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Conformable Granular Fractional Differentiability for Fuzzy Number Valued Functions

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Abstract. This paper deals with the utilization of the concept of the granular differentiability to establish a fractional derivative of the conformable type for fuzzy number valued functions. Subsequently, we introduce the notion of a conformable granular integral and provide evidence of its fundamental properties pertaining to differentiability and integrability through illustrative examples. Lastly, we delve into the discussion of the solution approach for the conformable granular initial value problem (CGIVP), as well as the solution of conformable granular differential equations (CGDEqs) associated with growth and decay.

1. Introduction

Since 1965, fuzzy theory had many applications in science and engineering fields to avoid uncertain process problems in real-world situations. The concept of fuzzy calculus has risen in recent times to cover a number of vague or imprecise situations. The applications of fuzzy derivatives are widely explored in various fields of research. Firstly L.A. Zadeh [16] presented a theory on fuzzy sets in his paper. The derivative for fuzzy function has been introduced by several authors, namely Hukuhara, Strongly generalized Hukuhara, Generalized Hukuhara derivatives and g-derivative, $\mathbb{H}_2$-differentiability [4, 5, 13–15]. Later granular derivative [10] using horizontal membership function (HMF) was introduced by Mazandarani, which is efficient derivative rather than derivatives listed above.

The fractional derivative has developed significantly in the past few decades. Fractional calculus is entered in the year 1695. The applications of fractional calculus are used in Mathematical

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models the real-world phenomenon. The most frequently used definition of derivatives is fractional Riemann- Liouville(R-L) and fractional Caputo derivatives [12]. The generalized Hukuhara R-L and Caputo fractional derivatives were introduced in [1], [2]. The Strongly generalized Hukuhara R-L and Caputo fuzzy non-integer order derivatives were discussed in [14]. In [3], the fuzzy fractional differential equations(DEqs) are discussed under Caputo fractional generalized differentiability. Marzieh Najariyan, in her paper [11], the granular Caputo and R-L fractional derivatives proposed and investigated the fuzzy non-integer order linear dynamic systems.

In 2014, R. Khalil [7] introduced a different type of derivative named conformable fractional derivative and proved fundamental properties that are distinct from those usual in other formulations. The fuzzy conformable fractional strongly generalized Hukuhara derivative is established [6]. In [11], the author describes the conformable fractional Laplace transforms and obtains the analytical solutions of non-integer order conformable DEqs under generalized Hukuhara derivative. The conformable fuzzy DEqs involved above mentioned derivatives had some drawbacks like no unique solution and the diameter of the solution being unbounded. The goal of this paper is to develop non-integer order fuzzy conformable calculus using the concept of HMF. Our approach is natural and has the same advantages as crisp functions. We obtain the solution to fuzzy conformable granular initial value growth and decay problems.

The current paper presents advancements in the field of conformable granular fractional differentiability and integrability for fuzzy number valued functions(FNVF). It is organized into several sections. Section 2, provides an overview of the basic theory of fractional calculus. In section 3, we introduce a novel concept called the conformable granular derivative and discuss its fundamental properties. The definition of the conformable granular integral is presented in section 4. Moving on to section 5, we derive CGDEqs and demonstrate graphical illustrations of several examples based on the previously introduced concepts.

2. Preliminaries

Basic needful concepts of fuzzy fractional calculus are given below, which will be applied throughout the paper. In this paper, $\mathbb{R}$ is denoted as a real number set, K_1 is denoted as fuzzy number set on $\mathbb{R}$.

Definition 2.1. *[7] Given $\psi : [0,\infty] \to \mathbb{R}$, be function then the conformable derivative(CD) of fractional order $\alpha \in (0,1]$ of ψ is defined as for all $z > 0$,*

$$T_\alpha \psi(z) = \lim_{h\to 0} \frac{\psi\Big[z + hz^{1-\alpha}\Big] - \psi[z]}{h}.$$

If the CD of ψ of fractional order α exists, then we simply say $\psi(z)$ is $\alpha-$differentiable.

Remark 2.1. *If ψ is CD in some $(0,c), c > 0$ and $\lim_{z\to 0^+} \psi^{(\alpha)}(z)$ exists, then define*

$$\psi^{(\alpha)}(0) = \lim_{z\to 0^+} \psi^{(\alpha)}(z).$$

Definition 2.2. *[7] Conformable Integral of fractional order of* α, $I^a_\alpha\psi(z) = I^a_1\Big[z^{\alpha-1}\psi(z)\Big] = \int_0^a \frac{\psi(s)}{s^{1-\alpha}}ds$, *where the integral is the usual Riemann improper integral and* $\alpha \in (0,1]$.

$\widetilde{n} : \mathbb{R} \to [0,1]$ is commonly referred to as a fuzzy number (FN) when it is a normal, fuzzy convex, upper semi-continuous, and compactly supported set of fuzzy number sets.

A function $\widetilde{\psi}$ is a FNVF if $\widetilde{\psi} : [c,d] \subseteq \mathbb{R} \to K_1$ and $[\widetilde{\psi}]^\theta = [\psi^\theta_L, \psi^\theta_R]$, where $\psi^\theta_L, \psi^\theta_R$ are left and right end points of $\theta-$level set $[\widetilde{\psi}]^\theta$.

Definition 2.3. *[10] A FN* $\widetilde{n}$ *whose HMF* $n^{gr} : [0,1] \times [0,1] \to [c,d]$ *is defined as* $n^{gr}(\theta, \beta_n) = z$ *and it is also defined by* $\mathbb{H}(\widetilde{n}) \overset{\Delta}{=} n^{gr}(\theta, \beta_n)$. *Moreover,* $n^{gr}(\theta, \beta_n) = n^\theta_L + (n^\theta_R - n^\theta_L)\beta_n$.

Note 2.1. *The inverse HMF is obtained by*

$$[\widetilde{n}(z)]^\theta = \left[\inf_{\theta\le\gamma}\min_{\beta_n} n^{gr}(\gamma, \beta_n), \sup_{\theta\le\gamma}\max_{\beta_n} n^{gr}(\gamma, \beta_n)\right]. \tag{2.1}$$

Here 'gr' stands for the information granule contained in $z \in [c,d], \theta \in [0,1]$ *is the membership degree of* z *in* $\widetilde{n}(z)$ *and* $\beta_n \in [0,1]$ *is read as the relative-distance-measure (RDM) variable.*

Definition 2.4. *[10] Let* $\widetilde{n}_1, \widetilde{n}_2$ *are two FNs and* $*$ *represents one of the arithmetic operators +, -,* $\div$ *and* $\times$. *Then* $\widetilde{n}_1 * \widetilde{n}_2$ *is a fuzzy number* $\widetilde{n}$ *such that* $\mathbb{H}(\widetilde{n}) = \mathbb{H}(\widetilde{n}_1) * \mathbb{H}(\widetilde{n}_2)$ *provided* $0 \notin \mathbb{H}(\widetilde{n}_2)$ *when* $*$ *denotes division operator.*

Note 2.2. *[10] If* $\widetilde{n}_1, \widetilde{n}_2, \widetilde{n}_3 \in K_1$ *then the following relations holds:*

(1) $\widetilde{n}_1 - \widetilde{n}_2 = -[\widetilde{n}_2 - \widetilde{n}_1]$
(2) $\widetilde{n}_1 - \widetilde{n}_1 = 0$
(3) $\widetilde{n}_1 \div \widetilde{n}_1 = 1$
(4) $[\widetilde{n}_1 + \widetilde{n}_2]\widetilde{n}_3 = \widetilde{n}_1\widetilde{n}_3 + \widetilde{n}_2\widetilde{n}_3$.

Note 2.3. *[10] The fuzzy numbers* $\widetilde{n}_1$ *and* $\widetilde{n}_2$ *are equal if and only if their HMFs are equal i.e.,* $\widetilde{n}_1 = \widetilde{n}_2 \iff \mathbb{H}(\widetilde{n}_1) = \mathbb{H}(\widetilde{n}_2)$ *and whenever* $\mathbb{H}(\widetilde{n}_1) \ge \mathbb{H}(\widetilde{n}_2)$ *then* $\widetilde{n}_1 \ge \widetilde{n}_2$ *for all* $\beta_{n_1}, \beta_{n_2} \in [0,1]$.

Note 2.4. *[11] Let* $\widetilde{\phi}$ *FNVF, then HMF of* $\widetilde{\psi}(\widetilde{\phi}(z))$ *is defined as* $\mathbb{H}\Big[\widetilde{\psi}\big(\mathbb{H}(\widetilde{\phi}(z))\big)\Big]$.

Definition 2.5. *[10] Let* $\widetilde{n}_1, \widetilde{n}_2$ *be two FNs, then the granular metric on* K_1 *is denoted by* $D^{gr} : K_1 \times K_1 \to \mathbb{R}^+ \cup 0$, *defined by*

$$D^{gr}[\widetilde{n}_1, \widetilde{n}_2] = \sup_{\theta\in[0,1]}\max_{\beta_{n_1},\beta_{n_2}\in[0,1]} |n^{gr}_1(\theta, \beta_{n_1}) - n^{gr}_2(\theta, \beta_{n_2})|.$$

Definition 2.6. *[10] Suppose that* $\widetilde{\psi} : [c,d] \subseteq \mathbb{R} \to K_1$ *be the FNVF is known as the granular differentiable (gr-differentiable) of* $\widetilde{\psi}$ *at* $z \in [c,d]$, *if there exists a FN* $\frac{d_{gr}\widetilde{\psi}(z)}{dz} \in K_1$ *such that the following limit exists*

$$\lim_{h\to0}\frac{\widetilde{\psi}[z+h] - \widetilde{\psi}[z]}{h} = \frac{d_{gr}\widetilde{\psi}(z)}{dz},$$

where the limit is exists in the (D^{gr}, K_1) *metric space.*

Theorem 2.1. *[10] Given a FNVF $\widetilde{\psi} : [c,d] \subseteq \mathbb{R} \to K_1$ is granularly differentiable at the point $z \in [c,d]$ if and only if its HMF is differentiable with respect to z at that point. Moreover,*

$$\mathbb{H}\Big(\frac{d_{gr}\widetilde{\psi}(z)}{dz}\Big) = \frac{\partial}{\partial z}\mathbb{H}(\widetilde{\psi}(z)\big).$$

Definition 2.7. *[10] Suppose FNVF $\widetilde{\psi} : [c,d] \subseteq \mathbb{R} \to K_1$ is continuous with HMF $\psi^{gr}(z,\theta,\beta_\psi)$ is integrable on $z \in [c,d]$ and $\int_c^d \widetilde{\psi}(z)dz$ is the integral of $\widetilde{\psi}$ on $[c,d]$. The FNVF $\widetilde{\psi}$ is known as granularly integrable in the interval $[c,d]$ if there exists a FN $\widetilde{n} = \int_c^d \widetilde{\psi}(z)dz$ such that $\mathbb{H}(\widetilde{n}) = \int_c^d \mathbb{H}[\widetilde{\psi}(z)]dz$.*

Theorem 2.2. *[10] Suppose the FNVF $\widetilde{\psi} : [c,d] \subseteq \mathbb{R} \to K_1$ is gr-differentiable, $\frac{d_{gr}\widetilde{\psi}(z)}{dz}$ be a continuous FNVF in interval [c,d], then $\int_c^d (\frac{d_{gr}\widetilde{\psi}(z)}{dz})dz = \widetilde{\psi}(d) - \widetilde{\psi}(c)$.*

Theorem 2.3. *[11] Let $\widetilde{\psi} : K^n \to K_1$ and $\widetilde{\phi}_j : [c,d] \subseteq \mathbb{R} \to K_1$ for j=1, 2, ... m. The function $\widetilde{\psi}\big(\widetilde{\phi}_1(z), \widetilde{\phi}_2(z), ...\widetilde{\phi}_j(z), ...\widetilde{\phi}_m(z)\big)$ is known as granular partial (gr-partial) differentiable with respect to z if there exists a FN $\frac{\partial_{gr}}{\partial \widetilde{\phi}_j}\widetilde{\psi}\big(\widetilde{\phi}_1(z), ..., \widetilde{\phi}_j(z), ...\widetilde{\phi}_m(z)\big)$, such that the below limit exists*

$$\lim_{h\to 0}\frac{1}{h}\Big(\widetilde{\psi}\big(\widetilde{\phi}_1(z), ..., \widetilde{\phi}_j(z) + h, ...\widetilde{\phi}_m(z)\big) - \widetilde{\psi}\big(\widetilde{\phi}_1(z), ..., \widetilde{\phi}_j(z), ...\widetilde{\phi}_m(z)\big)\Big)$$
$$= \frac{\partial_{gr}}{\partial\widetilde{\phi}_j}\widetilde{\psi}\big(\widetilde{\phi}_1(z), ...\widetilde{\phi}_j(z), ..\widetilde{\phi}_m(z)\big).$$

Theorem 2.4. *[11] Let $\widetilde{\phi} : [c,d] \subseteq \mathbb{R} \to K_1$ and $\widetilde{\psi} : B \subseteq A \to K_1$ be FNVFs. Suppose $\widetilde{\phi}$ is gr-differentiable at the point $z \in [c,d]$ and $\widetilde{\psi}$ is gr-partial differentiable at that point $\widetilde{\phi}(z)$. The gr-derivative of the composite function $\widetilde{\psi} \circ \widetilde{\phi}(z)$ at that point z is $\frac{d_{gr}}{dz}[\widetilde{\psi} \circ \widetilde{\phi}(z)] = \frac{\partial_{gr}}{\partial\widetilde{\phi}}\widetilde{\psi}\big(\widetilde{\phi}(z)\big)\frac{d_{gr}\widetilde{\phi}(z)}{dz}$.*

3. Conformable Granular Derivative

Definition 3.1. *The FNVF $\widetilde{\psi} : [c,d] \subseteq \mathbb{R} \to K_1$ is called the $\alpha-$ granular differentiable at z if $\forall \epsilon > 0, \delta > 0, |h| < \delta$ such that*

$$D_{gr}^{\alpha}\left[\frac{\widetilde{\psi}(z + hz^{1-\alpha}) - \widetilde{\psi}(z)}{h}, T_{gr}^{\alpha}\widetilde{\psi}(z)\right] < \epsilon,$$

where, $T_{gr}^{\alpha}\widetilde{\psi}(z) \in K_1$ (provided it exists) and $\alpha \in [0,1]$. We read $T_{gr}^{\alpha}\widetilde{\psi}(z)$ as $\alpha-$ granular derivative of $\widetilde{\psi}$ at z if its $\alpha-$ granular derivative exists at z, $\alpha-$granular derivative is also called conformable granular derivative(CGD).

Theorem 3.1. *If a FNVF$\widetilde{\psi} : [c,d] \subseteq \mathbb{R} \to K_1$ is a CGD at $z_0 > 0, \alpha \in [0,1]$, then $\widetilde{\psi}$ is continuous at z_0.*

Proof. Consider

$$\widetilde{\psi}(z_0 + hz^{1-\alpha}) - \widetilde{\psi}(z_0) = \frac{\widetilde{\psi}(z_0 + hz^{1-\alpha}) - \widetilde{\psi}(z_0)}{h}h.$$

Apply limit on both sides,

$$\lim_{h\to 0}[\widetilde{\psi}(z_0 + hz^{1-\alpha}) - \widetilde{\psi}(z_0)] = \left(\lim_{h\to 0}\frac{\widetilde{\psi}(z_0 + hz^{1-\alpha}) - \widetilde{\psi}(z_0)}{h}\right)\Big(\lim_{h\to 0} h\Big).$$

Let $hz^{1-\alpha} = \epsilon$, then

$$\lim_{\epsilon \to 0}(\widetilde{\psi}[z_0 + \epsilon] - \widetilde{\psi}[z_0]) = T^{\alpha}_{gr}\widetilde{\psi}(z_0) \times 0 = 0,$$

which implies that

$$\lim_{\epsilon \to 0} \widetilde{\psi}(z_0 + \epsilon) = \widetilde{\psi}(z_0).$$

Hence $\widetilde{\psi}$ is continuous at z_0. □

Theorem 3.2. *Given $\widetilde{\psi} : [c, d] \subseteq \mathbb{R} \to K_1$ is FNVF. If $\widetilde{\psi}$ is granular differentiable, then $T^{\alpha}_{gr}\widetilde{\psi} = z^{1-\alpha}\frac{d_{gr}}{dz}\widetilde{\psi}$.*

Proof. From definition 3.1, we know that

$$T^{\alpha}_{gr}\widetilde{\psi}(z) = \lim_{h \to 0} \frac{\widetilde{\psi}\Big(z + hz^{1-\alpha}\Big) - \widetilde{\psi}(z)}{h}.$$

Let $hz^{1-\alpha} = \epsilon$, then

$$\begin{aligned} T^{\alpha}_{gr}\widetilde{\psi}(z) &= \lim_{\epsilon \to 0} \frac{\widetilde{\psi}(z + \epsilon) - \widetilde{\psi}(z)}{\epsilon z^{\alpha-1}} \\ &= z^{1-\alpha} \lim_{\epsilon \to 0} \frac{\widetilde{\psi}(z + \epsilon) - \widetilde{\psi}(z)}{\epsilon}. \end{aligned}$$

Therefore,

$$T^{\alpha}_{gr}\widetilde{\psi}(z) = z^{1-\alpha}\frac{d_{gr}}{dz}\widetilde{\psi}.$$

□

Theorem 3.3. *Given a FNVF $\widetilde{\psi} : [c, d] \subseteq \mathbb{R} \to K_1$ is a CGD at the point z if and only if its HMF is differentiable with respect to z at the point. Furthermore, $\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}(z)\Big) = z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}(z, \theta, \beta_{\psi})$.*

Proof. From the Theorem 2.1 and 3.2, we have

$$\begin{aligned} \mathbb{H}\Big(\frac{d_{gr}\widetilde{\psi}}{dz}\Big) &= \frac{\partial}{\partial z}\psi^{gr}\Big(z, \theta, \beta_{\psi}\Big) and \\ T^{\alpha}_{gr}\widetilde{\psi}(z) &= z^{1-\alpha}\frac{d_{gr}}{dz}\widetilde{\psi}. \end{aligned}$$

Consider,

$$\begin{aligned} \mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}(z)\Big) &= \mathbb{H}\Big(z^{1-\alpha}\frac{dgr\widetilde{\psi}}{dz}\Big) \\ &= z^{1-\alpha}\mathbb{H}\Big(\frac{dgr\widetilde{\psi}}{dz}\Big) \\ &= z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}(z, \theta, \beta_{\psi}). \end{aligned}$$

Therefore,

$$\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}(z)\Big) = z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}(z, \theta, \beta_{\psi}).$$

□

Example 3.1. *Suppose that* $\widetilde{\psi}(z) = \Big(2.3,\ 5.6,\ 9.7\Big)\sin z, z \in [0,3]$.
Then its HMF is given by

$$\mathbb{H}[\widetilde{\psi}(z)] = \Big[2.3 + 3.3\theta + 7.4(1-\theta)\beta_{\psi}\Big]\sin z.$$

Now the HMF of $T^{\alpha}_{gr}\widetilde{\psi}(z)$ *is*

$$\begin{aligned}\mathbb{H}[T^{\alpha}_{gr}\widetilde{\psi}(z)] &= z^{1-\alpha}\frac{\partial}{\partial z}\mathbb{H}[\widetilde{\psi}(z)]\\ &= z^{1-\alpha}\Big[2.3 + 3.3\theta + 7.4(1-\theta)\beta_{\psi}\Big]\cos z.\end{aligned}$$

From Note 2.3, we have

$$\begin{aligned}[T^{\alpha}_{gr}\widetilde{\psi}(z)]^{\theta} = \Bigg[&\inf_{\theta\le\gamma}\min_{\beta_{\psi}} z^{1-\alpha}\cos z\Big(2.3 + 3.3\gamma + 7.4(1-\gamma)\beta_{\psi}\Big),\\ &\sup_{\theta\le\gamma}\max_{\beta_{\psi}} z^{1-\alpha}\cos z\Big(2.3 + 3.3\gamma + 7.4(1-\gamma)\beta_{\psi}\Big)\Bigg],\end{aligned} \tag{3.1}$$

where θ, $\beta_{\psi} \in [0,1]$. *Applying inverse HMF, we get* θ*-level sets of derivative* (3.1), *using MATLAB and is depicted in Figure 1.*

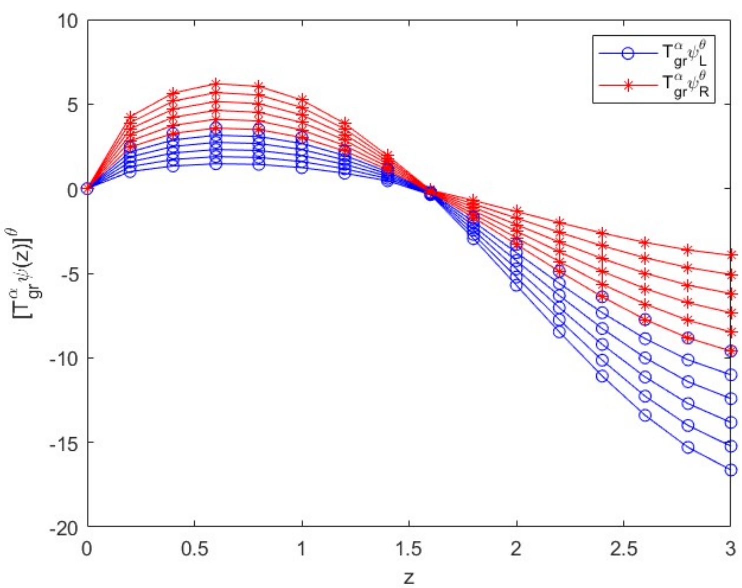

FIGURE 1. θ-level sets of CGD of the FNVF $\widetilde{\psi}(z)$ corresponding to the Example 3.1 with $\alpha = 0.5$.

Theorem 3.4. *If* $\alpha \in (0,1]$. *Given that if* $\widetilde{\psi}, \widetilde{\phi} : [c,d] \subseteq \mathbb{R} \to K_1$ *be conformable granular differentiable functions at a point* $z > 0$, *then*

1. $T^{\alpha}_{gr}(\widetilde{\psi} + \widetilde{\phi}) = T^{\alpha}_{gr}\widetilde{\psi} + T^{\alpha}_{gr}\widetilde{\phi}$;

2. $T^{\alpha}_{gr}(\widetilde{\psi}-\widetilde{\phi}) = T^{\alpha}_{gr}\widetilde{\psi} - T^{\alpha}_{gr}\widetilde{\phi}$;

3. $T^{\alpha}_{gr}(C\widetilde{\psi}) = CT^{\alpha}_{gr}\widetilde{\psi}$, *where* $C \in \mathbb{R}$;

4. $T^{\alpha}_{gr}(\widetilde{\psi\phi}) = \widetilde{\psi}T^{\alpha}_{gr}\widetilde{\phi} + \widetilde{\phi}T^{\alpha}_{gr}\widetilde{\phi}$;

5. $T^{\alpha}_{gr}\Big(\frac{\widetilde{\psi}}{\widetilde{\phi}}\Big) = \frac{\widetilde{\phi}T^{\alpha}_{gr}\widetilde{\psi}-\widetilde{\psi}T^{\alpha}_{gr}\widetilde{\phi}}{(\widetilde{\phi})^2}$, $\widetilde{\phi} \neq \widetilde{0}$;

6. $T^{\alpha}_{gr}(\lambda) = \widetilde{0}$, *where* λ *is a constant.*

Proof. (1) From Theorem 3.3, we have

$$\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}(z)\Big) = z^{1-\alpha}\tfrac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}).$$

Consider

$$\begin{aligned}
\mathbb{H}\Big(T^{\alpha}_{gr}\Big(\widetilde{\psi}\ +\widetilde{\phi}\Big)\Big) &= z^{1-\alpha}\frac{\partial}{\partial z}\Big[\psi^{gr}(z,\theta,\beta_{\psi}) + \phi^{gr}(z,\theta,\beta_{\phi})\Big] \\
&= z^{1-\alpha}\Big[\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\Psi}) + \frac{\partial}{\partial z}\phi^{gr}(z,\theta,\beta_{\phi})\Big] \\
&= z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}) + z^{1-\alpha}\frac{\partial}{\partial z}\phi^{gr}(z,\theta,\beta_{\phi}) \\
&= \mathbb{H}\Big[T^{\alpha}_{gr}\widetilde{\psi} + T^{\alpha}_{gr}\widetilde{\phi}\Big].
\end{aligned}$$

From Note 2.3, we have

$$T^{\alpha}_{gr}(\widetilde{\psi}+\widetilde{\phi}) = T^{\alpha}_{gr}\widetilde{\psi} + T^{\alpha}_{gr}\widetilde{\phi}.$$

(2) From Theorem 3.3, we have

$$\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}(z)\Big) = z^{1-\alpha}\tfrac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}).$$

Consider

$$\begin{aligned}
\mathbb{H}\Big(T^{\alpha}_{gr}\Big(\widetilde{\psi}-\widetilde{\phi}\Big)\Big) &= z^{1-\alpha}\frac{\partial}{\partial z}\Big[\psi^{gr}(z,\theta,\beta_{\psi}) - \phi^{gr}(z,\theta,\beta_{\phi})\Big] \\
&= z^{1-\alpha}\Big[\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}) - \frac{\partial}{\partial z}\phi^{gr}(z,\theta,\beta_{\phi})\Big] \\
&= z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}) - z^{1-\alpha}\frac{\partial}{\partial z}\phi^{gr}(z,\theta,\beta_{\phi}) \\
&= \mathbb{H}\Big[T^{\alpha}_{gr}\widetilde{\psi} - T^{\alpha}_{gr}\widetilde{\phi}\Big].
\end{aligned}$$

From Note 2.3, we have

$$T^{\alpha}_{gr}(\widetilde{\psi}-\widetilde{\phi}) = T^{\alpha}_{gr}\widetilde{\psi} - T^{\alpha}_{gr}\widetilde{\phi}.$$

(3) From Theorem 3.3, we have

$$\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}(z)\Big) = z^{1-\alpha}\tfrac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}).$$

Consider

$$\begin{aligned}
\mathbb{H}\left(T^{\alpha}_{gr}(C\widetilde{\psi})\right) &= z^{1-\alpha}\frac{\partial}{\partial z}(C\widetilde{\psi}^{gr}(z,\theta,\beta_{\psi}))\\
&= z^{1-\alpha}C\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi})\\
&= C\left[z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi})\right]\\
\mathbb{H}\left(T^{\alpha}_{gr}C\widetilde{\psi}\right) &= \mathbb{H}\left(CT^{\alpha}_{gr}\widetilde{\psi}\right)
\end{aligned}$$

From Note 2.3, we get

$$T^{\alpha}_{gr}(C\widetilde{\psi}) = CT^{\alpha}_{gr}\widetilde{\psi}.$$

(4) From Theorem 3.3, we have

$$\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}(z)\Big) = z^{1-\alpha}\tfrac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}).$$

Consider

$$\begin{aligned}
&\mathbb{H}\Big[T^{\alpha}_{gr}\Big(\widetilde{\psi\phi}\Big)\Big]\\
&= z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}\phi^{gr}\\
&= z^{1-\alpha}\Big[\psi^{gr}(z,\theta,\beta_{\psi})\frac{\partial}{\partial z}\phi^{gr}(z,\theta,\beta_{\phi}) + \phi^{gr}(z,\theta,\beta_{\phi})\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi})\Big]\\
&= \psi^{gr}(z,\theta,\beta_{\psi}).z^{1-\alpha}\frac{\partial}{\partial z}\phi^{gr}(z,\theta,\beta_{\phi}) + \phi^{gr}(z,\theta,\beta_{\phi}).z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi})\\
&= \mathbb{H}(\widetilde{\psi}).\mathbb{H}(T^{\alpha}_{gr}\widetilde{\phi}) + \mathbb{H}(\widetilde{\phi}).\mathbb{H}(T^{\alpha}_{gr}\widetilde{\psi})\\
&= \mathbb{H}\Big[\widetilde{\psi}T^{\alpha}_{gr}\widetilde{\phi} + \widetilde{\phi}T^{\alpha}_{gr}\widetilde{\psi}\Big].
\end{aligned}$$

From Note 2.3, we get

$$T^{\alpha}_{gr}(\widetilde{\psi\phi}) = \widetilde{\psi}T^{\alpha}_{gr}\widetilde{\phi} + \widetilde{\phi}T^{\alpha}_{gr}\widetilde{\psi}.$$

(5) From Theorem 3.3, we have

$$\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}(z)\Big) = z^{1-\alpha}\tfrac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}).$$

Consider

$$\mathbb{H}\Big[T^{\alpha}_{gr}\Big(\frac{\widetilde{\psi}}{\widetilde{\phi}}\Big)\Big]$$
$$= z^{1-\alpha}\frac{\partial}{\partial z}\Big[\frac{\psi^{gr}(z,\theta,\beta_{\psi})}{\phi^{gr}(z,\theta,\beta_{\phi})}\Big]$$
$$= z^{1-\alpha}.\frac{\phi^{gr}(z,\theta,\beta_{\phi})\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}) - \psi^{gr}(z,\theta,\beta_{\psi})\frac{\partial}{\partial z}\phi^{gr}(z,\theta,\beta_{\phi})}{\Big[\phi^{gr}(z,\theta,\beta_{\phi})\Big]^2}$$
$$= \frac{z^{1-\alpha}\phi^{gr}(z,\theta,\beta_{\phi})\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}) - z^{1-\alpha}\psi^{gr}(z,\theta,\beta_{\psi})\frac{\partial}{\partial z}\phi^{gr}(z,\theta,\beta_{\phi})}{\Big[\phi^{gr}(z,\theta,\beta_{\phi})\Big]^2}$$
$$= \frac{\phi^{gr}(z,\theta,\beta_{\phi}).z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}) - \psi^{gr}(z,\theta,\beta_{\phi})z^{1-\alpha}\frac{\partial}{\partial z}\phi^{gr}(z,\theta,\beta_{\phi})}{\Big[\phi^{gr}(z,\theta,\beta_{\phi})\Big]^2}$$
$$= \frac{\mathbb{H}(\widetilde{\phi})\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}\Big) - \mathbb{H}(\widetilde{\psi})\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\phi}\Big)}{\mathbb{H}\Big[\Big(\widetilde{\phi}\Big)^2\Big]}$$
$$= \mathbb{H}\Big[\frac{\widetilde{\phi}T^{\alpha}_{gr}\widetilde{\psi} - \widetilde{\psi}T^{\alpha}_{gr}\widetilde{\phi}}{(\widetilde{\phi})^2}\Big].$$

From note 2.3, we get

$$T^{\alpha}_{gr}\Big(\frac{\widetilde{\psi}}{\widetilde{\phi}}\Big) = \frac{\widetilde{\phi}T^{\alpha}_{gr}\widetilde{\psi} - \widetilde{\psi}T^{\alpha}_{gr}\widetilde{\phi}}{(\widetilde{\phi})^2}.$$

(6) From the Theorem 3.3, we have $\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}(z)\Big) = z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi})$.
Consider

$$\mathbb{H}\Big(T^{\alpha}_{gr}\lambda\Big) = z^{1-\alpha}\frac{\partial}{\partial z}\lambda$$
$$= z^{1-\alpha}0 = \widetilde{0}.$$

□

Example 3.2. *Consider* $\widetilde{\psi}(z) = \widetilde{3}(z^2 - z), \widetilde{\phi}(z) = \widetilde{3}z$*, for* $z \in [0,1]$ *and* $\widetilde{3}$ *= (2, 3, 4). Since* $\mathbb{H}[\widetilde{3}] = (2+\theta) + 2(1-\theta)\beta_3$*, then we have* $\mathbb{H}[\widetilde{\psi}(z)] = \Big[(2+\theta) + 2(1-\theta)\beta_3\Big](z^2 - z)$ *and* $\mathbb{H}[\widetilde{\phi}(z)] = \Big[(2+\theta) + 2(1-\theta)\beta_3\Big]z$*. Using Theorem 3.4, we get*

(1) $\mathbb{H}\Big(T^{\alpha}_{gr}(\widetilde{\psi}+\widetilde{\phi})\Big) = 2z^{2-\alpha}[(2+\theta) + 2(1-\theta)\beta_3]$
therefore $[T^{\alpha}_{gr}(\widetilde{\psi}+\widetilde{\phi})]^{\theta} = \mathbb{H}^{-1}\Big[2z^{2-\alpha}((2+\theta) + 2(1-\theta)\beta_3)\Big]$*, using MATLAB draws θ-sets and depicted in Figure 2.*

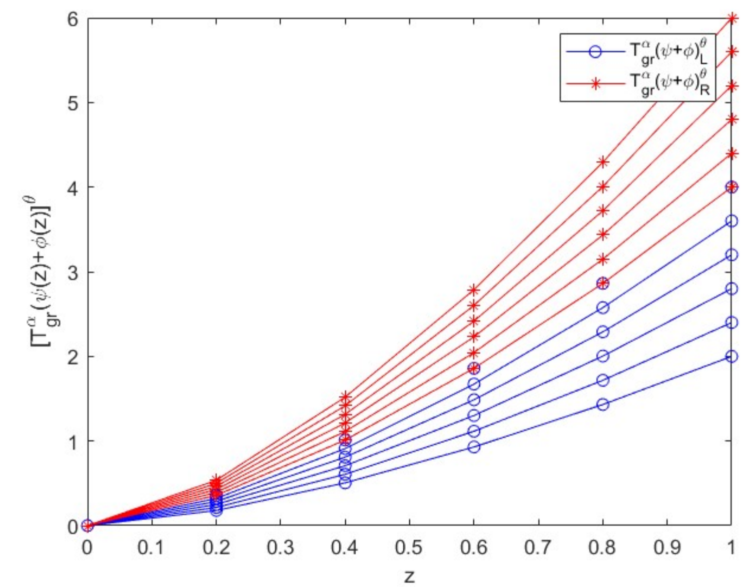

FIGURE 2. θ-level sets of CGD of the fuzzy function $(\widetilde{\psi}+\widetilde{\phi})$ corresponding to the Example 3.2 with $\alpha = 0.5$.

(2) $\mathbb{H}\Big(T^{\alpha}_{gr}(\widetilde{\psi}-\widetilde{\phi})\Big) = \Big(2(z-1)z^{1-\alpha}[(2+\theta)+2(1-\theta)\beta_3)\Big]$,
therefore $[T^{\alpha}_{gr}(\widetilde{\psi}-\widetilde{\phi})]^{\theta} = \mathbb{H}^{-1}\Big[2(z-1)z^{1-\alpha}((2+\theta)+2(1-\theta)\beta_3)\Big]$, *using MATLAB draws θ-sets and depicted in Figure 3.*

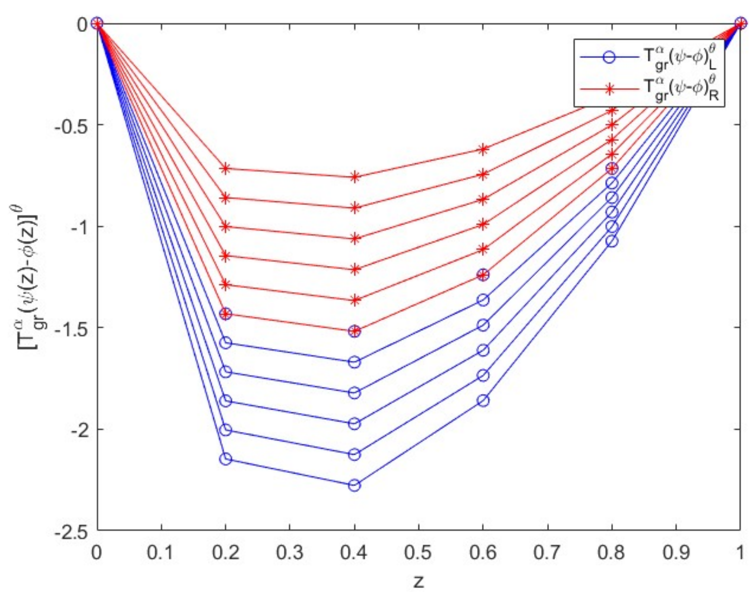

FIGURE 3. θ-level sets of CGD of the fuzzy function $(\widetilde{\psi}-\widetilde{\phi})$ corresponding to the Example 3.2 with $\alpha = 0.5$.

(3) $\mathbb{H}\Big[T^{\alpha}_{gr}(\widetilde{\psi\phi})\Big] = \Big[(2+\theta)+2(1-\theta)\beta_3\Big]^2 z^{1-\alpha}(3z^2-2z)$
therefore $[T^{\alpha}_{gr}(\widetilde{\psi\phi})]^{\theta} = \mathbb{H}^{-1}\Big[\Big((2+\theta)+2(1-\theta)\beta_3\Big)^2 z^{1-\alpha}(3z^2-2z)\Big]$, *using MATLAB draws θ-sets and depicted in Figure 4.*

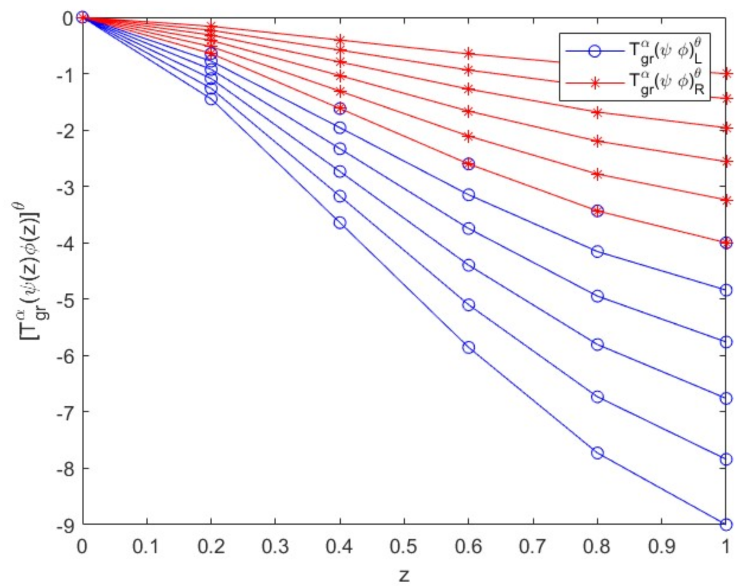

FIGURE 4. θ-level sets of CGD of the FNVF $\widetilde{\psi\phi}$ corresponding to the Example 3.2 with $\alpha = 0.5$.

Theorem 3.5. *: Let $\alpha \in [0, \infty]$. If $\widetilde{\psi}$ is granular differentiable at $\widetilde{\phi}(z)$ and $\widetilde{\phi}$ is CGD then*

$$T^{\alpha}_{gr}\Big[\widetilde{\psi}\Big(\widetilde{\phi}(z)\Big)\Big] = \frac{d_{gr}}{dz}\widetilde{\psi}\Big(\widetilde{\phi}(z)\Big)T^{\alpha}_{gr}\widetilde{\phi}(z).$$

Proof. From the Theorem 3.3, we have

$$\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}(z)\Big) = z^{1-\alpha}\tfrac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi}).$$

Consider,

$$\begin{aligned}\mathbb{H}\Big[T^{\alpha}_{gr}\Big[\widetilde{\psi}\Big(\widetilde{\phi}(z)\Big)\Big]\Big] &= z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}\Big(\phi^{gr}(z)\Big)\\ &= z^{1-\alpha}\frac{\partial}{\partial z}\psi^{gr}\Big(\phi^{gr}(z)\Big).\frac{\partial}{\partial z}\phi^{gr}(z)\\ &= \frac{\partial}{\partial z}\psi^{gr}\Big(\phi^{gr}(z)\Big)z^{1-\alpha}\frac{\partial}{\partial z}\phi^{gr}(z)\\ &= \mathbb{H}\Big[\frac{\partial}{\partial z}\widetilde{\psi}\Big(\widetilde{\phi}(z)\Big)\Big].\mathbb{H}\Big[T^{\alpha}_{gr}\widetilde{\phi}(z)\Big]\end{aligned}$$

From Note 2.3, we have

$$T^{\alpha}_{gr}\Big[\widetilde{\psi}\Big(\widetilde{\phi}(z)\Big)\Big] = \frac{d_{gr}}{dz}\widetilde{\psi}\Big(\widetilde{\phi}(z)\Big)T^{\alpha}_{gr}\widetilde{\phi}(z).$$

□

Example 3.3. *Consider* $\widetilde{\psi}(\widetilde{\phi}(z)) = \sin(\widetilde{\phi}(z))$, *where* $\widetilde{\phi}(z) = e^z + \widetilde{2}, \widetilde{2} = (1,2,3)$. *Since* $\mathbb{H}(\widetilde{\phi}(z)) = e^z + (1+\theta) + 2(1-\theta)\beta_2$ *then we have* $\mathbb{H}\Big(T^{\alpha}_{gr}\widetilde{\psi}\big(\widetilde{\phi}(z)\big)\Big) = z^{1-\alpha}e^z\cos(\big(e^z + (1+\theta) + 2(1-\theta)\beta_2\big))$. *Therefore*

$$T^{\alpha}_{gr}\widetilde{\psi}\Big(\widetilde{\phi}(z)\Big) = \mathbb{H}^{-1}\Big[z^{1-\alpha}e^z\cos(e^z + (1+\theta) + 2(1-\theta)\beta_2\Big].$$

4. Conformable Granular Integral

Definition 4.1. *Suppose* $\widetilde{\psi} : \mathbb{R} \to K_1$ *is continuous FNVF whose HMF* $\psi^{gr}(z,\theta,\beta_{\psi})$ *is integrable on* $z \in [c,d]$. *Let* $\int_c^d z^{\alpha-1}\widetilde{\psi}(z)dz$ *is conformable fractional integral on [c,d]. Then the fuzzy function* $\widetilde{\psi}$ *is said to be conformable granular integrable (CGI) on [c,d] if there exists a fuzzy number* $\widetilde{n} = \int_c^d z^{\alpha-1}\widetilde{\psi}(z)dz$ *such that* $\mathbb{H}(\widetilde{n}) = \int_c^d z^{\alpha-1}\widetilde{\psi}(z)dz$ *and this integral is denoted by* $I^{\alpha}_{gr}\widetilde{\psi}(z)$.

Theorem 4.1. *Given* $\widetilde{\psi}, \widetilde{\phi}$ *be FNVFs. Then*

$$I^{\alpha}_{gr}\Big[\widetilde{\psi}(z) + \widetilde{\phi}(z)\Big] = I^{\alpha}_{gr}\widetilde{\psi}(z) + I^{\alpha}_{gr}\widetilde{\phi}(z).$$

Proof. From the definition of CGI, we have

$$\begin{aligned}\mathbb{H}\Big[I^{\alpha}_{gr}\Big[\widetilde{\psi}(z) + \widetilde{\phi}(z)\Big]\Big] &= \int_0^z z^{\alpha-1}\mathbb{H}\Big[\widetilde{\psi}(z) + \widetilde{\phi}(z)\Big]dz \\ &= \int_0^z z^{\alpha-1}\mathbb{H}\Big(\widetilde{\psi}(z)dz\Big) + \int_0^z z^{\alpha-1}\mathbb{H}\Big(\widetilde{\phi}(z)dz\Big) \\ &= \mathbb{H}\Big[I^{\alpha}_{gr}\widetilde{\psi}(z)\Big] + \mathbb{H}\Big[I^{\alpha}_{gr}\widetilde{\phi}(z)\Big].\end{aligned}$$

From Note 2.3, we have

$$I^{\alpha}_{gr}\Big[\widetilde{\psi}(z) + \widetilde{\phi}(z)\Big] = I^{\alpha}_{gr}\widetilde{\psi}(z) + I^{\alpha}_{gr}\widetilde{\phi}(z).$$

□

Theorem 4.2. *If* $\widetilde{\psi} : [c,d] \subseteq \mathbb{R} \to K_1$ *is FNVF, then* $I^{\alpha}_{gr}\Big[\lambda\widetilde{\psi}(z)\Big] = \lambda I^{\alpha}_{gr}\widetilde{\psi}(z)$ *when* $\lambda \in \mathbb{R}$.

Proof. From the definition of CGI, we have

$$\begin{aligned}\mathbb{H}\Big[I^{\alpha}_{gr}\Big(\lambda\widetilde{\psi}(z)\Big)\Big] &= \int_0^z z^{\alpha-1}\mathbb{H}\Big(\lambda\widetilde{\psi}(z)\Big)dz \\ &= \lambda\int_0^z z^{\alpha-1}\mathbb{H}\Big(\widetilde{\psi}(z)\Big)dz \\ &= \lambda\mathbb{H}\Big[I^{\alpha}_{gr}\widetilde{\psi}(z)\Big].\end{aligned}$$

From Note 2.3, we get

$$I^{\alpha}_{gr}\Big[\lambda\widetilde{\psi}(z)\Big] = \lambda I^{\alpha}_{gr}\widetilde{\psi}(z).$$

□

Theorem 4.3. *If* $\widetilde{\psi} : [c,d] \subseteq \mathbb{R} \to K_1$, *be continuous FNVF then* $T^{\alpha}_{gr}I^{\alpha}_{gr}\widetilde{\psi}(z) = \widetilde{\psi}(z)$.

Proof. From the definition of CGI, we have

$$\begin{aligned}\mathbb{H}\Big[T^{\alpha}_{gr}I^{\alpha}_{gr}\widetilde{\psi}(z)\Big] &= z^{1-\alpha}\frac{\partial}{\partial z}I^{\alpha}\psi^{gr}(z,\theta,\beta_{\psi})\\ &= z^{1-\alpha}\frac{\partial}{\partial z}\int_0^z z^{\alpha-1}\psi^{gr}(z,\theta,\beta_{\psi})dz\\ &= z^{1-\alpha}z^{\alpha-1}\psi^{gr}(z,\theta,\beta_{\psi})\\ &= \mathbb{H}\Big[\widetilde{\psi}(z)\Big].\end{aligned}$$

From Note 2.3, we get

$$T^{\alpha}_{gr}I^{\alpha}_{gr}\widetilde{\psi}(z) = \widetilde{\psi}(z).$$

□

Theorem 4.4. *If $\widetilde{\psi} : [c,d] \subseteq \mathbb{R} \to K_1$ is $\alpha-$granular differentiable, then ${}_aI^{\alpha}_{gr}T^{\alpha}_{gr}\widetilde{\psi}(z) = \widetilde{\psi}(z) - \widetilde{\psi}(c)$.*

Proof. We know that

$${}_aI^{\alpha}_{gr}T^{\alpha}_{gr}\widetilde{\psi}(z) =_a I^{\alpha}_{gr}z^{1-\alpha}\widetilde{\psi}'(z).$$

Consider,

$$\begin{aligned}\mathbb{H}\Big[{}_aI^{\alpha}_{gr}T^{\alpha}_{gr}\widetilde{\psi}(z)\Big] &= \int_c^z z^{\alpha-1}z^{1-\alpha}\mathbb{H}\Big(\widetilde{\psi}'(z)\Big)dz\\ &= \int_c^z \frac{\partial}{\partial z}\psi^{gr}(z,\theta,\beta_{\psi})dz\\ &= \psi^{gr}|_c^z\\ &= \psi^{gr}(z) - \psi^{gr}(c)\\ &= \mathbb{H}\left(\widetilde{\psi}(z)\right) - \mathbb{H}\left(\widetilde{\psi}(c)\right).\end{aligned}$$

From Note 2.3, we get

$${}_cI^{\alpha}_{gr}T^{\alpha}_{gr}\widetilde{\psi}(z) = \widetilde{\psi}(z) - \widetilde{\psi}(c).$$

□

Theorem 4.5. *(Integration by parts): Let $\widetilde{\psi}, \widetilde{\phi} : [c,d] \subseteq \mathbb{R} \to K_1$ be FNVF. Then*

$$\int_c^d \widetilde{\psi}(z)T^{\alpha}_{gr}\widetilde{\phi}(z)dz = \widetilde{\psi}\widetilde{\phi}|_c^d - \int_c^d \widetilde{\phi}(z)T^{\alpha}_{gr}\widetilde{\psi}(z)d_{\alpha}z,$$

where $d_{\alpha}z = z^{\alpha-1}dz$.

Proof. From the definition of CGI, we have

$$\begin{aligned}
\mathbb{H}\Big[\int_c^d \widetilde{\psi}(z) T_{gr}^{\alpha}\widetilde{\phi}(z)dz\Big] &= \int_c^d z^{\alpha-1}\mathbb{H}\Big[\widetilde{\psi}(z)\Big]\mathbb{H}\Big[T_{gr}^{\alpha}\widetilde{\psi}(z)\Big]dz \\
&= \int_c^d z^{\alpha-1}\psi^{gr}z^{1-\alpha}\frac{\partial}{\partial z}\phi^{gr}dz \\
&= \int_c^d \psi^{gr}\frac{\partial}{\partial z}\phi^{gr}dz \\
&= \psi^{gr}\phi^{gr}|_c^d - \int_c^d \frac{d}{dz}\psi^{gr}\phi^{gr}dz \\
&= \mathbb{H}(\widetilde{\psi})\mathbb{H}\Big(\widetilde{\phi}\Big)|_c^d - \int_c^d z^{\alpha-1}\Big(z^{1-\alpha}\frac{d}{dz}\psi^{gr}\Big)\phi^{gr}dz \\
&= \mathbb{H}(\widetilde{\psi})\mathbb{H}(\widetilde{\phi})|_c^d - \int_c^d z^{\alpha-1}\mathbb{H}\Big[T_{gr}^{\alpha}\widetilde{\psi}(z)\Big]\mathbb{H}(\widetilde{\phi})dz.
\end{aligned}$$

From the Note 2.3, we have

$$\int_c^d \widetilde{\psi}(z)T_{gr}^{\alpha}\widetilde{\phi}(z)dz = \widetilde{\psi}(z)\widetilde{\phi}(z)|_c^d - \int_c^d z^{\alpha-1}T_{gr}^{\alpha}\widetilde{\psi}(z)\widetilde{\phi}(z)dz.$$

□

Example 4.1. *Consider* $\widetilde{\psi}(z) = \widetilde{2}z^2$ *and* $\widetilde{\phi}(z) = e^z$, *where* $\widetilde{2} = (1,2,3)$. *Since* $\mathbb{H}[\widetilde{2}] = (1+\theta) + 2(1-\theta)\beta_2$ *then* $\Big[\int z^{\alpha-1}\widetilde{\psi}(z)T_{gr}^{\alpha}\widetilde{\phi}(z)dz\Big] = [(1+\theta)+2(1-\theta)\beta_2]e^z z^2 - 2\Big[(1+\theta)+2(1-\theta)\beta_2\Big](ze^z - e^z)$. *Apply inverse HMF, therefore we get* $\int z^{\alpha-1}\widetilde{\psi}(z)z_{gr}^{\alpha}\widetilde{\phi}(z)dz = \mathbb{H}^{-1}\Big[\Big\{(1+\theta)+2(1-\theta)\beta_2\Big\}e^z(z^2-2z+2)\Big]$.

Example 4.2. *Let* $\widetilde{\psi}(z) = e^z + \widetilde{1}$, *where* $\widetilde{1} = (0,2,3)$. *We know that* $I_{gr}^{\alpha}\widetilde{\psi}(z) = \int_0^z z^{\alpha-1}\widetilde{\psi}(z)dz$ *then* $\mathbb{H}\Big[I_{gr}^{\alpha}\widetilde{\psi}(z)\Big] = \int_0^z z^{\alpha-1}\mathbb{H}\Big[\widetilde{\psi}(z)\Big]dz$. *Since,* $\mathbb{H}\Big[\widetilde{1}\Big] = 2\theta + 3(1-\theta)\beta_1$, *where* $\beta_1 \in [0,1]$ *therefore,* $\mathbb{H}\Big[I_{gr}^{\alpha}\widetilde{\psi}(z)\Big] = \frac{z^{2\alpha-1}}{2\alpha-1} + \frac{z^{\alpha}}{\alpha}\Big[2\theta + 3(1-\theta)\beta_1\Big]$. *Applying inverse HMF, we get* $I_{gr}^{\alpha}\widetilde{\psi}(z) = \mathbb{H}^{-1}\Big[\frac{z^{2\alpha-1}}{2\alpha-1} + \frac{t}{\alpha}\Big(2\theta + 3(1-\theta)\beta_1\Big)\Big]$.

5. Conformable Granular Initial Value Problem and Applications

Consider the following CGIVP,

$$T_{gr}^{\alpha}\widetilde{\phi}(z) = \widetilde{\psi}(z, \widetilde{\phi}(z)) \tag{5.1}$$

$$\widetilde{\phi}(z_0) = \widetilde{\phi}_0, \tag{5.2}$$

where $\widetilde{\phi} : [c,d] \subseteq \mathbb{R} \to K_1$, $\widetilde{\psi} : [c,d] \times K_1 \to K_1$ is called fuzzy mapping and $\widetilde{\phi_0} \in K_1$ is initial condition.

The solution of CGIVP (5.1) is obtained using HMF according to the steps given below:

- Take HMF on both sides of equation (5.1), we get

$$z^{1-\alpha}\frac{\partial}{\partial z}\phi^{gr}(z,\theta,\beta_\phi) = \psi^{gr}(z,\phi^{gr},\beta_\psi) \tag{5.3}$$

$$\phi^{gr}(z_0,\theta,\beta_\phi) = \phi_0^{gr}(\theta,\beta_{\phi_0}), \tag{5.4}$$

  where $\beta_\phi \in (0,1)$. Hence equation (5.3) is a partial differential equation in one independent variable z. Therefore the equation (5.3) is taken as conformable fractional DEq.
- Solving equation (5.3), we get

$$\mathbb{H}(\Big(\widetilde{\phi}(z)\Big)) = \phi^{gr}\Big(z,\theta,\beta_\phi\Big). \tag{5.5}$$

- Applying inverse HMF on both sides of equation (5.5), we get

$$\Big[\widetilde{\phi}(z)\Big]^\theta = \Big[\inf_{\theta\le\gamma\le 1}\min_{\beta_\phi}\phi^{gr}(z,\gamma,\beta_\phi),\ \sup_{\theta\le\gamma\le 1}\max_{\beta_\phi}\phi^{gr}(z,\gamma,\beta_\phi)\Big]$$

  which is the $\theta-$cut solution of CGIVP equation (5.1).

The application problems of natural growth and decay models are discussed for FNVFs based on HMF with triangular FN.

Example 5.1. *Suppose the CGIVP of the growth model with the initial condition as triangular FN.*

$$T^\alpha_{gr}\widetilde{\phi}(z) = 0.5\widetilde{\phi}(z),\ z \in [0,5] \tag{5.6}$$

$$sbject\ to,\ \widetilde{\phi}(0) = (0.4, 0.6, 0.9). \tag{5.7}$$

Now $[\widetilde{\phi}(0)]^\theta = [0.4+0.2\theta, 0.9-0.3\theta]$ *where* $\theta \in [0,1]$.
Apply HMF on both sides of equation (5.6) *and* (5.7), *we get*

$$z^{1-\alpha}\frac{\partial}{\partial z}\phi^{gr} = 0.5\phi^{gr},\ with\ [\widetilde{\phi}(0)]^{gr}(\theta,\beta_\phi) = 0.4+0.2\theta+0.5(1-\theta)\beta_\phi, \tag{5.8}$$

where $\beta_\phi \in [0,1]$.
Solving (5.8), *we get*

$$\phi^{gr}(z,\theta,\beta_\phi) = [0.4+0.2\theta+0.5(1-\theta)\beta_\phi]e^{\frac{0.5z^\alpha}{\alpha}}, \tag{5.9}$$

which is the solution of equation (5.8).
Using inverse HMF on (5.9), *we get*

$$\Big[\widetilde{\phi}(z)\Big]^\theta = \Big[\inf_{\theta\le\gamma\le 1}\min_{\beta_\phi}\phi^{gr}(z,\gamma,\beta_\phi),\ \sup_{\theta\le\gamma\le 1}\max_{\beta_\phi}\phi^{gr}(z,\gamma,\beta_\phi)\Big],$$

which is the θ-cuts solution of equation (5.6), *using MATLAB the θ-sets are depicted in Figure 5.*

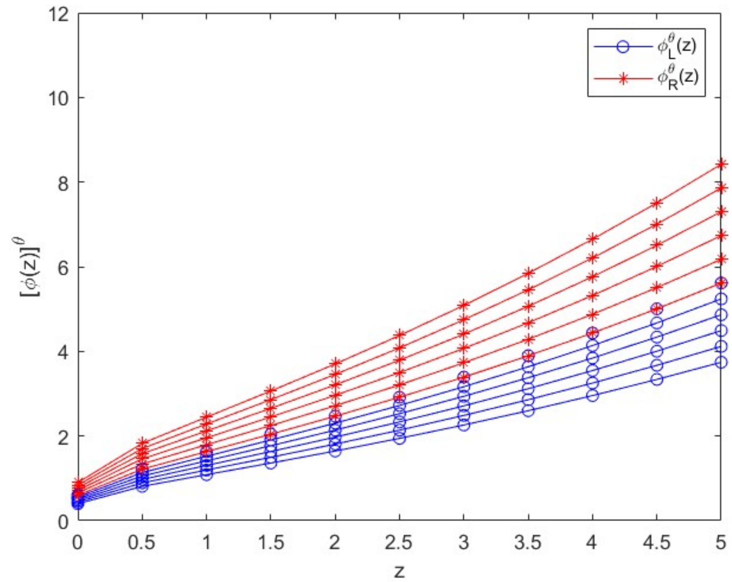

Figure 5. θ-level sets of solution of the IVP (5.6) corresponding to the Example 5.1 with $\alpha = 0.5$.

Example 5.2. *Suppose the CGIVP of the decay model with the initial condition as triangular FN.*

$$T_{gr}^{\alpha}\widetilde{\phi}(z) = -0.3\widetilde{\phi}(z),\ z \in [0,1] \tag{5.10}$$

$$\textit{sbject to, } \widetilde{\phi}(0) = (3.97, 4.3, 5.1). \tag{5.11}$$

Now $[\widetilde{\phi}(0)]^{\theta} = [3.97 + 0.33\theta, 5.1 - 0.8\theta]$, *where* $\theta \in [0,1]$.
Apply HMF on both sides of equation (5.10) *and* (5.11), *we get*

$$z^{1-\alpha}\frac{\partial}{\partial z}\phi^{gr} = -0.3\phi^{gr},\ \textit{with } [\widetilde{\phi}(0)]^{gr}(\theta, \beta_{\phi}) = 3.97 + 0.33\theta + 1.13(1-\theta)\beta_{\phi}, \tag{5.12}$$

where $\beta_{\phi} \in [0,1]$.
Solving (5.12), *we get*

$$\phi^{gr}(z, \theta, \beta_{\phi}) = [3.97 + 0.33\theta + 1.13(1-\theta)\beta_{\phi}]e^{\frac{-0.3z^{\alpha}}{\alpha}}, \tag{5.13}$$

which is the solution of equation (5.12).
Using inverse HMF on (5.13), *we get*

$$\left[\widetilde{\phi}(z)\right]^{\theta} = \left[\inf_{\theta \le \gamma \le 1} \min_{\beta_{\phi}} \phi^{gr}(z, \gamma, \beta_{\phi}),\ \sup_{\theta \le \gamma \le 1} \max_{\beta_{\phi}} \phi^{gr}(z, \gamma, \beta_{\phi})\right],$$

which is the θ-cuts solution of equation (5.10), *using MATLAB the θ-cut sets are depicted in Figure 6.*

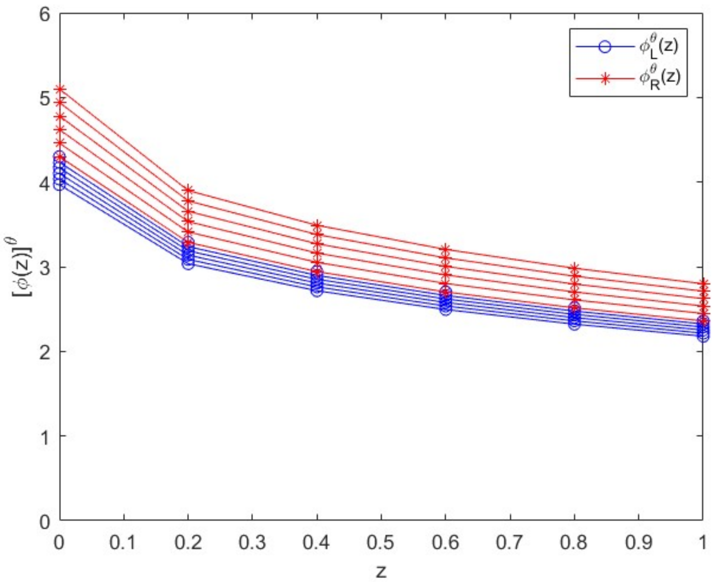

FIGURE 6. θ-level sets of solution of the IVP (5.10) corresponding to the Example 5.2 with $\alpha = 0.5$.

Conclusions. We have proposed a definition for the conformable granular derivative within the framework of HMF, incorporating a relative-distance-measure variable for FNVF. Additionally, we have established the fundamental properties of this novel derivative and have provided graphical examples for illustration. Furthermore, we have introduced the definition of the conformable granular integral and have developed a solution method for CGIVPs using HMF, which incorporates a relative-distance-measure variable. This approach proves beneficial in obtaining unique and bounded solutions, similar to IVPs involving crisp functions. We have also explored the applications of this framework in growth and decay problems. Moving forward, our future objectives include the development of a conformable granular system of DEqs as well as CGDEqs with boundary conditions.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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