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PREŠIĆ-BOYD-WONG TYPE RESULTS IN ORDERED METRIC SPACES

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Abstract. The purpose of this paper is to prove some Prešić-Boyd-Wong type fixed point theorems in ordered metric spaces. The results of this paper generalize the famous results of Prešić and Boyd-Wong in ordered metric s-paces. We also initiate the homotopy result in product spaces. Some examples are provided which illustrate the results proved herein.

1. Introduction and preliminaries

In 1922 Banach [26] proved the following theorem known as Banach contraction mapping theorem.

Theorem 1. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that*

$$d(fx, fy) \leq \lambda d(x, y) \tag{1}$$

for all $x, y \in X$, where $0 \leq \lambda < 1$, then there exists a unique $x \in X$ such that $fx = x$. This point x is called the fixed point of mapping f.

Due to simplicity and usefulness, several authors generalized the Banach contraction mapping theorem. One such generalization is given by Prešić [24, 25]. Prešić generalized the Banach contraction mapping theorem in product spaces and proved the following theorem.

Theorem 2. *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a mapping satisfying the following contractive type condition:*

$$d(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1}), \tag{2}$$

for every $x_1, x_2, \dots, x_k, x_{k+1} \in X$, where $q_1, q_2, \dots, q_k$ are nonnegative constants such that $q_1 + q_2 + \cdots + q_k < 1$. Then there exists a unique point $x \in X$ such that $f(x, x, \dots, x) = x$. Moreover if $x_1, x_2, \dots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = f(\lim x_n, \lim x_n, \dots, \lim x_n)$.

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Condition (2) in the case $k = 1$ reduces to the condition (1). So, Theorem 1 is a generalization of the Banach fixed point theorem. The results of Prešić is useful in proving the convergence of some particular sequences and in proving the existence of solutions of differences equations, for example, see [15,24,25,40]. For more on the generalizations of Prešić type operators the reader is referred to [12,16–19,21,28–36].

On the other hand Boyd and Wong [4] generalized the Banach contraction mapping theorem and proved the following theorem.

Theorem 3. *Let (X, d) be a complete metric space and $f : X \to X$ a mapping that satisfies*

$$d(fx, fy) \leq \psi(d(x, y)) \quad \text{for all} \quad x, y \in X, \tag{3}$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is upper semi-continuous function from the right (i.e., $\lambda_i \downarrow \lambda \geq 0 \Rightarrow \limsup_{i\to\infty} \psi(\lambda_i) \leq \psi(\lambda)$) such that $\psi(t) < t$ for each $t > 0$. Then f has a unique fixed point $u \in X$. Moreover, for each $x \in X$, $\lim_{n\to\infty} f^n x = u$.

Note that the condition (3) in the case $\psi(t) = \lambda t$ reduces to condition (1). So, Theorem 3 is a generalization of the Banach fixed point theorem. Some generalization of the Boyd-Wong theorem can be found in [9, 10, 13, 14, 20, 23, 38].

The existence of fixed point in partially ordered sets was investigated by Ran and Reurings [1] and then by Nieto and Lopez [7, 8]. Some applications of fixed point theorems in ordered metric spaces to differential equations can be seen in [7, 8]. Several authors generalized the results of these papers in different directions for example, see [2, 3, 5, 6, 22, 27, 37, 39, 41]. The following version of the fixed point theorem was proved, among others, in these papers.

Theorem 4. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that the following conditions hold:*

(i) *there exists $k \in (0, 1)$ such that $d(fx, fy) \leq kd(x, y)$ for all $x, y \in X$ with $y \preceq x$;*
(ii) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;*
(iii) *if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \leq x$, for all $n \in \mathbb{N}$.*

Then f has a fixed point $x^* \in X$.

Recently, in [36] Malhotra et al. defined the ordered Prešić type contraction mappings the setting of cone metric spaces (see also [19, 33]) and generalized the result of Prešić in ordered case. In the present paper, we generalize the results of Prešić, Boyd and Wong, Theorem 4 and several known results in and prove some Prešić-Boyd-Wong type fixed point theorems in ordered metric spaces. A homotopy result in the product spaces is also proved. Examples are included which illustrate the results.

Following definitions will be needed in sequel.

Definition 1. *Let X be any nonempty set, k a positive integer and $f : X^k \to X$ be a mapping. An element $x \in X$ is called a fixed point of f if $f(x, x, \ldots, x) = x$.*

Definition 2. *Let X be a nonempty set, k a positive integer, $f : X^k \to X$ and $g : X \to X$ be mappings.*

(i) *An element $x \in X$ said to be a coincidence point of f and g if $gx = f(x, x, \dots, x)$.*
(ii) *If $w = gx = f(x, x, \dots, x)$, then w is called a point of coincidence of f and g.*
(iii) *If $x = gx = f(x, x, \dots, x)$, then x is called a common fixed point of f and g.*
(iv) *Mappings f and g are said to be commuting if $g(f(x, x, \dots, x)) = f(gx, gx, \dots, gx)$ for all $x \in X$.*
(v) *Mappings f and g are said to be weakly compatible if they commute at their coincidence points.*

Definition 3. *Let X be a nonempty set equipped with a partial order relation "$\preceq$", k a positive integer and $f : X^k \to X$ be a mapping. A sequence $\{x_n\}$ in X is said to be nondecreasing with respect to "$\preceq$", if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. The mapping f is said to be nondecreasing with respect to "$\preceq$" if for any finite nondecreasing sequence $\{x_n\}_{n=1}^{k+1}$ we have $f(x_1, x_2, \dots, x_k) \preceq f(x_2, x_3, \dots, x_{k+1})$. Let $g : X \to X$ be a mapping. f is said to be g-nondecreasing with respect to "$\preceq$" if for any finite nondecreasing sequence $\{gx_n\}_{n=1}^{k+1}$ we have $f(x_1, x_2, \dots, x_k) \preceq f(x_2, x_3, \dots, x_{k+1})$.*

Note that for $k = 1$, above definitions reduce to usual definitions of nondecreasing and g-nondecreasing mappings.

Definition 4. *Let X be a nonempty set equipped with a partial order relation "$\preceq$", and $g : X \to X$ be a mapping. A nonempty subset $\mathcal{A}$ of X is said to be well ordered if every two elements of $\mathcal{A}$ are comparable. The elements $a, b \in \mathcal{A}$ are called g-comparable if ga and gb are comparable. The set $\mathcal{A}$ is called g-well ordered if for all $a, b \in \mathcal{A}$, a and b are g-comparable i.e. ga and gb are comparable.*

Example 1. *Let $X = [0, \infty)$, $\mathcal{A} = [0, 1]$ and define a relation "$\preceq$" on X by*

$$x \preceq y \Leftrightarrow \{(x = y) \text{ or } (x, y \in [0, \frac{1}{2}] \text{ with } x \leq y)\}.$$

Then $\preceq$ is a partial order relation on X. Define $g : X \to X$ by $gx = \frac{x}{2}$ for all $x, y \in X$.
Note that $\mathcal{A}$ is not well ordered. Indeed if $x, y \in (\frac{1}{2}, \infty), x \neq y$ then neither $x \preceq y$ nor $y \preceq x$. But $g(\mathcal{A}) = [0, \frac{1}{2}]$, therefore $\mathcal{A}$ is g-well ordered.

Let (X, d) be a metric space equipped with partial order relation "$\preceq$", then $(X, \preceq, d)$ is called an ordered metric space. Let k be a positive integer and $f : X^k \to X$ be a mapping. f is called ordered Prešić contraction if

$$d(f(x_1, x_2 \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \leq \sum_{i=1}^{k} \alpha_i d(x_i, x_{i+1}), \tag{4}$$

for all $x_1, x_2, \dots, x_k, x_{k+1} \in X$ with $x_1 \preceq x_2 \preceq \cdots \preceq x_k \preceq x_{k+1}$, where α_i are nonnegative constants such that $\sum_{i=1}^{k} \alpha_i < 1$.
If (5) is satisfied for all $x_1, x_2, \dots, x_k, x_{k+1} \in X$, then f is called Prešić contraction. Note that in ordered metric spaces a Prešić contraction is necessarily an ordered

Prešić contraction, but converse may not be true (see examples 3.1 and 3.2 of [36]).

Let $\psi : \mathbb{R}_+^k \to \mathbb{R}_+$ be a function satisfying the following conditions:

(1) for $t_n \in \mathbb{R}_+$, $n \in \mathbb{N}$ and $t_n \downarrow t \geq 0$ implies $\limsup_{n\to\infty} \psi(t_n, t_n, \dots, t_n) \leq \psi(t, t, \dots, t)$;

(2) $\psi(t, t, \dots, t) < t$ for each $t > 0$;

(3) $\psi(t, 0, \dots, 0) + \psi(0, t, 0, \dots, 0) + \dots + \psi(0, \dots, 0, t) \leq \psi(t, t, \dots, t)$ for each $t \in \mathbb{R}_+$.

We denote the class of all such functions by Ψ, i.e., $\psi \in \Psi$ if and only if ψ satisfies the all above conditions.

Example 2. *Let $\psi : \mathbb{R}_+^k \to \mathbb{R}_+$ be defined by $\psi(t_1, t_2, \dots, t_k) = \sum_{i=1}^{k} \alpha_i t_i$, where α_i are nonnegative constants such that $\sum_{i=1}^{k} \alpha_i < 1$. Then $\psi \in \Psi$.*

Mapping $f : X^k \to X$ is said to be an ordered Prešić-Boyd-Wong contraction if

$$d(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \leq \psi(d(x_1, x_2), d(x_2, x_3), \dots, d(x_k.x_{k+1})) \tag{5}$$

for all $x_1, x_2, \dots, x_k, x_{k+1} \in X$ with $x_1 \preceq x_2 \preceq \dots \preceq x_k \preceq x_{k+1}$, where $\psi \in \Psi$. If (5) is satisfied for all $x_1, x_2, \dots, x_k, x_{k+1} \in X$, then f is called Prešić-Boyd-Wong contraction.

Now we can state our main results.

2. Main results

Theorem 5. *Let $(X, \preceq, d)$ be a complete ordered metric space, k a positive integer. Let $f : X^k \to X$, $g : X \to X$ be two mappings such that $f(X^k) \subset g(X)$ and $g(X)$ is a closed subspace of X. Suppose following conditions hold:*

(I)
$$d(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \leq \psi(d(gx_1, gx_2), d(gx_2, gx_3), \dots, d(gx_k, gx_{k+1})), \tag{6}$$
for all $x_1, x_2, \dots, x_k, x_{k+1} \in X$ with $gx_1 \preceq gx_2 \preceq \dots \preceq gx_k \preceq gx_{k+1}$, where $\psi \in \Psi$;

(II) *there exist $x_1 \in X$ such that $gx_1 \preceq f(x_1, x_1, \dots, x_1)$;*

(III) *f is g-nondecreasing;*

(IV) *if a nondecreasing sequence $\{gx_n\}$ converges to $gu \in X$, then $gx_n \preceq gu$ for all $n \in \mathbb{N}$ and $gu \preceq ggu$.*

Then f and g have a point of coincidence. If in addition f and g are weakly compatible, then f and g have a common fixed point $v \in X$. Moreover, the set of common fixed points of f and g is g-well ordered if and only if f and g have a unique common fixed point.

Proof. Starting with given $x_1 \in X$, we define a sequence $\{y_n\}$ as follows: let $y_1 = gx_1$, as $f(X^k) \subset g(X)$ and $gx_1 \preceq f(x_1, x_1, \dots, x_1)$, define $y_{n+1} = gx_{n+1} = f(x_n, x_n, \dots, x_n)$, $n \in \mathbb{N}$. Then $gx_1 \preceq gx_2$ i.e. $y_1 \preceq y_2$ and f is g-nondecreasing, so

$$\begin{aligned} y_2 &= f(x_1, x_1, \dots, x_1) \preceq f(x_1, x_1, \dots, x_1, x_2) \preceq f(x_1, x_1, \dots, x_1, x_2, x_2) \\ &\preceq \dots \preceq f(x_1, x_2, \dots, x_2) \preceq f(x_2, x_2, \dots, x_2) = gx_3 = y_3 \end{aligned}$$

i.e. $y_2 = gx_2 \preceq y_3 = gx_3$.
Continuing this procedure, we obtain

$$gx_1 \preceq gx_2 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots,$$

i.e.

$$y_1 \preceq y_2 \preceq \cdots \preceq y_n \preceq y_{n+1} \preceq \cdots.$$

Thus $\{y_n\} = \{gx_n\}$ is a nondecreasing sequence with respect to "$\preceq$".
For simplicity set $d_n = d(y_n, y_{n+1}), n \in \mathbb{N}$. We may assume that $d_n > 0$ for all $n \in \mathbb{N}$, otherwise coincidence point and point of coincidence of f and g exist trivially.
We shall show that $\lim\limits_{n\to\infty} d_n = 0$.
Note that

$$\begin{aligned} d_{n+1} &= d(y_{n+1}, y_{n+2}) \\ &= d(f(x_n, x_n, \ldots, x_n), f(x_{n+1}, x_{n+1}, \ldots, x_{n+1})) \\ &\leq d(f(x_n, x_n, \ldots, x_n), f(x_n, \ldots, x_n, x_{n+1})) \\ &\quad + d(f(x_n, \ldots, x_n, x_{n+1}), f(x_n, \ldots, x_n, x_{n+1}, x_{n+1})) \\ &\quad + \cdots + d(f(x_n, x_{n+1}, \ldots, x_{n+1}), f(x_{n+1}, \ldots, x_{n+1})) \end{aligned}$$

and $gx_n \preceq gx_{n+1}$, $\psi \in \Psi$, so it follows from (6) that

$$\begin{aligned} d_{n+1} &\leq \psi(0, \ldots, 0, d(gx_n, gx_{n+1})) + \psi(0, \ldots, 0, d(gx_n, gx_{n+1}), 0) \\ &\quad + \cdots + \psi(d(gx_n, gx_{n+1}), 0, \ldots, 0) \\ &= \psi(0, \ldots, 0, d_n) + \psi(0, \ldots, 0, d_n, 0) + \cdots + \psi(d_n, 0, \ldots, 0) \\ &\leq \psi(d_n, d_n, \ldots, d_n) \\ &< d_n, \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore $\{d_n\}$ is a monotonic nondecreasing sequence and bounded below, so $\lim\limits_{n\to\infty} d_n$ exists. Let $\lim\limits_{n\to\infty} d_n = \delta \geq 0$. Assume $\delta > 0$, then as $\psi \in \Psi$ we obtain

$$\delta = \lim_{n\to\infty} d_{n+1} \leq \lim_{n\to\infty} \psi(d_n, d_n, \ldots, d_n) \leq \psi(\delta, \delta, \ldots, \delta) < \delta,$$

a contradiction, so $\delta = 0$. We shall show that $\{y_n\}$ is Cauchy sequence.
Assume that $\{y_n\}$ is not Cauchy, then there exists $\epsilon > 0$ and integers $m_l, n_l, l \in \mathbb{N}$ such that $m_l > n_l \geq l$ and

$$d(y_{n_l}, y_{m_l}) \geq \epsilon \quad \text{for} \quad l \in \mathbb{N}.$$

Also, choosing m_l as small as possible, it may be assumed that

$$d(y_{m_l-1}, y_{n_l}) < \epsilon.$$

So for each $l \in \mathbb{N}$, we have

$$\begin{aligned} \epsilon \leq d(y_{m_l}, y_{n_l}) &\leq d(y_{m_l}, y_{m_l-1}) + d(y_{m_l-1}, y_{n_l}) \\ &\leq d_{m_l-1} + \epsilon \end{aligned}$$

and it follows from the fact $\lim_{n\to\infty} d_n = 0$ that $\lim_{l\to\infty} d(y_{m_l}, y_{n_l}) = \epsilon$. Observe that

$$\begin{aligned}\epsilon \le d(y_{m_l}, y_{n_l}) &\le d(y_{m_l}, y_{m_l+1}) + d(y_{m_l+1}, y_{n_l+1}) + d(y_{n_l+1}, y_{n_l})\\ &= d_{m_l} + d_{n_l} + d(f(x_{n_l}, \dots, x_{n_l}), f(x_{m_l}, \dots, x_{m_l}))\\ &\le d_{m_l} + d_{n_l} + d(f(x_{n_l}, \dots, x_{n_l}), f(x_{n_l}, \dots, x_{n_l}, x_{m_l}))\\ &\quad + d(f(x_{n_l}, \dots, x_{n_l}, x_{m_l}), f(x_{n_l}, \dots, x_{n_l}, x_{m_l}, x_{m_l}))\\ &\quad + \dots + d(f(x_{n_l}, x_{m_l}, \dots, x_{m_l}), f(x_{m_l}, \dots, x_{m_l})).\end{aligned}$$

As $m_l > n_l$ and $\{y_n\}$ is nondecreasing with respect to " $\preceq$ ", so $y_{n_l} \preceq y_{m_l}$ i.e., $gx_{n_l} \preceq gx_{m_l}$, therefore it follows from (6) and the above inequality that

$$\begin{aligned}\epsilon \le d(y_{m_l}, y_{n_l}) &\le d_{m_l} + d_{n_l} + \psi(0, \dots, 0, d(y_{n_l}, y_{m_l})) + \psi(0, \dots, 0, d(y_{n_l}, y_{m_l}), 0)\\ &\quad + \dots + \psi(d(y_{n_l}, y_{m_l}), 0, \dots, 0)\\ &\le d_{m_l} + d_{n_l} + \psi(d(y_{n_l}, y_{m_l}), \dots, d(y_{n_l}, y_{m_l})).\end{aligned}$$

Letting $l \to \infty$ and using the facts that $\lim_{n\to\infty} d_n = 0$ and $\psi \in \Psi$, we have

$$\epsilon = \lim_{l\to\infty} d(y_{m_l}, y_{n_l}) \le \lim_{l\to\infty} \psi(d(y_{m_l}, y_{n_l}), \dots, d(y_{m_l}, y_{n_l})) \le \psi(\epsilon, \dots, \epsilon) < \epsilon,$$

which is a contradiction. Therefore $\{y_n\} = \{gx_n\}$ is a Cauchy sequence in $g(X)$. As $g(X)$ is closed, there exist $u, v \in X$ such that $v = gu$ and

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = gu = v. \tag{7}$$

We shall show that u is a coincidence point and v is a point of coincidence of f and g. Note that

$$\begin{aligned}d(v, f(u, u, \dots, u)) &\le d(v, y_{n+1}) + d(y_{n+1}, f(u, u, \dots, u))\\ &= d(v, y_{n+1}) + d(f(x_n, x_n, \dots, x_n), f(u, u, \dots, u))\\ &\le d(v, y_{n+1}) + d(f(x_n, x_n, \dots, x_n), f(x_n, \dots, x_n, u))\\ &\quad + d(f(x_n, \dots, x_n, u)), f(x_n, \dots, x_n, u, u))\\ &\quad + \dots + d(f(x_n, u, \dots, u)), f(u, \dots, u)).\end{aligned}$$

If $v \neq f(u, u, \dots, u)$, then by (IV) we have $gx_n \preceq gu, gu \preceq ggu$, so using (6) it follows from the above inequality that

$$\begin{aligned}d(v, f(u, u, \dots, u)) &\le d(v, y_{n+1}) + \psi(0, \dots, 0, d(gx_n, gu)) + \psi(0, \dots, 0, d(gx_n, gu), 0)\\ &\quad + \dots + \psi(d(gx_n, gu), 0, \dots, 0)\\ &\le d(v, y_{n+1}) + \psi(d(gx_n, gu), \dots, d(gx_n, gu))\\ &< d(v, y_{n+1}) + d(gx_n, gu),\end{aligned}$$

letting $n \to \infty$ and using (7) we obtain

$$d(v, f(u, u, \dots, u)) = 0 \quad \text{i.e.} \quad f(u, u, \dots, u) = gu = v.$$

Thus u is a coincidence point and v is a point of coincidence of f and g.
Suppose f and g are weakly compatible, so

$$f(v, v, \dots, v) = f(gu, gu, \dots, gu) = g(f(u, u, \dots, u)) = gv.$$

Again if $d(gu, gv) > 0$ then as $gu \preceq ggu = gv$ we obtain from (6) that

$$\begin{aligned} d(v, f(v, v, \dots, v)) &= d(f(u, u, \dots, u), f(v, v, \dots, v)) \\ &\leq d(f(u, u, \dots, u), f(u, \dots, u, v)) + d(f(u, \dots, u, v), f(u, \dots, u, v, v)) \\ &\quad + \dots + d(f(u, v, \dots, v), f(v, \dots, v)) \\ &\leq \psi(0, \dots, 0, d(gu, gv)) + \psi(0, \dots, 0, d(gu, gv), 0) \\ &\quad + \dots + \psi(d(gu, gv), 0, \dots, 0) \\ &\leq \psi(d(gu, gv), \dots, d(gu, gv)) < d(gu, gv) = d(v, f(v, v, \dots, v)), \end{aligned}$$

a contradiction, therefore $v = gv = f(v, v, \dots, v)$. Thus v is the common fixed point of f and g.
Suppose the set of common fixed points of f and g is g-well ordered. We shall show that common fixed point is unique. Assume on contrary that v' is another common fixed point of f and g i.e. $v' = gv' = f(v', v', \dots, v')$ and $v \neq v'$. As v and v' are g-comparable, let e.g. $gv \preceq gv'$. From (6), it follows that

$$\begin{aligned} d(v, v') &= d(f(v, v, \dots, v), f(v', v', \dots, v')) \\ &\leq d(f(v, v, \dots, v), f(v, \dots, v, v')) + d(f(v, \dots, v, v'), f(v, \dots, v, v', v')) \\ &\quad + \dots + d(f(v, v', \dots, v'), f(v', v', \dots, v')) \\ &\leq \psi(0, \dots, 0, d(gv, gv')) + \psi(0, \dots, 0, d(gv, gv'), 0) \\ &\quad + \dots + \psi(d(gv, gv'), 0, \dots, 0) \\ &\leq \psi(d(gv, gv'), \dots, d(gv, gv')) < d(v, v') \end{aligned}$$

a contradiction. Therefore, $v = v'$, i.e., the common fixed point is unique. For converse, if common fixed point of f and g is unique then the set of common fixed points of f and g being singleton therefore g-well ordered. □

Remark 1. *For $k = 1$ the above theorem is a generalization and extension of result of Boyd and Wong in ordered metric spaces.*

Following is a simple example which illustrate the above result.

Example 3. *Let $X = [0, \infty)$ with the usual metric and partial order $\preceq = \{(x, y) : x, y \in X, y \leq x\}$. For $k = 2$, define $f : X^2 \to X$ and $g : X \to X$ by*

$$f(x_1, x_2) = \frac{x_1 + x_2}{3 + x_1 + x_2} \quad \text{for all } x_1, x_2 \in X \text{ and } gx = x \quad \text{for all } x \in X.$$

Define $\psi : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$\psi(t_1, t_2) = \frac{t_1 + t_2}{3 + |t_1 - t_2|} \quad \text{for all } t_1, t_2 \in \mathbb{R}_+.$$

Then it easy to see that all the conditions of Theorem 5 are satisfied and 0 is the unique common fixed point of f and g in X.

Taking $g = I_X$ i.e. identity mapping of X in Theorem 5, we get the following fixed point result for ordered Prešić-Boyd-Wong contraction.

Corollary 6. *Let $(X, \preceq, d)$ be a complete ordered metric space, k a positive integer. Let $f : X^k \to X$ be a mapping such that the following conditions hold:*

(I) *f is ordered Prešić-Boyd-Wong contraction;*
(II) *there exist $x_1 \in X$ such that $x_1 \preceq f(x_1, x_1, \dots, x_1)$;*
(III) *f is nondecreasing;*

(IV) *if a nondecreasing sequence* $\{x_n\}$ *converges to* $u \in X$*, then* $x_n \preceq u$ *for all* $n \in \mathbb{N}$.

Then f *has a fixed point* $v \in X$*. Moreover, the set of fixed points of* f *is well ordered if and only if* f *has a unique fixed point.*

The following example illustrate the case when the known results are not applicable but the Corollary 6 of this paper is applicable.

Example 4. *Let* $X = [0,2]$ *and* d *is the usual metric on* X*, then* (X,d) *is a complete metric space. For* $k = 2$*, define a mapping* $f\colon X^2 \to X$ *by*

$$f(x,y) = \begin{cases} \dfrac{x}{1+x}, & \text{if } (x,y) \in [0,1) \times [0,1) \cup [1,2] \times [0,1); \\ \dfrac{y}{1+y}, & \text{if } (x,y) \in [0,1) \times [1,2]; \\ 0, & \text{otherwise,} \end{cases}$$

and a function $\psi\colon \mathbb{R}_+^k \to \mathbb{R}_+$ *by*

$$\psi(t_1,t_2) = \frac{t_1}{1+|t_1/2 - t_2|} \text{ for all } t_1,t_2 \in \mathbb{R}_+.$$

Let $\preceq$ *be a partial order define on* X *by*

$$\begin{aligned} \preceq \quad = \quad & \{(x,y)\colon (x,y) \in [0,1) \times [0,1) \text{ with } y \le x\} \cup \{(x,y)\colon (x,y) \in [1,2] \times (0,1)\} \\ & \cup \{(x,x)\colon x \in X\}, \end{aligned}$$

then $\psi \in \Psi$*. Now by careful calculations one can see that all the conditions of Corollary 6 are satisfied and* 0 *is the unique fixed point of* f*. Note that,* f *is not an ordered Prešić type contraction, therefore it is not a Prešić type contraction. To see this, take arbitrary values* $x, y = z \in [0,1)$ *and then condition (5) is not satisfied.*

Following theorem is a generalization of the result of Prešić and Boyd and Wong in metric spaces.

Theorem 7. *Let* (X,d) *be a complete metric space,* k *a positive integer. Let* $f : X^k \to X$*,* $g : X \to X$ *be two mappings such that* $f(X^k) \subset g(X)$ *and* $g(X)$ *is a closed subspace of* X*. Suppose following conditions hold:*

(8)
$$d(f(x_1,x_2,\ldots,x_k), f(x_2,x_3,\ldots,x_{k+1})) \le \psi(d(gx_1,gx_2), d(gx_2,gx_3),\ldots,d(gx_k,gx_{k+1})),$$

for all $x_1,x_2,\ldots,x_k,x_{k+1} \in X$*, where* $\psi \in \Psi$*. Then* f *and* g *have a point of coincidence. If in addition* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point* $v \in X$*.*

Proof. We note that the inequality (8) is true for all $x_1,x_2,\ldots,x_k,x_{k+1} \in X$, therefore the proof of theorem follows from similar process as used in the proof of Theorem 5. □

Taking $g = I_X$ i.e. identity mapping of X in Theorem 7, we get the following fixed point result for Prešić-Boyd-Wong contraction.

Corollary 8. *Let* (X,d) *be a complete metric space,* k *a positive integer. Let* $f : X^k \to X$ *be a Prešić-Boyd-Wong contraction. Then* f *has a unique fixed point* $v \in X$*.*

Remark 2. *Note that, for* $\psi(t_1, t_2, \ldots, t_k) = \sum_{i=1}^{k} \alpha_i t_i$, *where* α_i *are nonnegative constants such that* $\sum_{i=1}^{k} \alpha_i < 1$, *Corollary 8 reduces to the Prešić theorem.*

3. A homotopy result

In this section we prove a homotopy result for Prešić type mapping on product space.

Theorem 9. *Let* (X, d) *be any complete metric space,* U *an open subset of* X. *Suppose* $H : (\overline{U})^k \times [0, 1] \to X$ *be a function such that the following conditions hold:*

(i) *for every* $x \in \partial U$ *(here* ∂U *is the boundary of* U*) and* $\lambda \in [0, 1]$, $x \neq H(x, x, \ldots, x, \lambda)$;

(ii) *for all* $x_1, x_2, \ldots, x_k, x_{k+1} \in \overline{U}$ *and* $\lambda \in [0, 1]$

$$d(H(x_1, x_2, \ldots, x_k, \lambda), H(x_2, x_3, \ldots, x_{k+1}, \lambda)) \leq \sum_{i=1}^{k} \alpha_i d(x_i, x_{i+1}), \tag{9}$$

where α_i *are nonnegative constants such that* $\sum_{i=1}^{k} \alpha_i < \frac{1}{k}$;

(iii) *for all* $x_1, x_2, \ldots, x_k \in \overline{U}$ *and* $\lambda, \mu \in [0, 1]$ *there exists* $M \geq 0$ *such that*

$$d(H(x_1, x_2, \ldots, x_k, \lambda), H(x_1, x_2, \ldots, x_k, \mu)) \leq M|\lambda - \mu|. \tag{10}$$

If $H_{\lambda=\lambda'}$ *has a fixed point in* U *for at least one* $\lambda' \in [0, 1]$, *then* H_λ *has a fixed point in* U *for all* $\lambda \in [0, 1]$. *Furthermore, for any fixed* $\lambda \in [0, 1]$, *the fixed point of* H_λ *is unique.*

Proof. Define

$$\mathcal{F} = \{\lambda \in [0, 1] : x = H(x, x, \ldots, x, \lambda) \text{ for some } x \in U\}.$$

As $H_{\lambda=\lambda'}$ for at least one $\lambda' \in [0, 1]$, has a fixed point in U, i.e., there exists $x \in U$ such that $H(x, x, \ldots, x, \lambda') = x$, so $\lambda' \in \mathcal{F}$ and $\mathcal{F}_x \neq \emptyset$. We shall show that $\mathcal{F}$ is both open and closed in $[0, 1]$ and therefore by connectedness $\mathcal{F} = [0, 1]$.

(I) $\mathcal{F}$ **is closed:** Let $\{\lambda_n\}$ be any sequence in $\mathcal{F}$ and $\lim_{n\to\infty} \lambda_n = \lambda \in [0, 1]$. As $\lambda_n \in \mathcal{F}$ for all $n \in \mathbb{N}$ so there exists $x_n \in U$ such that $x_n = H(x_n, x_n, \ldots, x_n, \lambda_n)$ for all $n \in \mathbb{N}$.

Note that, for all $n, m \in \mathbb{N}$ with $m > n$ we have

$$\begin{aligned}
d(x_n, x_m) &= d(H(x_n, x_n, \ldots, x_n, \lambda_n), H(x_m, x_m, \ldots, x_m, \lambda_m)) \\
&\leq d(H(x_n, \ldots, x_n, \lambda_n), H(x_n, \ldots, x_n, x_m, \lambda_n)) \\
&\quad + d(H(x_n, \ldots, x_n, x_m, \lambda_n), H(x_n, \ldots, x_n, x_m, x_m, \lambda_n)) \\
&\quad + \cdots + d(H(x_n, x_m, \ldots, x_m, \lambda_n), H(x_m, \ldots, x_m, \lambda_n)) \\
&\quad + d(H(x_m, \ldots, x_m, \lambda_n), H(x_m, \ldots, x_m, \lambda_m)).
\end{aligned}$$

Using (9) and (10) it follows that

$$\begin{aligned}
d(x_n, x_m) &\leq \alpha_k d(x_n, x_m) + \alpha_{k-1} d(x_n, x_m) + \cdots + \alpha_1 d(x_n, x_m) + M|\lambda_n - \lambda_m| \\
&= [\sum_{i=1}^{k} \alpha_i] d(x_n, x_m) + M|\lambda_n - \lambda_m|
\end{aligned}$$

i.e.

$$d(x_n, x_m) \leq \frac{M}{1 - \sum\limits_{i=1}^{k} \alpha_i} |\lambda_n - \lambda_m|.$$

Letting $n \to \infty$ and using the fact that $\lim\limits_{n\to\infty} \lambda_n = \lambda$ it follows from the above inequality that $\lim\limits_{n\to\infty} d(x_n, x_m) = 0$, therefore $\{x_n\}$ is a Cauchy sequence. As X is complete, there exists $u \in \overline{U}$ such that

$$\lim_{n\to\infty} x_n = u.$$

Now for any $n \in \mathbb{N}$ we obtain

$$\begin{aligned} d(x_n, H(u, u, \ldots, u, \lambda)) &= d(H(x_n, x_n, \ldots, x_n, \lambda_n), H(u, u, \ldots, u, \lambda)) \\ &\leq d(H(x_n, \ldots, x_n, \lambda_n), H(x_n, \ldots, x_n, u, \lambda_n)) \\ &\quad + d(H(x_n, \ldots, x_n, u, \lambda_n), H(x_n, \ldots, x_n, u, u, \lambda_n)) \\ &\quad + \cdots + d(H(x_n, u, \ldots, u, \lambda_n), H(u, \ldots, u, \lambda_n)) \\ &\quad + d(H(u, u, \ldots, u, \lambda_n), H(u, u, \ldots, u, \lambda)), \end{aligned}$$

using (9) and (10) it follows that

$$\begin{aligned} d(x_n, H(u, u, \ldots, u, \lambda)) &\leq \alpha_k d(x_n, u) + \alpha_{k-1} d(x_n, u) + \cdots + \alpha_1 d(x_n, u) + M|\lambda_n - \lambda| \\ &= [\sum_{i=1}^{k} \alpha_i] d(x_n, u) + M|\lambda_n - \lambda|. \end{aligned}$$

As $\lim\limits_{n\to\infty} \lambda_n = \lambda$ and $\lim\limits_{n\to\infty} x_n = u$, we obtain

$$\lim_{n\to\infty} d(x_n, H(u, u, \ldots, u, \lambda)) = d(u, H(u, u, \ldots, u, \lambda)) = 0$$

i.e. $u = H(u, u, \ldots, u, \lambda)$ and $u \in \overline{U}$. As (i) holds, therefore $u \in U$ so $\lambda \in \mathcal{F}$. Thus $\mathcal{F}$ is closed.

(II) $\mathcal{F}$ is open: Let $\lambda_0 \in \mathcal{F}$, then there exists $u_0 \in U$ such that $u_0 = H(u_0, \ldots, u_0, \lambda_0)$. As U is open, there exists $\delta > 0$ such that $B(u_0, \delta) = \{x \in X : d(x, u_0) < \delta\} \subset U$. Fix $\epsilon > 0$ with

$$\epsilon < \frac{1 - k \sum\limits_{i=1}^{k} \alpha_i}{M} \delta. \tag{11}$$

Let $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, then for all $x_1, x_2, \ldots, x_k \in \overline{B(u_0, \delta)} = \{x \in X : d(x, u_0) \leq \delta\}$, we have

$$\begin{aligned} d(u_0, H(x_1, x_2, \ldots, x_k, \lambda)) &= d(H(u_0, u_0, \ldots, u_0, \lambda_0), H(x_1, x_1, \ldots, x_k, \lambda)) \\ &\leq d(H(u_0, u_0, \ldots, u_0, \lambda_0), H(u_0, \ldots, u_0, x_1, \lambda_0)) \\ &\quad + d(H(u_0, \ldots, u_0, x_1, \lambda_0), H(u_0, \ldots, u_0, x_1, x_2, \lambda_0)) \\ &\quad + \cdots + d(H(u_0, x_1, \ldots, x_{k-1}, \lambda_0), H(x_1, \ldots, x_k, \lambda_0)) \\ &\quad + d(H(x_1, x_2, \ldots, x_k, \lambda_0), H(x_1, x_2, \ldots, x_k, \lambda)). \end{aligned}$$

It follows from (9), (10) and the above inequality that

$$\begin{aligned} d(u_0, H(x_1, x_2, \dots, x_k, \lambda)) &\leq [\sum_{i=1}^{k} \alpha_i] d(u_0, x_1) + [\sum_{i=2}^{k} \alpha_i] d(x_1, x_2) + \cdots \\ &\quad + [\sum_{i=k-1}^{k} \alpha_i] d(x_{k-2}, x_{k-1}) + \alpha_k d(x_{k-1}, x_k) + M|\lambda_0 - \lambda| \\ &< [k \sum_{i=1}^{k} \alpha_i] \delta + M\epsilon. \end{aligned}$$

Using (11) in the above inequality we obtain

$$\begin{aligned} d(u_0, H(x_1, x_2, \dots, x_k, \lambda)) &< [k \sum_{i=1}^{k} \alpha_i] \delta + [1 - k \sum_{i=1}^{k} \alpha_i] \delta \\ &= \delta. \end{aligned}$$

Therefore

$$d(u_0, H(x_1, x_2, \dots, x_k, \lambda)) < \delta \text{ i.e. } H(x_1, x_2, \dots, x_k, \lambda) \in \overline{B(u_0, \delta)}.$$

Thus, for each fixed $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, H_λ is a self map of $\overline{B(u_0, \delta)}$, so we can apply Corollary 8 and Remark 2, to deduce that H_λ has a fixed point in $\overline{U}$, and as (i) holds, this fixed point must be in U. Thus $\lambda \in \mathcal{F}$ for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. Therefore $\mathcal{F}$ is open in $[0, 1]$ i.e. $\mathcal{F} = [0, 1]$. Thus H_λ has a fixed point in U for all $\lambda \in [0, 1]$.

For uniqueness, let $\lambda \in [0, 1]$ be fixed and for this fixed λ, u and v be two fixed points of H_λ in U i.e. $u = H(u, u, \dots, u, \lambda)$ and $v = H(v, v, \dots, v, \lambda)$ and $u \neq v$. Then it follows from (9) that

$$\begin{aligned} d(u, v) &= d(H(u, u, \dots, u, \lambda), H(v, v, \dots, v, \lambda)) \\ &\leq d(H(u, \dots, u, \lambda), H(u, \dots, u, v, \lambda)) + d(H(u, \dots, u, v, \lambda), H(u, \dots, u, v, v, \lambda)) \\ &\quad + \cdots + d(H(u, v, \dots, v, \lambda), H(v, v, \dots, v, \lambda)) \\ &\leq \alpha_k d(u, v) + \alpha_{k-1} d(u, v) + \cdots + \alpha_1 d(u, v) \\ &= [\sum_{i=1}^{k} \alpha_i] d(u, v) \\ &< d(u, v), \end{aligned}$$

a contradiction. Thus fixed point is unique. □

For $k = 1$ in the above theorem, we obtain following Homotopy result.

Corollary 10. *Let (X, d) be any complete metric space, U an open subset of X. Suppose $H : \overline{U} \times [0, 1] \to X$ be a function such that the following conditions hold:*

(i) *for every $x \in \partial U$ (here ∂U is the boundary of U) and $\lambda \in [0, 1]$, $x \neq H(x, \lambda)$;*

(ii) *for all $x, y \in \overline{U}$ and $\lambda \in [0, 1]$*

$$d(H(x, \lambda), H(y, \lambda)) \leq \alpha d(x, y),$$

where $0 \leq \alpha < 1$;

(iii) *for all $x \in \overline{U}$ and $\lambda, \mu \in [0, 1]$ there exists $M \geq 0$ such that*

$$d(H(x, \lambda), H(x, \mu)) \leq M|\lambda - \mu|.$$

If $H_{\lambda=\lambda'}$ has a fixed point in U for at least one $\lambda' \in [0,1]$, then H_λ has a fixed point in U for all $\lambda \in [0,1]$. Furthermore, for any fixed $\lambda \in [0,1]$, the fixed point of H_λ is unique.

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