	$([0.3, 0.48], [0.0004, 0.0017])$	$(0.39, 0.00002)$
I_3	$([0.57, 0.68], [0.0024, 0.0092])$	$(0.54, 0.014)$

Case 1
$Sc(F_1) = 0.085$
$Sc(F_2) = 0.075$
$Sc(F_3) = 0.098$

Table 8. Now rank all the alternatives

	Score	*Ranking*
CPyHFPWA	$Sc(F_3) > Sc(F_1) > Sc(F_2)$	$I_3 > I_1 > I_2$

7. Conclusion

In our present work we developed cubic pythagorean hesitant fuzzy linear spaces and defined $P(R)$ intersection, $P(R)$ union of internal and external cubic Pythagorean hesitant fuzzy linear spaces and proved different results with few examples. We defined a series of operators like cubic pythagorean hesitant fuzzy weighted averaging aggregation operators, cubic pythagorean hesitant fuzzy order weighted averaging aggregation operators and cubic pythagorean hesitant fuzzy hybrid order weighted averaging aggregation operators are developed. These aggregation operators are further extended to cubic pythagorean hesitant fuzzy prioritized weighted averaging aggregation operators, cubic pythagorean hesitant fuzzy prioritized order weighted averaging aggregation operators and cubic pythagorean hesitant fuzzy prioritized hybrid order weighted averaging aggregation operators. We presented score functions to find scores of cubic pythagorean hesitant fuzzy linear spaces. We solved multi criteria decision making(MCDM) problem on cubic Pythagorean hesitant fuzzy linear spaces using series of operators defined above. The criteria given by decision makers are prioritized and further solved the MCDM problem using the above defined prioritized operators. The result in both the cases are compared to choose the best criteria among the given. We also observe that the same problem when solved under Picture Hesitant Fuzzy Linear Spaces [11] F_1 was proved to be the best platform which is different from the Pythagorean Hesitant Fuzzy Linear Spaces.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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