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FRACTIONAL OSTROWSKI INEQUALITIES FOR s-GODUNOVA-LEVIN FUNCTIONS

MUHAMMAD ASLAM NOOR, KHALIDA INAYAT NOOR, MUHAMMAD UZAIR AWAN*

Abstract. In this paper, we derive some new fractional Ostrowski type inequalities for s-Godunova-Levin functions introduced by Dragomir [3, 4]. Some special cases are also discussed.

1. Introduction

Recently much attention has been given to theory of convex functions by many researchers. Consequently the classical concept of convex functions has been extended and generalized in different directions using various novel ideas, see [1, 2, 3, 4, 6, 9, 15, 16, 17, 18, 24]. Recently Dragomir has introduced the notion of s-Godunova-Levin functions. This class of convex function generalizes the class of Godunova-Levin functions and the class of P-functions.

It is known that convex functions play an important role in the development of many famous inequalities. Thus many researchers have generalized the classical version of famous inequalities such as Hermite-Hadamard inequality, Ostroski inequality, Simpson inequality etc for different classes of convex functions, see [2, 3, 4, 5, 6, 7, 8, 12, 14, 15, 16, 17, 18, 21, 23, 24, 25].

Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I, the interior of the interval I, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right], \tag{1.1}$$

holds. This result is known in the literature as the Ostrowski inequality [19].

In this paper, we derive some new inequalities of Ostrowski type for s-Godunova-Levin functions via fractional integrals. We also discuss some special cases. This is the main motivation of this paper.

2. Preliminaries

In this section, we recall some preliminary concepts. First of all let $I = [a, b] \subseteq \mathbb{R}$ be the interval and $\mathbb{R}$ be the set of real numbers.

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Definition 2.1 ([6]). *A nonnegative function $f : I \to \mathbb{R}$ is said to be P-function, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \forall x, y \in I, t \in [0,1]. \tag{2.2}$$

Definition 2.2 ([9]). *A function $f : I \to \mathbb{R}$ is said to be Godunova-Levin function, if*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}, \forall x, y \in I, t \in (0,1). \tag{2.3}$$

For some useful details and extensions of Godunova-Levin functions, see [3, 4, 7, 9, 13, 16, 17, 20]

Definition 2.3 ([16]). *A function $f : I \to \mathbb{R}$ is said to be s-Godunova-Levin functions of first kind, if $s \in (0,1]$, we have*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{1-t^s}, \forall x, y \in I, t \in (0,1). \tag{2.4}$$

It is obvious that for $s = 1$ the definition of s-Godunova-Levin functions of first kind collapses to the definition of Godunova-Levin functions.
Our next definition is another due to Dragomir [3, 4].

Definition 2.4 ([3, 4]). *A function $f : I \to \mathbb{R}$ is said to be s-Godunova-Levin functions of second kind, if $s \in [0,1]$, we have*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s}, \forall x, y \in I, t \in (0,1). \tag{2.5}$$

It is obvious that for $s = 0$, s-Godunova-Levin functions of second kind reduces to the definition of P-functions. If $s = 1$, it then reduces to Godunova-Levin functions.
The following result plays a key role in deriving our main results.

Lemma 2.1 ([22]). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $f' \in L_1[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$, we have*

$$\Big(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\Big) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \Big[J^\alpha_{x^-} f(a) + J^\alpha_{x^+} f(b)\Big]$$
$$= \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f'(tx + (1-t)a)dt - \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f'(tx + (1-t)b)dt.$$

3. Main Results

In this section, we derive our main results.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$ and $f' \in L_1[a,b]$ for all $x \in [a,b]$ and $\alpha > 0$. If $|f'|$ is s-Godunova-Levin function of second kind and $|f'(x)| \leq M$, then, we have*

$$\left| \Big(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\Big) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \Big[J^\alpha_{x^-} f(a) + J^\alpha_{x^+} f(b)\Big] \right|$$
$$\leq \Big(\frac{1}{1+\alpha-s} + \frac{\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}\Big) \Big(\frac{M\big[(x-a)^{\alpha+1} + (b-a)^{\alpha+1}\big]}{b-a}\Big).$$

Proof. Using Lemma 2.1 and the fact that $|f'|$ is s-Godunova-Levin function of second kind, we have

$$\begin{aligned}
&\left|\Big(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\Big)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^+}f(b)\right]\right|\\
&=\left|\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)a)dt-\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)b)dt\right|\\
&\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}|f'(tx+(1-t)a)|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}|f'(tx+(1-t)b)|dt\\
&\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\Big[\frac{1}{t^s}|f'(x)|+\frac{1}{(1-t)^s}|f'(a)|\Big]dt\\
&+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\Big[\frac{1}{t^s}|f'(x)|+\frac{1}{(1-t)^s}|f'(b)|\Big]dt\\
&\le\Big(\frac{1}{1+\alpha-s}+\frac{\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}\Big)\Big(\frac{M\big[(x-a)^{\alpha+1}+(b-a)^{\alpha+1}\big]}{b-a}\Big).
\end{aligned}$$

This completes the proof. □

Note that for $\alpha=1$, Theorem 3.1 collapses to following result for s-Godunova-Levin function of second kind.

Corollary 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ and $f'\in L_1[a,b]$ for all $x\in[a,b]$. If $|f'|$ is s-Godunova-Levin function of second kind and $|f'(x)|\le M$, then, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\le\frac{M[(x-a)^2+(b-x)^2]}{(b-a)(1-s)}. \tag{3.6}$$

Also, we have some special cases of Corollary 3.1.

I. If we take $x=\frac{a+b}{2}$ in (3.6), then we have the following mid-point inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\le\frac{M(b-a)}{2(1-s)}.$$

II. If we take $x=a$ in (3.6), then we have the following inequality

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(u)du\right|\le M(b-a)\Big(\frac{1}{1-s}\Big).$$

III. If we take $x=b$ in (3.6), then we have the following inequality

$$\left|f(b)-\frac{1}{b-a}\int_a^b f(u)du\right|\le M(b-a)\Big(\frac{1}{1-s}\Big).$$

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$ and $f' \in L_1[a, b]$ for all $x \in [a, b]$ and $\alpha > 0$. If $|f'|^q$ is s-Godunova-Levin function of second kind on $[a, b]$, $p, q > 1, 1/p + 1/q = 1$ and $|f'(x)| \leq M$, then, we have*

$$\left|\Big(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\Big)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\Big[J^\alpha_{x^-}f(a) + J^\alpha_{x^+}f(b)\Big]\right|$$
$$\leq M\Big(\frac{1}{1-s}\Big)^{\frac{1}{q}}\Big(\frac{1}{1+p\alpha}\Big)^{\frac{1}{p}}\left[\frac{(x-a)^{\alpha+1} + (b-a)^{\alpha+1}}{b-a}\right].$$

Proof. Using Lemma 2.1, well-known Holder's inequality and the fact that $|f'|^q$ is s-Godunova-Levin function of second kind, we have

$$\left|\Big(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\Big)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\Big[J^\alpha_{x^-}f(a) + J^\alpha_{x^+}f(b)\Big]\right|$$
$$= \left|\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx+(1-t)a)dt - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx+(1-t)b)dt\right|$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\Big(\int_0^1 t^{p\alpha}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 |f'(tx+(1-t)a)|^q dt\Big)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^{\alpha+1}}{b-a}\Big(\int_0^1 t^{p\alpha}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 |f'(tx+(1-t)b)|^q dt\Big)^{\frac{1}{q}}$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\Big(\int_0^1 t^{p\alpha}dt\Big)^{\frac{1}{p}}\left(\int_0^1 \Big(\frac{1}{t^s}|f'(x)|^q + \frac{1}{(1-t)^s}|f'(a)|^q\Big)dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^{\alpha+1}}{b-a}\Big(\int_0^1 t^{p\alpha}dt\Big)^{\frac{1}{p}}\left(\int_0^1 \Big(\frac{1}{t^s}|f'(x)|^q + \frac{1}{(1-t)^s}|f'(b)|^q\Big)dt\right)^{\frac{1}{q}}$$
$$\leq M\Big(\frac{1}{1-s}\Big)^{\frac{1}{q}}\Big(\frac{1}{1+p\alpha}\Big)^{\frac{1}{p}}\left[\frac{(x-a)^{\alpha+1} + (b-a)^{\alpha+1}}{b-a}\right].$$

This completes the proof. □

For $\alpha = 1$, Theorem 3.2 collapses to following result for s-Godunova-Levin function of second kind.

Corollary 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$ and $f' \in L_1[a, b]$ for all $x \in [a, b]$. If $|f'|^q$ is s-Godunova-Levin function of second kind on $[a, b]$, $p, q > 1, 1/p + 1/q = 1$ and $|f'(x)| \leq M$, then, we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq M\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{1-s}\Big)^{\frac{1}{q}}\left[\frac{(x-a)^2 + (b-x)^2}{b-a}\right]. \tag{3.7}$$

Also, we have

I. If we take $x = \frac{a+b}{2}$ in (3.7), then we have the following mid-point inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du \right| \le \frac{M(b-a)}{2}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{1-s}\Big)^{\frac{1}{q}}.$$

II. If we take $x = a$ in (3.7), then we have the following inequality

$$\left| f(a) - \frac{1}{b-a}\int_a^b f(u)du \right| \le M(b-a)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{1-s}\Big)^{\frac{1}{q}}.$$

III. If we take $x = b$ in (3.7), then we have the following inequality

$$\left| f(b) - \frac{1}{b-a}\int_a^b f(u)du \right| \le M(b-a)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{1-s}\Big)^{\frac{1}{q}}.$$

Theorem 3.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$ and $f' \in L_1[a, b]$ for all $x \in [a, b]$ and $\alpha > 0$. If $|f'|^q$ is s-Godunova-Levin function of second kind on $[a, b]$, $q > 1$ and $|f'(x)| \le M$, then, we have*

$$\begin{aligned}&\left|\Big(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\Big)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\Big[J^\alpha_{x^-}f(a) + J^\alpha_{x^+}f(b)\Big]\right|\\ &\le M\Big(\frac{1}{1-s}\Big)^{\frac{1}{q}}\Big(\frac{1}{1+p\alpha}\Big)^{\frac{1}{p}}\left[\frac{(x-a)^{\alpha+1} + (b-a)^{\alpha+1}}{b-a}\right].\end{aligned}$$

Proof. Using Lemma 2.1, well-known power mean inequality and the fact that $|f'|^q$ is s-Godunova-Levin function of second kind, we have

$$\begin{aligned}&\left|\Big(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\Big)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\Big[J^\alpha_{x^-}f(a) + J^\alpha_{x^+}f(b)\Big]\right|\\
&= \left|\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx+(1-t)a)dt - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx+(1-t)b)dt\right|\\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\Big(\int_0^1 t^\alpha dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^\alpha|f'(tx+(1-t)a)|^q dt\Big)^{\frac{1}{q}}\\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\Big(\int_0^1 t^\alpha dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^\alpha|f'(tx+(1-t)b)|^q dt\Big)^{\frac{1}{q}}\\
&\le M\Big(\frac{1}{\alpha+1}\Big)^{1-\frac{1}{q}}\Big(\frac{1}{1+\alpha-s} - \frac{\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}\Big)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1} + (b-a)^{\alpha+1}}{b-a}\right].\end{aligned}$$

This completes the proof. □

When $\alpha = 1$ in Theorem 3.3, we have the following result.

Corollary 3.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$ and $f' \in L_1[a, b]$ for all $x \in [a, b]$. If $|f'|^q$ is s-Godunova-Levin function of second* *kind on* $[a,b]$, $q>1$ *and* $|f'(x)|\leq M$, *then, we have*

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(u)du\right|\leq M^q\Big(\frac{1}{2}\Big)^{1-1/q}\Big(\frac{1}{1-s}\Big)^{1/q}\Big(\frac{(x-a)^2+(b-x)^2}{b-a}\Big). \tag{3.8}$$

We now discuss some special cases.

I. If we take $x=\frac{a+b}{2}$ in (3.8). Then we have the following mid-point inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(u)du\right|\leq M^q\Big(\frac{1}{2}\Big)^{1-1/q}\Big(\frac{1}{1-s}\Big)^{1/q}\Big(\frac{b-a}{2}\Big).$$

II. If we take $x=a$ in (3.8). Then we have the following inequality

$$\left|f(a)-\frac{1}{b-a}\int\limits_a^b f(u)du\right|\leq M^q(b-a)\Big(\frac{1}{2}\Big)^{1-1/q}\Big(\frac{1}{1-s}\Big)^{1/q}.$$

III. If we take $x=b$ in (3.8). Then we have the following inequality

$$\left|f(b)-\frac{1}{b-a}\int\limits_a^b f(u)du\right|\leq M^q(b-a)\Big(\frac{1}{2}\Big)^{1-1/q}\Big(\frac{1}{1-s}\Big)^{1/q}.$$

Remark 3.1. We would like to mention here that one can extend the main results established in section 3 for the class of s-Godunova-Levin function of first kind.

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Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan

*Corresponding author