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Solvability and Dynamical Analysis of Difference Equations

Mensah Folly-Gbetoula*

School of Mathematics, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa

**Corresponding author:* Mensah.Folly-Gbetoula@wits.ac.za

Abstract. We obtain symmetries of a family of difference equations and we prove a relationship between these symmetries and similarity variables. We proceed with reduction and eventually derive formula solutions of the difference equations. Furthermore, we discuss the periodic nature of the solutions and analyze the stability of the fixed points. We use Lie point symmetry analysis as our tool in obtaining the solutions. Though we have analyzed a specific family of difference equations in this paper, the algorithmic techniques presented can be utilized to tackle many other difference equations.

1. Introduction

The study of differential and difference equations is of a great importance because they describe real life phenomena when the variable involved (usually time) is continuous and discrete, respectively. There are various mathematical methods for solving differential equations. One of the mathematical tools is Lie symmetry analysis. Lie symmetry analysis has recently been applied to difference equations and much progress has been made (see [4,6,11]). The idea behind Lie symmetry analysis is to find the group of transformations that leave the difference equation invariant. When the equation is of lower order, the calculations are not as tedious as compared to the case when the equation has a higher order. A common observation in many papers is that authors present the formula solutions and, then prove generally by induction that the results are correct. Frequently, they use proof by induction as their tool to show that their results are correct. The beauty of Lie symmetry method is that it exhibits the algorithm that leads one to the invariants and similarity variables necessary for the obtention of the solutions. The method was first proposed by Maeda in [9] where the author showed, for first

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order difference equations, that symmetries give analytic expressions of solutions of one-dimensional equations.

Recently, Hydon proposed a systematic algorithm for solving difference equations of any order but mainly applied it to relatively lower order difference equations. Later, some authors successfully derived solutions of higher order using this method [3, 10].

In this paper, we performed an invariance analysis of the recurrence equation

$$x_{n+6k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+4k} x_{n+5k} (A_n + B_n x_n x_{n+k} x_{n+2k})}, \tag{1.1}$$

where A_n and B_n are random sequences; and x_i, $i = 0, 1, \ldots, 6k-1$ are the initial conditions. Eventually, symmetries are derived and formula solutions are obtained. We also investigate the periodic nature of these solutions and discuss the stability of the fixed points admitted by the equation in concern. Finally, we explain how one can use our results to deduce the solutions to the equivalent difference equation

$$x_{n+1} = \frac{x_{n-6k+1} x_{n-5k+1} x_{n-4k+1}}{x_{n-2k+1} x_{n-k+1} (a_n + b_n x_{n-6k+1} x_{n-5k+1} x_{n-4k+1})}, \tag{1.2}$$

a form preferred by some authors. For similar work on difference equations from different approaches, refer to [1, 2, 7, 8, 12].

2. Groundwork

The notation and definitions used in this section are from [6] and [5]. Consider the equation $\mathbb{E}(\mathbf{x}) = 0$ where $\mathbf{x} = (x_1, \ldots, x_N)$ are the continuous variables. The group transformations

$$\Theta_\varepsilon : \mathbf{x} \to \hat{\mathbf{x}}(\mathbf{x}; \varepsilon) \tag{2.1}$$

are said to be a one-parameter (local) Lie group of transformations provided that the following three conditions are satisfied:

(1) Θ_0 is the identity map, so that $\hat{\mathbf{x}} = \mathbf{x}$ when $\varepsilon = 0$.
(2) $\Theta_\gamma \Theta_\varepsilon = \Theta_{\gamma+\varepsilon}$ for every γ, ε sufficiently close to 0.
(3) Every $\hat{x}_i$ can be represented as a Taylor series in ε, that is,

$$\hat{x}_j(\mathbf{x}; \varepsilon) = x_j + \varepsilon \xi_j(\mathbf{x}) + O(\varepsilon^2), \qquad j = 0, 1, \ldots, N.$$

Definition 2.1. *The infinitesimal generator of the one-parameter Lie group of point transformations* (2.1) *is the operator*

$$X = X(x) = \xi(x) \cdot \Delta = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \tag{2.2}$$

and Δ is the gradient operator.

Theorem 2.1. *$F(x)$ is invariant under the Lie group of transformations* (2.1) *if and only if $XF(x) = 0$.*

Given a $6k$th-order difference equation of the form

$$x_{n+6k} = \omega(n, x_n, x_{n+k}, x_{n+2k}, x_{n+4k}, x_{n+5k}) \tag{2.3}$$

for some function ω with the condition that $\partial\omega/\partial x_n \neq 0$, we look for a one-parameter Lie group of point transformations

$$\hat{x}_n = x_n + \varepsilon Q(n, x_n) \tag{2.4}$$

where ε is the group parameter and $Q = Q(n, x_n)$ is the characteristic function. Let

$$X = Q(n, x_n)\frac{\partial}{\partial x_n} + S^k Q(n, x_n)\frac{\partial}{\partial x_{n+k}} + S^{2k} Q(n, x_n)\frac{\partial}{\partial x_{n+2k}} + S^{4k} Q(n, x_n)\frac{\partial}{\partial x_{n+4k}} + S^{5k} Q(n, x_n)\frac{\partial}{\partial x_{n+5k}}$$

be the prolonged generator admitted by the group of point transformations (2.4), where the operator

$$S^i : n \to n + i$$

is referred to as the forward shift operator. Then, the infinitesimal condition for invariance is given by $S^{6k}Q(n, x_n) - X\omega = 0$, that is,

$$S^{6k}Q - \frac{\partial\omega}{\partial x_{n+5k}} S^{5k}Q - \frac{\partial\omega}{\partial x_{n+4k}} S^{4k}Q - \frac{\partial\omega}{\partial x_{n+2k}} S^{2k}Q - \frac{\partial\omega}{\partial x_{n+k}} S^{k}Q - \frac{\partial\omega}{\partial x_{n}} Q = 0 \tag{2.5}$$

as long as (2.3) is satisfied. The functional equation (2.5) is solvable via a proper differential operator and a number of derivations. Next, we introduce the following definitions and theorems indispensable for the study of stability of equilibrium points.

Definition 2.2. *The equilibrium point $\bar{x}$ of (2.3) is locally stable if, for any $\epsilon > 0$ such that if $\{x_n\}_{n=0}^{\infty}$ is a solution of (2.3) with $|x_0 - \bar{x}| + |x_1 - \bar{x}| + \cdots + |x_{6k-2} - \bar{x}| + |x_{6k-1} - \bar{x}| < \delta$, then $|x_n - \bar{x}| < \epsilon$ for all $n \geq 0$.*

Definition 2.3. *The equilibrium point $\bar{x}$ of (2.3) is a global attractor if, for any solution $\{x_n\}_{n=0}^{\infty}$ of (2.3), the limit of x_n is $\bar{x}$ as n approaches infinity.*

Definition 2.4. *The equilibrium point $\bar{x}$ of (2.3) is globally asymptotically stable if $\bar{x}$ is locally stable and is a global attractor of (2.3).*

Letting

$$p_i = \frac{\partial\omega}{\partial x_{n+i}}(\bar{x}, \ldots, \bar{x}),\ \ i = 0, k, 2k, 4k, 5k, \tag{2.6}$$

we obtain the equation

$$\lambda^{6k} - p_{5k}\lambda^{5k} - p_{4k}\lambda^{4k} - p_{2k}\lambda^{2k} - p_k\lambda^k - p_0 = 0 \tag{2.7}$$

known as the characteristic equation of (2.3) about the fixed point $\bar{x}$.

Theorem 2.2. *Suppose ω is a smooth function defined on some open neighborhood of equilibrium point $\bar{x}$. Then the following statements are true:*

(i) *$\bar{x}$ is locally asymptotically stable if all the roots of (2.7) have absolute value less than one.*

(ii) *$\bar{x}$ is unstable if at least one root of (2.7) has absolute value greater than one.*

Definition 2.5. *The equilibrium point $\bar{x}$ of (2.3) is called non-hyperbolic if there exists a root of (2.7) with absolute value equal to one.*

Theorem 2.3. *Suppose that p_0, p_k, p_{2k}, p_{4k} and p_{5k} are real numbers such that*

$$|p_0| + |p_k| + |p_{2k}| + |p_{4k}| + |p_{5k}| < 1.$$

Then, the roots of (2.7) lie inside the open unit disk $|\lambda| < 1$.

3. Symmetries and solutions

In this section, we consider the difference equation

$$x_{n+6k} = \omega = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+4k} x_{n+5k}(A_n + B_n x_n x_{n+k} x_{n+2k})}. \tag{3.1}$$

Imposing the invariance criterion (2.5) to (3.1) yields

$$Q(n+6k, \omega) + \frac{x_n x_{n+k} x_{n+2k} Q(n+5k, x_{n+5k})}{x_{n+4k} x_{n+5k}^2 (B_n x_n x_{n+k} x_{n+2k} + A_n)} + \frac{x_n x_{n+k} x_{n+2k} Q(n+4k, x_{n+4k})}{x_{n+4k}^2 x_{n+5k} (B_n x_n x_{n+k} x_{n+2k} + A_n)} - \frac{A_n}{x_{n+4k} x_{n+5k}} \left[\frac{x_n x_{n+k} Q(n+2k, x_{n+2k})}{(B_n x_n x_{n+k} x_{n+2k} + A_n)^2} + \frac{x_n x_{n+2k} Q(n+k, x_{n+k})}{(B_n x_n x_{n+k} x_{n+2k} + A_n)^2} + \frac{x_{n+k} x_{n+2k} Q(n, x_n)}{(B_n x_n x_{n+k} x_{n+2k} + A_n)^2} \right] = 0. \tag{3.2}$$

Applying the differential operator $\frac{\partial}{\partial x_n} + \frac{A_n x_{n+4k}}{x_n (A_n + B_n x_n x_{n+k} x_{n+2k})} \frac{\partial}{\partial x_{n+4k}}$ to (3.2) and multiplying the resulting equation by $(A_n + B_n x_n x_{n+k} x_{n+2k})^3 / (A_n x_{n+k} x_{n+2k} x_{n+4k} x_{n+5k}^2)$, we get

$$\begin{aligned} & x_{n+4k}(A_n + B_n x_n x_{n+k} x_{n+2k}) Q'(n+4k, x_{n+4k}) - (A_n + B_n x_n x_{n+k} x_{n+2k}) Q(n+4k, x_{n+4k}) \\ & + B_n x_n x_{n+k} x_{n+4k} Q(n+2k, x_{n+2k}) + B_n x_n x_{n+2k} x_{n+4k} Q(n+k, x_{n+k}) - \\ & x_{n+4k}(A_n + B_n x_n x_{n+k} x_{n+2k}) Q'(n, x_n) + x_{n+4k} \left(\frac{A_n}{x_n} + 2 B_n x_{n+k} x_{n+2k} \right) Q(n, x_k) = 0. \end{aligned} \tag{3.3}$$

The notation $'$ stands for the derivative with respect to the continuous variable. The derivation of (3.3) with respect to x_n twice yields

$$-x_{n+k} x_{n+2k} B_n x_n Q'''(n, x_n) + A_n \left(-Q'''(n, x_n) + \frac{1}{x_n} Q''(n, x_n) - \frac{2}{x_n^2} Q'(n, x_n) + \frac{2}{x_n^3} Q(n, x_n) \right) = 0. \tag{3.4}$$

Remembering that the function $Q(n, x_n)$ is independent of the shifts of x_n, we apply the method of separation to get the system of determining equations

$$x_{n+k} x_{n+2k} \text{ terms} : -B_n x_n Q'''(n, x_n) = 0 \tag{3.5}$$

$$1 \text{ terms} : -A_n Q'''(n, x_n) + \frac{A_n}{x_n} Q''(n, x_n) - \frac{2A_n}{x_n^2} Q'(n, x_n) + \frac{2A_n}{x_n^3} Q(n, x_n) = 0 \tag{3.6}$$

that leads to the differential equation

$$x_n^2 Q''(n,x_n) - 2x_n Q'(n,x_n) + 2Q(n,x_n) = 0 \tag{3.7}$$

whose solution is

$$Q(n,u_n) = \beta_n u_n + \gamma_n {u_n}^2 \tag{3.8}$$

for some functions β_n and γ_n which depend on n and are arbitrary. Substituting (3.8) and its shifts in (3.2), and then replacing the expression of u_{n+6k} given in (3.1) in the resulting equation with some bit of simplification leads to

$$\begin{aligned}
&x_n x_{n+k} x_{n+2k} x_{n+4k}^2 x_{n+5k} B_n \gamma_{n+4k} + x_n x_{n+k} x_{n+2k} x_{n+4k} x_{n+5k}^2 B_n \gamma_{n+5k} + x_n x_{n+k} x_{n+2k} x_{n+4k} x_{n+5k} B_n (\beta_{n+4k} \\
&+ \beta_{n+5k} + \beta_{n+6k}) + x_{n+4k} x_{n+5k}^2 A_n (\gamma_{n+4k} + \gamma_{n+5k}) - x_n x_{n+4k} x_{n+5k} A_n \gamma_n - x_{n+k} x_{n+4k} x_{n+5k} A_n \gamma_{n+k} - \\
&x_{n+2k} x_{n+4k} x_{n+5k} A_n \gamma_{n+2k} - x_{n+4k} x_{n+5k} A_n (\beta_n + \beta_{n+k} + \beta_{n+2k} - \beta_{n+4k} - \beta_{n+5k} - \beta_{n+6k}) \\
&+ x_n x_{n+k} x_{n+2k} A_n \gamma_{n+6k} = 0.
\end{aligned} \tag{3.9}$$

Equating the coefficients of all products of powers of shifts of x_n to zero leads to the reduced constraints

$$\begin{aligned}
&\gamma_n = 0, \\
&\beta_n + \beta_{n+k} + \beta_{n+2k} = 0.
\end{aligned} \tag{3.10}$$

The solutions of the $(2k)$th-order linear difference equation above are given by

$$\beta_n = \exp\left\{\frac{i2\pi n(3p \pm 1)}{3k}\right\}, \quad p = 0,1,\ldots,k-1, \tag{3.11}$$

and the characteristics are given by

$$Q(n,x_n) = x_n\beta_n = x_n \exp\left\{\frac{i2\pi n(3p \pm 1)}{3k}\right\}, \quad p = 0,1,\ldots,k-1. \tag{3.12}$$

Thus, the $2k$ symmetry generators are:

$$X_{1p} = x_n \exp\left(\frac{i2\pi n(3p+1)}{3k}\right)\partial_{x_n}, \quad X_{2p} = x_n \exp\left(\frac{i2\pi n(3p-1)}{3k}\right)\partial_{x_n}, \quad p = 0,1,\ldots,k-1. \tag{3.13}$$

The reduction is done using the characteristic $Q(n,x_n) = \beta_n x_n$ given in (3.12) together with the canonical coordinate

$$s_n = \int \frac{dx_n}{\beta_n x_n} = \frac{1}{\beta_n}\ln|x_n|. \tag{3.14}$$

We come up with the invariant function $\tilde{V}_n$ obtained as follows:

$$\tilde{V}_n = s_n\beta_n + s_{n+k}\beta_{n+k} + s_{n+2k}\beta_{n+2k}. \tag{3.15}$$

It is easy to verify that $X_{1p}\tilde{V}_n = X_{2p}\tilde{V}_n = 0$. For the sake of convenience, we introduce the compatible variable

$$|V_n| = \exp\{-\tilde{V}_n\}. \tag{3.16}$$

That is to say, $V_n = \pm 1/(x_n x_{n+k} x_{n+2k})$. Using the plus sign, we have that

$$V_n = \frac{1}{x_n x_{n+k} x_{n+2k}} \tag{3.17}$$

and one can show that

$$V_{n+4k} = A_n V_n + B_n \tag{3.18}$$

and

$$x_{n+3k} = \frac{V_n}{V_{n+k}} x_n. \tag{3.19}$$

Furthermore, by simple iterations, equations (3.18) and (3.19) take the forms

$$V_{4kn+j} \quad = V_j \left(\prod_{k_1=0}^{n-1} A_{4kk_1+j} \right) + \sum_{m=0}^{n-1} \left(B_{4km+j} \prod_{k_2=m+1}^{n-1} A_{4kk_2+j} \right), j = 0, 1, \ldots, 4k-1, \tag{3.20}$$

and

$$x_{3kn+j} \quad = x_j \left(\prod_{s=0}^{n-1} \frac{V_{3ks+j}}{V_{3ks+k+j}} \right), \quad j = 0, 1, \ldots, 3k-1, \tag{3.21}$$

respectively. It follows from (3.21) that

$$\begin{aligned}
x_{12kn+j} &= x_j \left(\prod_{s=0}^{4n-1} \frac{V_{3ks+j}}{V_{3ks+k+j}} \right) \\
&= x_j \prod_{s=0}^{n-1} \frac{V_{12ks+j}}{V_{12ks+k+j}} \frac{V_{12ks+3k+j}}{V_{12ks+3k+k+j}} \frac{V_{12ks+6k+j}}{V_{12ks+6k+k+j}} \frac{V_{12ks+9k+j}}{V_{12ks+9k+k+j}} \\
&= x_j \prod_{s=0}^{n-1} \prod_{r=0}^{3} \frac{V_{12ks+3kr+j}}{V_{12ks+3kr+k+j}} \\
&= x_j \prod_{s=0}^{n-1} \prod_{r=0}^{3} \frac{V_{4k\left(3s+\lfloor \frac{3kr+j}{4k} \rfloor\right)+\tau(3kr+j)}}{V_{4k\left(3s+\lfloor \frac{3kr+k+j}{4k} \rfloor\right)+\tau(3kr+k+j)}},
\end{aligned} \tag{3.22}$$

$j = 0, 1, \ldots, 12k-1$. Note that $\lfloor \cdot \rfloor$ is the floor function and $\tau(a)$ represents the remainder when a is divided by $4k$. Obviously, $0 \leq \tau(a) \leq 4k-1$. Invoking (3.20) in (3.22), we get

$$x_{12kn+j} = x_j \prod_{s=0}^{n-1} \prod_{r=0}^{3} \frac{V_{\tau(3kr+j)} \left(\prod_{k_1=0}^{3s-1+\lfloor \frac{3kr+j}{4k} \rfloor} A_{4kk_1+\tau(3kr+j)} \right) + \sum_{m=0}^{3s-1+\lfloor \frac{3kr+j}{4k} \rfloor} \left(B_{4km+\tau(3kr+j)} \prod_{k_2=m+1}^{3s-1+\lfloor \frac{3kr+j}{4k} \rfloor} A_{4kk_2+\tau(3kr+j)} \right)}{V_{\tau(3kr+k+j)} \left(\prod_{k_1=0}^{3s-1+\lfloor \frac{3kr+k+j}{4k} \rfloor} A_{4kk_1+\tau(3kr+k+j)} \right) + \sum_{m=0}^{3s-1+\lfloor \frac{3kr+k+j}{4k} \rfloor} \left(B_{4km+\tau(3kr+k+j)} \prod_{k_2=m+1}^{3s-1+\lfloor \frac{3kr+k+j}{4k} \rfloor} A_{4kk_2+\tau(3kr+k+j)} \right)}, \tag{3.23}$$

where $V_i = 1/(x_i x_{i+k} x_{i+2k})$. Thus, our solution to the difference equation (1.2) is given by the equation (3.23) as long as the denominators are non-zero.

3.1. **The case A_n and B_n are 1-periodic sequences.** In the special case where the sequences A_n and B_n are 1-periodic sequences, in other words, $A_n = A$ and $B_n = B$, the formula solution (3.23) simplifies considerably to:

$$\begin{aligned} x_{12kn+j} = x_j \prod_{s=0}^{n-1} & \frac{V_{\tau(j)} A^{3s+\lfloor \frac{j}{4k} \rfloor} + B \sum_{m=0}^{3s+\lfloor \frac{j}{4k} \rfloor -1} A^m}{V_{\tau(j+k)} A^{3s+\lfloor \frac{j+k}{4k} \rfloor} + B \sum_{m=0}^{3s+\lfloor \frac{j+k}{4k} \rfloor -1} A^m} \frac{V_{\tau(j+3k)} A^{3s+\lfloor \frac{j+3k}{4k} \rfloor} + B \sum_{m=0}^{3s+\lfloor \frac{j+3k}{4k} \rfloor -1} A^m}{V_{\tau(j)} A^{3s+\lfloor \frac{j}{4k} \rfloor +1} + B \sum_{m=0}^{3s+\lfloor \frac{j}{4k} \rfloor} A^m} \\ & \frac{V_{\tau(j+2k)} A^{3s+\lfloor \frac{j+2k}{4k} \rfloor +1} + B \sum_{m=0}^{3s+\lfloor \frac{j+2k}{4k} \rfloor} A^m}{V_{\tau(j+3k)} A^{3s+\lfloor \frac{j+3k}{4k} \rfloor +1} + B \sum_{m=0}^{3s+\lfloor \frac{j+3k}{4k} \rfloor} A^m} \frac{V_{\tau(j+k)} A^{3s+\lfloor \frac{j+k}{4k} \rfloor +2} + B \sum_{m=0}^{3s+\lfloor \frac{j+k}{4k} \rfloor +1} A^m}{V_{\tau(j+2k)} A^{3s+\lfloor \frac{j+2k}{4k} \rfloor +2} + B \sum_{m=0}^{3s+\lfloor \frac{j+2k}{4k} \rfloor +1} A^m}, \end{aligned} \tag{3.24}$$

for $j = 0, 1, \dots, 12k-1$. It can be shown that $j = 0, 1, \dots, 12k-1$ can be written as $j = 4kr + pk + j_1$ with $r = 0, 1, 2$; $p = 0, 1, 2, 3$; $j_1 = 0, 1, \dots, k-1$. So,

$$\begin{aligned} x_{12kn+4kr+j_1} = x_{4kr+j_1} \prod_{s=0}^{n-1} & \frac{V_{j_1} A^{3s+r} + B \sum_{m=0}^{3s+r-1} A^m}{V_{j_1+k} A^{3s+r} + B \sum_{m=0}^{3s+r-1} A^m} \frac{V_{j_1+3k} A^{3s+r} + B \sum_{m=0}^{3s+r-1} A^m}{V_{j_1} A^{3s+r+1} + B \sum_{m=0}^{3s+r} A^m} \\ & \frac{V_{j_1+2k} A^{3s+r+1} + B \sum_{m=0}^{3s+r} A^m}{V_{j_1+3k} A^{3s+r+1} + B \sum_{m=0}^{3s+r} A^m} \frac{V_{j_1+k} A^{3s+r+2} + B \sum_{m=0}^{3s+r+1} A^m}{V_{j_1+2k} A^{3s+r+2} + B \sum_{m=0}^{3s+r+1} A^m}, \end{aligned} \tag{3.25a}$$

$$\begin{aligned} x_{12kn+4kr+k+j_1} = x_{4kr+k+j_1} \prod_{s=0}^{n-1} & \frac{V_{k+j_1} A^{3s+r} + B \sum_{m=0}^{3s+r-1} A^m}{V_{j_1+2k} A^{3s+r} + B \sum_{m=0}^{3s+r-1} A^m} \frac{V_{j_1} A^{3s+r+1} + B \sum_{m=0}^{3s+r} A^m}{V_{j_1+k} A^{3s+r+1} + B \sum_{m=0}^{3s+r} A^m} \\ & \frac{V_{j_1+3k} A^{3s+r+1} + B \sum_{m=0}^{3s+r} A^m}{V_{j_1} A^{3s+r+2} + B \sum_{m=0}^{3s+r+1} A^m} \frac{V_{j_1+2k} A^{3s+r+2} + B \sum_{m=0}^{3s+r+1} A^m}{V_{j_1+3k} A^{3s+r+2} + B \sum_{m=0}^{3s+r+1} A^m}, \end{aligned} \tag{3.25b}$$

$$x_{12kn+4kr+2k+j_1} = x_{4kr+2k+j_1}\prod_{s=0}^{n-1}\frac{V_{j_1+2k}A^{3s+r}+B\sum_{m=0}^{3s+r-1}A^m}{V_{j_1+3k}A^{3s+r}+B\sum_{m=0}^{3s+r-1}A^m}\frac{V_{j_1+k}A^{3s+r+1}+B\sum_{m=0}^{3s+r}A^m}{V_{j_1+2k}A^{3s+r+1}+B\sum_{m=0}^{3s+r}A^m}$$
$$\frac{V_{j_1}A^{3s+r+2}+B\sum_{m=0}^{3s+r+1}A^m}{V_{j_1+k}A^{3s+r+2}+B\sum_{m=0}^{3s+r+1}A^m}\frac{V_{j_1+3k}A^{3s+r+2}+B\sum_{m=0}^{3s+r+1}A^m}{V_{j_1}A^{3s+r+3}+B\sum_{m=0}^{3s+r+2}A^m}, \tag{3.25c}$$

$$x_{12kn+4kr+3k+j_1} = x_{4kr+3k+j_1}\prod_{s=0}^{n-1}\frac{V_{j_1+3k}A^{3s+r}+B\sum_{m=0}^{3s+r-1}A^m}{V_{j_1}A^{3s+r+1}+B\sum_{m=0}^{3s+r}A^m}\frac{V_{j_1+2k}A^{3s+r+1}+B\sum_{m=0}^{3s+r}A^m}{V_{j_1+3k}A^{3s+r+1}+B\sum_{m=0}^{3s+r}A^m}$$
$$\frac{V_{j_1+k}A^{3s+r+2}+B\sum_{m=0}^{3s+r+1}A^m}{V_{j_1+2k}A^{3s+r+2}+B\sum_{m=0}^{3s+r+1}A^m}\frac{V_{j_1}A^{3s+r+3}+B\sum_{m=0}^{3s+r+2}A^m}{V_{j_1+k}A^{3s+r+3}+B\sum_{m=0}^{3s+r+2}A^m}, \tag{3.25d}$$

for $j_1 = 0, 1, \ldots, k-1$ with $V_i = 1/(x_i x_{i+k} x_{i+2k})$. The above solutions are expressed in terms of x_i, $i = 0, 1, \ldots, 12k-1$ where the first $6k$ terms are the initial conditions. The other $6k$ terms are readily obtained using (3.1) and are given as follows:

$$\begin{aligned}
x_{6k+j_1} &= \frac{x_{j_1}x_{j_1+k}x_{j_1+2k}}{x_{j_1+4k}x_{j_1+5k}(A+Bx_{j_1}x_{j_1+k}x_{j_1+2k})},\\
x_{7k+j_1} &= \frac{x_{3k+j_1}x_{j_1+4k}(A+Bx_{j_1}x_{j_1+k}x_{j_1+2k})}{x_{j_1}(A+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k})},\\
x_{8k+j_1} &= \frac{x_{4k+j_1}x_{j_1+5k}(A+Bx_{k+j_1}x_{j_1+2k}x_{j_1+3k})}{x_{k+j_1}(A+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k})},\\
x_{9k+j_1} &= \frac{x_{j_1}x_{j_1+k}(A+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k})}{x_{j_1+4k}(A+Bx_{j_1}x_{j_1+k}x_{j_1+2k})(A+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k})},\\
x_{10k+j_1} &= \frac{x_{j_1+k}x_{j_1+2k}(A+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k})(A+Bx_{j_1}x_{j_1+k}x_{j_1+2k})}{x_{j_1+5k}(A+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k})(A^2+B(A+1)x_{j_1}x_{j_1+k}x_{j_1+2k})},\\
x_{11k+j_1} &= \frac{x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}(A+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k})(A^2+B(A+1)x_{j_1}x_{j_1+k}x_{j_1+2k})}{x_{j_1}x_{j_1+k}(A+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k})(A^2+B(A+1)x_{j_1+k}x_{j_1+2k}x_{j_1+3k})},
\end{aligned} \tag{3.26}$$

$j_1 = 0, 1, \ldots k-1$. We further split the above equations into various cases.

3.1.1. *The case when $A \neq 1$.* By substitution, one obtains the solution given by the equations:

$$x_{12kn+4kr+j_1} =$$
$$x_{4kr+j_1}\prod_{s=0}^{n-1}\frac{A^{3s+r}+Bx_{j_1}x_{j_1+k}x_{j_1+2k}\left(\frac{1-A^{3s+r}}{1-A}\right)}{A^{3s+r}+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k}\left(\frac{1-A^{3s+r}}{1-A}\right)}\frac{A^{3s+r}+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k}\left(\frac{1-A^{3s+r}}{1-A}\right)}{A^{3s+r+1}+Bx_{j_1}x_{j_1+k}x_{j_1+2k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}$$
$$\frac{A^{3s+r+1}+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}{A^{3s+r+1}+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}\frac{A^{3s+r+2}+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k}\left(\frac{1-A^{3s+r+2}}{1-A}\right)}{A^{3s+r+2}+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k}\left(\frac{1-A^{3s+r+2}}{1-A}\right)}, \tag{3.27}$$

$$x_{12kn+4kr+k+j_1} =$$
$$x_{4kr+k+j_1}\prod_{s=0}^{n-1}\frac{A^{3s+r}+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k}\left(\frac{1-A^{3s+r}}{1-A}\right)}{A^{3s+r}+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k}\left(\frac{1-A^{3s+r}}{1-A}\right)}\frac{A^{3s+r+1}+Bx_{j_1}x_{j_1+k}x_{j_1+2k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}{A^{3s+r+1}+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}$$
$$\frac{A^{3s+r+1}+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}{A^{3s+r+2}+Bx_{j_1}x_{j_1+k}x_{j_1+2k}\left(\frac{1-A^{3s+r+2}}{1-A}\right)}\frac{A^{3s+r+2}+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k}\left(\frac{1-A^{3s+r+2}}{1-A}\right)}{A^{3s+r+2}+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k}\left(\frac{1-A^{3s+r+2}}{1-A}\right)}, \tag{3.28}$$

$$x_{12kn+4kr+2k+j_1} =$$
$$x_{4kr+2k+j_1}\prod_{s=0}^{n-1}\frac{A^{3s+r}+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k}\left(\frac{1-A^{3s+r}}{1-A}\right)}{A^{3s+r}+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k}\left(\frac{1-A^{3s+r}}{1-A}\right)}\frac{A^{3s+r+1}+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}{A^{3s+r+1}+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}$$
$$\frac{A^{3s+r+2}+Bx_{j_1}x_{j_1+k}x_{j_1+2k}\left(\frac{1-A^{3s+r+2}}{1-A}\right)}{A^{3s+r+2}+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k}\left(\frac{1-A^{3s+r+2}}{1-A}\right)}\frac{A^{3s+r+2}+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k}\left(\frac{1-A^{3s+r+2}}{1-A}\right)}{A^{3s+r+3}+Bx_{j_1}x_{j_1+k}x_{j_1+2k}\left(\frac{1-A^{3s+r+3}}{1-A}\right)}, \tag{3.29}$$

$$x_{12kn+4kr+3k+j_1} =$$
$$x_{4kr+3k+j_1}\prod_{s=0}^{n-1}\frac{A^{3s+r}+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k}\left(\frac{1-A^{3s+r}}{1-A}\right)}{A^{3s+r+1}+Bx_{j_1}x_{j_1+k}x_{j_1+2k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}\frac{A^{3s+r+1}+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}{A^{3s+r+1}+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k}\left(\frac{1-A^{3s+r+1}}{1-A}\right)}$$
$$\frac{A^{3s+r+2}+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k}\left(\frac{1-A^{3s+r+2}}{1-A}\right)}{A^{3s+r+2}+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k}\left(\frac{1-A^{3s+r+2}}{1-A}\right)}\frac{A^{3s+r+3}+Bx_{j_1}x_{j_1+k}x_{j_1+2k}\left(\frac{1-A^{3s+r+3}}{1-A}\right)}{A^{3s+r+3}+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k}\left(\frac{1-A^{3s+r+3}}{1-A}\right)}. \tag{3.30}$$

The case $A=-1$: In this case, one obtains the solution defined by the following solution equations

$$x_{12kn+4kr+j_1} =$$
$$\begin{cases} x_{j_1+4kr} & \text{if } n \text{ is even} \\ \dfrac{x_{4kr+j_1}\left((-1)^r+\frac{B}{V_{j_1}}\left(\frac{1-(-1)^r}{2}\right)\right)}{(-1)^{r+1}+\frac{B}{V_{j_1}}\left(\frac{1-(-1)^{r+1}}{2}\right)}\dfrac{(-1)^r+\frac{B}{V_{j_1+3k}}\left(\frac{1-(-1)^r}{2}\right)}{(-1)^{r+1}+\frac{B}{V_{j_1+3k}}\left(\frac{1-(-1)^{r+1}}{2}\right)}\dfrac{(-1)^{r+1}+\frac{B}{V_{j_1+2k}}\left(\frac{1-(-1)^{r+1}}{2}\right)}{(-1)^r+\frac{B}{V_{j_1+2k}}\left(\frac{1-(-1)^r}{2}\right)} & \text{if } n \text{ is odd} \end{cases}$$

$$x_{12kn+4kr+k+j_1} =$$
$$\begin{cases} x_{j_1+4kr+k} & \text{if } n \text{ is even} \\ \dfrac{x_{4kr+k+j_1}\left((-1)^r+\frac{B}{V_{j_1+k}}\left(\frac{1-(-1)^r}{2}\right)\right)}{(-1)^{r+1}+\frac{B}{V_{j_1+k}}\left(\frac{1-(-1)^{r+1}}{2}\right)}\dfrac{(-1)^{r+1}+\frac{B}{V_{j_1}}\left(\frac{1-(-1)^{r+1}}{2}\right)}{(-1)^r+\frac{B}{V_{j_1}}\left(\frac{1-(-1)^r}{2}\right)}\dfrac{(-1)^{r+1}+\frac{B}{V_{j_1+3k}}\left(\frac{1-(-1)^{r+1}}{2}\right)}{(-1)^r+\frac{B}{V_{j_1+3k}}\left(\frac{1-(-1)^r}{2}\right)} & \text{if } n \text{ is odd} \end{cases}$$

$$x_{12kn+4kr+2k+j_1} =$$
$$\begin{cases} x_{j_1+4kr+2k} & \text{if } n \text{ is even} \\ \dfrac{x_{4kr+2k+j_1}\left((-1)^r+\frac{B}{V_{j_1+2k}}\left(\frac{1-(-1)^r}{2}\right)\right)}{(-1)^{r+1}+\frac{B}{V_{j_1+2k}}\left(\frac{1-(-1)^{r+1}}{2}\right)}\dfrac{(-1)^{r+1}+\frac{B}{V_{j_1+k}}\left(\frac{1-(-1)^{r+1}}{2}\right)}{(-1)^r+\frac{B}{V_{j_1+k}}\left(\frac{1-(-1)^r}{2}\right)}\dfrac{(-1)^r+\frac{B}{V_{j_1}}\left(\frac{1-(-1)^r}{2}\right)}{(-1)^{r+1}+\frac{B}{V_{j_1}}\left(\frac{1-(-1)^{r+1}}{2}\right)} & \text{if } n \text{ is odd} \end{cases}$$

$$x_{12kn+4kr+3k+j_1}=$$

$$\begin{cases} x_{j_1+4kr+3k} & \text{if } n \text{ is even} \\ \dfrac{x_{4kr+3k+j_1}\left[(-1)^r+\frac{B}{V_{j_1+3k}}\left(\frac{1-(-1)^r}{2}\right)\right]\left[(-1)^{r+1}+\frac{B}{V_{j_1+2k}}\left(\frac{1-(-1)^{r+1}}{2}\right)\right]\left[(-1)^r+\frac{B}{V_{j_1+k}}\left(\frac{1-(-1)^r}{2}\right)\right]}{\left[(-1)^{r+1}+\frac{B}{V_{j_1+3k}}\left(\frac{1-(-1)^{r+1}}{2}\right)\right]\left[(-1)^r+\frac{B}{V_{j_1+2k}}\left(\frac{1-(-1)^r}{2}\right)\right]\left[(-1)^{r+1}+\frac{B}{V_{j_1+k}}\left(\frac{1-(-1)^{r+1}}{2}\right)\right]} & \text{if } n \text{ is odd.} \end{cases} \tag{3.31}$$

More explicitly, for $j_1=0,1,\ldots,k-1$, we have:

$$x_{6k+j_1}=\frac{x_{j_1}x_{j_1+k}x_{j_1+2k}}{x_{j_1+4k}x_{j_1+5k}(-1+\frac{B}{V_{j_1}})},\qquad x_{6k+k+j_1}=\frac{x_{3k+j_1}x_{j_1+4k}(-1+\frac{B}{V_{j_1}})}{x_{j_1}(-1+\frac{B}{V_{j_1+k}})},$$

$$x_{6k+2k+j_1}=\frac{x_{4k+j_1}x_{j_1+5k}(-1+\frac{B}{V_{j_1+k}})}{x_{j_1+k}(-1+\frac{B}{V_{j_1+2k}})},\qquad x_{6k+3k+j_1}=\frac{x_{j_1}x_{j_1+k}(-1+\frac{B}{V_{j_1+2k}})}{x_{j_1+4k}(-1+\frac{B}{V_{j_1}})(-1+\frac{B}{V_{j_1+3k}})},$$

$$x_{6k+4k+j_1}=\frac{x_{j_1+k}x_{j_1+2k}(-1+\frac{B}{V_{j_1+3k}})(-1+\frac{B}{V_{j_1}})}{x_{j_1+5k}(-1+\frac{B}{V_{j_1+k}})},\qquad x_{6k+5k+j_1}=\frac{x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}(-1+\frac{B}{V_{j_1+k}})}{x_{j_1}x_{j_1+k}(-1+\frac{B}{V_{j_1+2k}})},$$

$$x_{12k+j_1}=\frac{x_{j_1}(-1+\frac{B}{V_{j_1+2k}})}{(-1+\frac{B}{V_{j_1}})(-1+\frac{B}{V_{j_1+3k}})},\qquad x_{12k+k+j_1}=\frac{x_{k+j_1}(-1+\frac{B}{V_{j_1}})(-1+\frac{B}{V_{j_1+3k}})}{(-1+\frac{B}{V_{j_1+k}})},$$

$$x_{12k+2k+j_1}=\frac{x_{2k+j_1}(-1+\frac{B}{V_{j_1+k}})}{(-1+\frac{B}{V_{j_1+2k}})(-1+\frac{B}{V_{j_1}})},\qquad x_{12k+3k+j_1}=\frac{x_{3k+j_1}(-1+\frac{B}{V_{j_1+2k}})}{(-1+\frac{B}{V_{j_1+k}})(-1+\frac{B}{V_{j_1+3k}})},$$

$$x_{12k+4k+j_1}=\frac{x_{4k+j_1}(-1+\frac{B}{V_{j_1+3k}})(-1+\frac{B}{V_{j_1}})}{(-1+\frac{B}{V_{j_1+2k}})},\qquad x_{12k+5k+j_1}=\frac{x_{5k+j_1}(-1+\frac{B}{V_{j_1+k}})}{(-1+\frac{B}{V_{j_1}})(-1+\frac{B}{V_{j_1+3k}})},$$

$$x_{12k+6k+j_1}=\frac{x_{6k+j_1}(-1+\frac{B}{V_{j_1+2k}})(-1+\frac{B}{V_{j_1}})}{(-1+\frac{B}{V_{j_1+k}})},\qquad x_{12k+7k+j_1}=\frac{x_{7k+j_1}(-1+\frac{B}{V_{j_1+k}})(-1+\frac{B}{V_{j_1+3k}})}{(-1+\frac{B}{V_{j_1+2k}})^{-1}},$$

$$x_{12k+8k+j_1}=x_{12k+j_1},\ x_{12k+9k+j_1}=x_{12k+k+j_1},\ x_{12k+10k+j_1}=x_{12k+2k+j_1},\ x_{12k+11k+j_1}=x_{12k+3k+j_1},$$

$$x_{12(2n)k+i}=x_i,\ i=0,1,\ldots,12k-1;\qquad x_{12(2n+1)k+j_1}=x_{12k+i},\ i=0,1,\ldots,12k-1. \tag{3.32}$$

3.1.2. *The case when $A=1$.* Using (3.25) and (3.26), the solution equations are as follows:

$$x_{6k+j_1}=\frac{x_{j_1}x_{j_1+k}x_{j_1+2k}}{x_{j_1+4k}x_{j_1+5k}(1+Bx_{j_1}x_{j_1+k}x_{j_1+2k})},$$

$$x_{7k+j_1}=\frac{x_{3k+j_1}x_{j_1+4k}(1+Bx_{j_1}x_{j_1+k}x_{j_1+2k})}{x_{j_1}(1+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k})},$$

$$x_{8k+j_1}=\frac{x_{4k+j_1}x_{j_1+5k}(1+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k})}{x_{j_1+k}(1+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k})},$$

$$
\begin{aligned}
x_{9k+j_1} =&\frac{x_{j_1}x_{j_1+k}(1+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k})}{x_{j_1+4k}(1+Bx_{j_1}x_{j_1+k}x_{j_1+2k})(1+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k})},\\
x_{10k+j_1} =&\frac{x_{j_1+k}x_{j_1+2k}(1+Bx_{j_1+3k}x_{j_1+4k}x_{j_1+5k})(1+Bx_{j_1}x_{j_1+k}x_{j_1+2k})}{x_{j_1+5k}(1+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k})(1+2Bx_{j_1}x_{j_1+k}x_{j_1+2k})},\\
x_{11k+j_1} =&\frac{x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}(1+Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k})(1+2Bx_{j_1}x_{j_1+k}x_{j_1+2k})}{x_{j_1}x_{j_1+k}(1+Bx_{j_1+2k}x_{j_1+3k}x_{j_1+4k})(1+2Bx_{j_1+k}x_{j_1+2k}x_{j_1+3k})},
\end{aligned}
$$

$$
\begin{aligned}
x_{12kn+4kr+j_1} =&x_{4kr+j_1}\prod_{s=0}^{n-1}\frac{1+B(3s+r)x_{j_1}x_{j_1+k}x_{j_1+2k}}{1+B(3s+r)x_{j_1+k}x_{j_1+2k}x_{j_1+3k}}\frac{1+B(3s+r)x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}}{1+B(3s+r+1)x_{j_1}x_{j_1+k}x_{j_1+2k}}\\
&\frac{1+B(3s+r+1)x_{j_1+2k}x_{j_1+3k}x_{j_1+4k}}{1+B(3s+r+1)x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}}\frac{1+B(3s+r+2)x_{j_1+k}x_{j_1+2k}x_{j_1+3k}}{1+B(3s+r+2)x_{j_1+2k}x_{j_1+3k}x_{j_1+4k}},
\end{aligned}
$$

$$
\begin{aligned}
x_{12kn+4kr+k+j_1} =&x_{4kr+k+j_1}\prod_{s=0}^{n-1}\frac{1+B(3s+r)x_{j_1+k}x_{j_1+2k}x_{j_1+3k}}{1+B(3s+r)x_{j_1+2k}x_{j_1+3k}x_{j_1+4k})}\frac{1+B(3s+r+1)x_{j_1}x_{j_1+k}x_{j_1+2k}}{1+B(3s+r+1)x_{j_1+k}x_{j_1+2k}x_{j_1+3k}}\\
&\frac{1+B(3s+r+1)x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}}{1+B(3s+r+2)x_{j_1}x_{j_1+k}x_{j_1+2k}}\frac{1+B(3s+r+2)x_{j_1+2k}x_{j_1+3k}x_{j_1+4k}}{1+B(3s+r+2)x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}},
\end{aligned}
$$

$$
\begin{aligned}
x_{12kn+4kr+2k+j_1} =&x_{4kr+2k+j_1}\prod_{s=0}^{n-1}\frac{1+B(3s+r)x_{j_1+2k}x_{j_1+3k}x_{j_1+4k}}{1+B(3s+r)x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}}\frac{1+B(3s+r+1)x_{j_1+k}x_{j_1+2k}x_{j_1+3k}}{1+B(3s+r+1)x_{j_1+2k}x_{j_1+3k}x_{j_1+4k}}\\
&\frac{1+B(3s+r+2)x_{j_1}x_{j_1+k}x_{j_1+2k}}{1+B(3s+r+2)x_{j_1+k}x_{j_1+2k}x_{j_1+3k}}\frac{1+B(3s+r+2)x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}}{1+B(3s+r+3)x_{j_1}x_{j_1+k}x_{j_1+2k}},
\end{aligned}
$$

$$
\begin{aligned}
x_{12kn+4kr+3k+j_1} =&x_{4kr+3k+j_1}\prod_{s=0}^{n-1}\frac{1+B(3s+r)x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}}{1+B(3s+r+1)x_{j_1}x_{j_1+k}x_{j_1+2k}}\frac{1+B(3s+r+1)x_{j_1+2k}x_{j_1+3k}x_{j_1+4k}}{1+B(3s+r+1)x_{j_1+3k}x_{j_1+4k}x_{j_1+5k}}\\
&\frac{1+B(3s+r+2)x_{j_1+k}x_{j_1+2k}x_{j_1+3k}}{1+B(3s+r+2)x_{j_1+2k}x_{j_1+3k}x_{j_1+4k}}\frac{1+B(3s+r+3)x_{j_1}x_{j_1+k}x_{j_1+2k}}{1+B(3s+r+3)x_{j_1+k}x_{j_1+2k}x_{j_1+3k}}, \quad (3.33)
\end{aligned}
$$

for $j_1 = 0, 1, \ldots, k-1$.

4. Periodic nature and behavior of the solutions

In this section, we show the existence of periodic solutions and we analyze the stability of the equilibrium points.

Theorem 4.1. *Let x_n be a solution of*

$$x_{n+6k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+4k}x_{n+5k}(-1+Bx_n x_{n+k}x_{n+2k})}, \tag{4.1}$$

with initial conditions x_i, $i = 0, \ldots, 6k-1$. Then, the solution to (4.1) is periodic with period $24k$.

Proof. The proof follows from (3.32). □

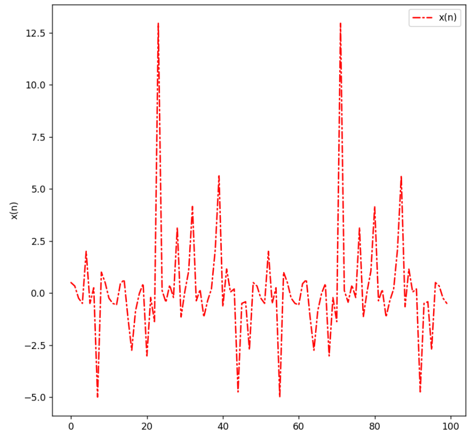

Figure 1. Graph of $x_{n+6k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+4k} x_{n+5k}(-1+Bx_n x_{n+k} x_{n+2k})}$, with $x_0 = 1/2, x_1 = 1/3, x_2 = -1/4, x_3 = -1/2, x_4 = 2, x_5 = -1/2, x_6 = 1/4, x_7 = -5, x_8 = 1, x_9 = 1/2, x_{10} = -1/4, x_{11} = -1/2$.

Figure 1 demonstrates Theorem 4.1. As expected, we have a 48-periodic solution, regardless of the value of B.

Theorem 4.2. *Let x_n be a solution of*

$$x_{n+6k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+4k} x_{n+5k}(A + Bx_n x_{n+k} x_{n+2k})}, \tag{4.2}$$

where $A \neq 1$ and B are real constants. Assume that the initial conditions x_i and x_{i+3k} satisfy $x_i = x_{i+3k}$, $x_i \neq x_{i+k}$; and $x_i x_{i+k} x_{i+2k} = (1-A)/B$. Then, the solution to (4.2) is $3k$-periodic.

Proof. Assuming that the initial conditions satisfy $x_i = x_{i+3k}$, $x_i \neq x_{i+k}$; and $x_i x_{i+k} x_{i+2k} = (1-A)/B$. It follows from (3.26) and (3.27)-(3.30) that

$$x_{6k+i} = x_i, \quad x_{12kn+i} = x_i, \tag{4.3}$$

for all $i = 0, 1, \ldots, 12k-1$. The condition $x_i = x_{i+3k}$ together with (4.3) imply that the solution is $3k$-periodic. $\square$

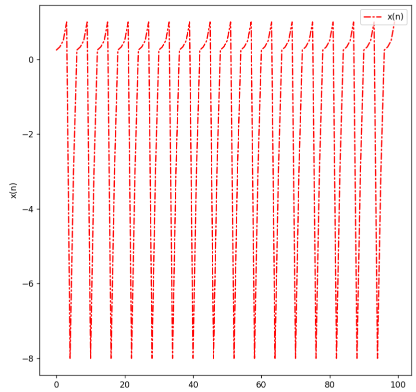

Figure 2. Graph of $x_{n+6k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+4k} x_{n+5k}(3+2x_n x_{n+k} x_{n+2k})}$, with $x_0 = 1/4, x_1 = 1/3, x_2 = 1/2, x_3 = 1, x_4 = -8, x_5 = -3, x_6 = 1/4, x_7 = 1/3, x_8 = 1/2, x_9 = 1, x_{10} = -8, x_{11} = -3$.

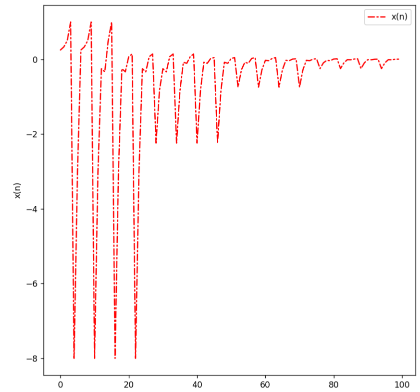

Figure 3. Graph of $x_{n+6k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+4k} x_{n+5k}(3+4x_n x_{n+k} x_{n+2k})}$, with $x_0 = 1/4, x_1 = 1/3, x_2 = 1/2, x_3 = 1, x_4 = -8, x_5 = -3, x_6 = 1/4, x_7 = 1/3, x_8 = 1/2, x_9 = 1, x_{10} = -8, x_{11} = -3$.

Figure 2 demonstrates Theorem 4.2, as expected, we have a 6-periodic solution. Remark that the initial conditions in Figure 3 do not satisfy the condition $x_i x_{i+k} x_{i+2k} = (1-A)/B$. We note that in Theorem 4.2, the restriction $x_i = x_{i+3k}$ is not sufficient for the solution to be $3k$-periodic. If the restriction $x_i x_{i+k} x_{i+2k} = (1-A)B$ on the initial conditions x_i is not added, the solution may not be $3k$-periodic. We illustrate this with Figure 3.

Theorem 4.3. *Consider the equation*

$$x_{n+6k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+4k} x_{n+5k}(1 + B x_n x_{n+k} x_{n+2k})}, \tag{4.4}$$

where $B \neq 0$ is a constant. The only equilibrium point $\bar{x} = 0$ is non hyperbolic.

Proof. It is easy to check that the equilibrium point of (4.4) is $x = 0$. Invoking (2.7), the characteristic equation of (4.4) about 0 is given by $\lambda^{6k} + \lambda^{5k} + \lambda^{4k} - \lambda^{2k} - \lambda^k - 1 = 0$. Obviously, $\lambda = 1$ is a solution of this equation. So, the fixed point zero is non-hyperbolic. □

Theorem 4.4. *Assuming that $|A| > 5$, the equilibrium point $\bar{x} = 0$ of (4.2) is locally asymptotically stable. Moreover, the non zero equilibrium points of (4.2) are non-hyperbolic for all $A \neq 1$.*

Proof. One obtains the equilibrium points of (4.2) by solving $\bar{x}^3(A + B\bar{x}^3 - 1) = 0$.
For the first part of the proof, we consider the characteristic equation of (4.2) about $\bar{x} = 0$ given by

$$\lambda^{6k} + \frac{1}{A}\lambda^{5k} + \frac{1}{A}\lambda^{4k} - \frac{1}{A}\lambda^{2k} - \frac{1}{A}\lambda^k - \frac{1}{A} = 0. \tag{4.5}$$

We have that

$$\left|\frac{1}{A}\right| + \left|\frac{1}{A}\right| + \left|-\frac{1}{A}\right| + \left|-\frac{1}{A}\right| + \left|-\frac{1}{A}\right| = \frac{5}{|A|} < 1 \tag{4.6}$$

for $|A| > 5$. By Theorem 2.3, the roots of (4.5) have magnitude less than 1. Consequently, the zero equilibrium point is locally asymptotically stable if $|A| > 5$.

For the second part, recall that the non-zero equilibrium points $\bar{x}$ satisfy $A + B\bar{x}^3 - 1 = 0$.

We now consider the characteristic equation of (4.2) about a non-zero equilibrium point given by

$$\lambda^{6k} + \lambda^{5k} + \lambda^{4k} - A\lambda^{2k} - A\lambda^{k} - A = (\lambda^{2k} + \lambda^{k} + 1)(\lambda^{4k} - A) = 0. \tag{4.7}$$

The solutions $\lambda_p = e^{\frac{i}{k}(\pm\frac{2\pi}{3}+2p\pi)}$, $p = 0, 1, \ldots, k-1$, of $\lambda^{2k} + \lambda^{k} + 1 = 0$ are such that $|\lambda_i| = 1$. Hence, any equilibrium point obtained from the equation $A + B\bar{x}^3 - 1 = 0$ is non-hyperbolic. $\square$

The case where $k = 1$, $A = -1$ and $B = 1$:

Employing the equations in (3.32), the solution of the sixth order difference equation

$$x_{n+6} = \frac{x_n x_{n+1} x_{n+2}}{x_{n+4} x_{n+5}(-1 + x_n x_{n+1} x_{n+2})}, \tag{4.8}$$

with initial conditions $x_0, x_1, x_2, x_3, x_4, x_5$, will be given by:

$$x_6 = \frac{x_0x_1x_2}{x_4x_5(-1+x_0x_1x_2)}, \quad x_7 = \frac{x_3x_4(-1+x_0x_1x_2)}{x_0(-1+x_1x_2x_3)},$$

$$x_8 = \frac{x_4x_5(-1+x_1x_2x_3)}{x_1(-1+x_2x_3x_4)}, x_9 = \frac{x_0x_1(-1+x_2x_3x_4)}{x_4(-1+x_0x_1x_2)(-1+Bx_3x_4x_5)},$$

$$x_{10} = \frac{x_1x_2(-1+x_3x_4x_5)(-1+x_0x_1x_2)}{x_5(-1+x_1x_2x_3)}, \quad x_{11} = \frac{x_3x_4x_5(-1+x_1x_2x_3)}{x_0x_1(-1+x_2x_3x_4)},$$

$$x_{12} = \frac{x_0(-1+x_2x_3x_4)}{(-1+x_0x_1x_2)(-1+x_3x_4x_5)}, \quad x_{13} = \frac{x_1(-1+x_0x_1x_2)(-1+x_3x_4x_5)}{(-1+x_1x_2x_3)},$$

$$x_{14} = \frac{x_2(-1+x_1x_2x_3)}{(-1+x_2x_3x_4)(-1+x_0x_1x_2)}, \quad x_{15} = \frac{x_3(-1+x_2x_3x_4)}{(-1+x_1x_2x_3)(-1+x_3x_4x_5)},$$

$$x_{16} = \frac{x_4(-1+x_3x_4x_5)(-1+x_0x_1x_2)}{(-1+x_2x_3x_4)}, \quad x_{17} = \frac{x_5(-1+x_1x_2x_3)}{(-1+x_0x_1x_2)(-1+x_3x_4x_5)},$$

$$x_{18} = \frac{x_0x_1x_2(-1+x_2x_3x_4)}{x_4x_5(-1+x_1x_2x_3)}, \quad x_{19} = \frac{x_3x_4(-1+x_0x_1x_2)(-1+x_3x_4x_5)}{x_0(-1+x_2x_3x_4)},$$

$$\begin{aligned}
&x_{20} = x_{12}, \ x_{21} = x_{13}, \ x_{22} = x_{14}, \ x_{23} = x_{15}, \\
&x_{12(2n)+i} = x_i, \quad i = 0, 1, \ldots, 11, \\
&x_{12(2n+1)+i} = x_{12+i}, \quad i = 0, 1, \ldots, 11.
\end{aligned} \tag{4.9}$$

We can see that we have a 24 periodic solution, confirming Theorem 4.1. If we force the conditions in Theorem 4.2, that is, $x_0 = x_3, x_1 = x_4, x_2 = x_5$ and $x_0x_1x_2 = 2$, then we have the following 3-periodic solution $\{x_n\}_{n\geq 0} = \{x_0,\ x_1,\ x_2,\ x_0,\ x_1,\ x_2,\ \ldots, x_0,\ x_1,\ x_2, \ldots\}$. Note that the solution will still be 24-periodic if $x_0x_1x_2 \neq 2$, even though $x_0 = x_3, x_1 = x_4, x_2 = x_5$.

5. Concluding remarks

Certain authors prefer the equivalent form

$$x_{n+1} = \frac{x_{n-5}x_{n-4}x_{n-3}}{x_{n-1}x_n(-1 + x_{n-5}x_{n-4}x_{n-3})} \tag{5.1}$$

of (4.8). It is noteworthy that the solution of (5.1) is derived from those of (4.8) by back shifting the solution of (4.8) five times. Hence, thanks to (4.9), the 24-periodic solution of (5.1) is given by

$$\begin{aligned}
\{x_n\}_{n\geq -5} = \Bigg\{ & x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ \frac{x_{-5}x_{-4}x_{-3}}{x_{-1}x_0(-1+x_{-5}x_{-4}x_{-3})},\ \frac{x_{-2}x_{-1}(-1+x_{-5}x_{-4}x_{-3})}{x_{-5}(-1+x_{-4}x_{-3}x_{-2})},\\
& \frac{x_{-1}x_0(-1+x_{-4}x_{-3}x_{-2})}{x_{-4}(-1+x_{-3}x_{-2}x_{-1})},\ \frac{x_{-5}x_{-4}(-1+x_{-3}x_{-2}x_{-1})}{x_{-1}(-1+x_{-5}x_{-4}x_{-3})(-1+x_{-2}x_{-1}x_0)},\ \frac{x_{-4}x_{-3}(-1+x_{-2}x_{-1}x_0)(-1+x_{-5}x_{-4}x_{-3})}{x_0(-1+x_{-4}x_{-3}x_{-2})},\\
& \frac{x_{-2}x_{-1}x_0(-1+x_{-4}x_{-3}x_{-2})}{x_{-5}x_{-4}(-1+x_{-3}x_{-2}x_{-1})},\ \frac{x_{-5}(-1+x_{-3}x_{-2}x_{-1})}{(-1+x_{-5}x_{-4}x_{-3})(-1+x_{-2}x_{-1}x_0)},\ \frac{x_{-4}(-1+x_{-5}x_{-4}x_{-3})(-1+x_{-2}x_{-1}x_0)}{(-1+x_{-4}x_{-3}x_{-2})},\\
& \frac{x_{-3}(-1+x_{-4}x_{-3}x_{-2})}{(-1+x_{-3}x_{-2}x_{-1})(-1+x_{-5}x_{-4}x_{-3})},\ \frac{x_{-2}(-1+x_{-3}x_{-2}x_{-1})}{(-1+x_{-4}x_{-3}x_{-2})(-1+x_{-2}x_{-1}x_0)},\ \frac{x_{-1}(-1+x_{-2}x_{-1}x_0)(-1+x_{-5}x_{-4}x_{-3})}{(-1+x_{-3}x_{-2}x_{-1})},\\
& \frac{x_0(-1+x_{-4}x_{-3}x_{-2})}{(-1+x_{-5}x_{-4}x_{-3})(-1+x_{-2}x_{-1}x_0)},\ \frac{x_{-5}x_{-4}x_{-3}(-1+x_{-3}x_{-2}x_{-1})}{x_{-1}x_0(-1+x_{-4}x_{-3}x_{-2})},\ \frac{x_{-2}x_{-1}(-1+x_{-5}x_{-4}x_{-3})(-1+x_{-2}x_{-1}x_0)}{x_{-5}(-1+x_{-3}x_{-2}x_{-1})},\\
& x_{15} = x_7,\ x_{17} = x_9,\ x_{18} = x_{10}, \cdots \Bigg\}.
\end{aligned} \tag{5.2}$$

This important remark is applicable to (1.1) and (1.2) in the sense that one obtains the solution of (1.2) by back shifting the solutions of (1.1) $6k - 1$ times.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. Aloqeili, Dynamics of a rational difference equation, Appl. Math. Comput. 176 (2006), 768–774. https://doi.org/10.1016/j.amc.2005.10.024.

[2] E.M. Elabbasy, H. El-Metwally, E.M. Elsayed, On the difference equation $x_{n+1} = \alpha x_{n-k}/(\beta + \gamma \prod_{i=0}^{k} x_{n-i}$, J. Concr. Appl. Math. 5 (2007), 101–113.

[3] M. Folly-Gbetoula, D. Nyirenda, On some sixth-order rational recursive sequences, J. Comput. Anal. Appl. 27 (2019), 1057–1069.

[4] M. Folly-Gbetoula, D. Nyirenda, A generalized two-dimensional system of higher order recursive sequences, J. Differ. Equ. Appl. 26 (2020), 244–260. https://doi.org/10.1080/10236198.2020.1718667.

[5] E.A. Grove, G.E. Ladas, Periodicities in nonlinear difference equations, Chapman & Hall/CRC, Boca Raton, 2005.

[6] P.E. Hydon, Difference equations by differential equation methods, Cambridge University Press, Cambridge, 2014.

[7] T.F. Ibrahim, On the third order rational difference equation $x_{n+1} = x_n x_{n-2}/x_{n-1}(a + bx_n x_{n-2})$, Int. J. Contemp. Math. Sci. 4 (2009), 1321–1334.

[8] R. Karatas, Global behavior of a higher order difference equation, Comput. Math. Appl. 60 (2010), 830–839. https://doi.org/10.1016/j.camwa.2010.05.030.

[9] S. Maeda, The similarity method for difference equations, IMA J. Appl. Math. 38 (1987), 129–134. https://doi.org/10.1093/imamat/38.2.129.

[10] K. Mkhwanazi, M. Folly-Gbetoula, Symmetries and solvability of a class of higher order systems of ordinary difference equations, Symmetry. 14 (2022), 108. https://doi.org/10.3390/sym14010108.

[11] N. Mnguni, D. Nyirenda, M. Folly-Gbetoula, Symmetry Lie algebra and exact solutions of some fourth-order difference equation, J. Nonlinear Sci. Appl. 11 (2018), 1262–1270.

[12] I. Yalcinkaya, On the global attractivity of positive solutions of a rational difference equation, Selcuk J. Appl. Math. 9 (2008), 3–8.