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Fear and Hunting Cooperation's Impact on the Eco-Epidemiological Model's Dynamics

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ABSTRACT. Due to the fact that living organisms do not exist individually, but rather exist in clusters interacting with each other, which helps to spread epidemics among them. Therefore, the study of the prey-predator system in the presence of an infectious disease is an important topic because the disease affects the system's dynamics and its existence. The presence of the hunting cooperation characteristic and the induced fear in the prey community impairs the growth rate of the prey and therefore affects the presence of the predator as well. Therefore, this research is interested in studying an eco-epidemiological system that includes the above factors. Therefore, an eco-epidemiological prey-predator model incorporating predation fear and cooperative hunting is built and examined. It is considered that the disease in the predator is of the SIS kind, which means that the infected predator can recover and become susceptible through medical treatment. All possible equilibrium points have been found. The solution's positivity and boundedness are examined. Local and global stability analyses are performed. The uniform persistence conditions are established. The local bifurcation around the equilibrium points is studied. Finally, numerical simulation is performed to validate the obtained results and comprehend the parameter impact on system dynamics.

1. INTRODUCTION

Cooperation during hunting is a common habit among several large predators that improves the predator's skill to catch prey and can cause fear, which lowers the prey's birth rate [1]. Understanding the predation connections among species in an ecosystem largely depends on examining prey-predator models [2-4]. Predators must consume food in order to survive, hence they frequently work to improve their skills to catch and kill prey since it is more beneficial to

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their long-term survival. Some animals frequently employ the method of cooperative hunting to increase their capacity to acquire and kill prey [5-10]. Several cooperative hunting tactics are explored [11], including the advantages of living in a social group, and how predators create cooperative groups with individuals who are actively engaged in prey capture. Although animals of the same species typically compete with one another for resources, some predators cooperate with one another and offer assistance to one another, because of a certain hunting process.

The study of the various mechanisms relating to the prey-predator relationship is one of the main subjects in ecology and evolutionary biology, which are considered by many researchers, see for example [12-15] and the references mentioned therein. Therefore, cooperative hunting and their induced prey's fear during the predation process is the most potent factor in a prey-predator relationship, especially when it comes to changing behavior that can affect both prey and predator features. Understanding these complicated situations has benefited greatly from the use of mathematical models. Since direct predation is very simple to observe in nature, most existing prey-predator models rely on the traditional Lotka-Volterra paradigm [16]. It usually presupposes that predators can only affect the prey population by direct killing. However, the presence of a predator may drastically alter the prey's physiological and behavioral characteristics to the point where it may have a greater impact on the prey population than direct predation [17] and the mentioned references therein. There are many research studies using mathematical modeling considered the impact of hunting cooperation [1,17-25] and the impact of fear [16,26-41] in the prey-predator models separately. Numerous prey species alter their behavior when predators are around because of the possibility of being eaten. In fact, the researchers have noted that prey that are afraid of predators have lower reproductive rates [42]. Prey clustering is a common antipredator activity in many prey species, and some studies have looked at how group size changes in response to the risk of predation. When the prey are in a group, the greatest advantage is enhanced predator detection. Predators can sometimes find larger groups easily, though, see [43].

Similar to how Kermack-McKendrick's groundbreaking research on SIRS (susceptible-infective-removal-susceptible) epidemiological models has drawn significant interest from experts. Because infectious diseases are primarily spread through contact between species, there is a high risk of disease transmission between prey species, predator species, or both because of their frequent interactions, particularly when hunting cooperation and fear are present. As a result, eco-epidemiology emerged as a brand-new field of study. Anderson and May [44] were the first to introduce eco-epidemiological models. Numerous ecologists and eco-epidemiologists have been familiar with eco-epidemiology due to the importance of protecting wild creatures [45]. Controlling the extension of diseases when disease and therapy coexist simultaneously is one of the key goals of the study of the eco-epidemic model [46-53].

Recent research has merged eco-epidemiological prey-predator models with hunting cooperation and fear. For example, an eco-epidemiological model with disease in the prey population that incorporates the fear effect of predators on prey and cooperative hunting among predators was proposed and researched by Liu et al [23]. They demonstrated the occurrence of several limit cycles for low disease transmission rates and predator mortality rates, and they also demonstrated that the system becomes stable at high disease transmission rates and predator mortality rates. An eco-epidemiological model that Fakhry and Naji [24] devised and researched included diseased prey devoured by a predator with a fear cost and hunting cooperation property. It is assumed that the predator couldn't tell the difference between healthy and sick prey, therefore it ate both. They discovered that the persistence of the system and the spread of sickness are both influenced by the presence of fear. The created fear, however, might stop the spread of disease in the event that hunting cooperation rates increase. Finally, the development of a mathematical model by AL-Jubouri and Naji [25] was adopted in order to explain how the interaction between the prey and the predator changes in the presence of infectious disease as well as the predator community's propensity for cooperative hunting, which instills enormous anxiety in the prey community. Additionally, the existence of a period of incubation for the illness delays the spread of the illness from sick predators to healthy predators. They discovered that the dynamics of the system are stabilized by the fear rate and destabilized by the delay. On the other hand, until a specific value is achieved, at which point the infected predator dies, the hunting cooperation rate has a destabilizing effect on the dynamics of the system.

The novelty of the topic investigated in this research, in contrast to the studies previously mentioned, lies in the existence of infectious diseases inside the predator community with the potential for curing the sickness based on treatment. In order to ascertain the dynamic behavior of the suggested epidemic ecology, it is crucial to investigate the effects of predator-hunting cooperation and the fear that is produced in the prey population. Therefore, the goal of the current work is to develop and examine an eco-epidemiological prey-predator model that includes predation fear and cooperative hunting. It is thought that the sickness in the predator is of the SIS kind, meaning that the sick predator can recover and revert to a vulnerable state with the help of medical treatment. We present the mathematical formulation in the section that follows. In Section 3, we look at the characteristics of the model's solution; in Section 4, we look at the viability of equilibria and their stability; in Section 5, we look at persistence; in Section 6, we talk about global stability; and in Section 7, we talk about the occurrence of local bifurcations. In Section 8, a numerical simulation is carried out. The paper concludes properly with a biological conclusion in Section 9.

2. MODEL FORMULATION

In the following, the adopted assumptions to build the mathematical model that describes the eco-epidemiological system are stated.

1. Let $X(T)$ be the density of the prey biomass at time T, and $Y(T) = S(T) + I(T)$ be the density biomass of the predator at time T, where $Y(T)$ is separated into two compartments due to the presence of the disease: susceptible population $S(T)$ and infected population $I(T)$.
2. In the absence of the predator, the number of prey increases logistically. In the lack of food, the predator decays exponentially.
3. The disease is thought to be of the SIS kind, and it is transmitted only between predator individuals by contact between an infected predator and a healthy predator, rather than genetically. The medicine administered to the infected predator also cures the sickness.
4. Because the predator has a hunting cooperation behavior, it attacks the prey in a group. Fear of predation is generated in a prey population as a result of this.

Accordingly, the stated eco-epidemiological system's dynamic can be represented by the following set of nonlinear first-order differential equations.

$$
\begin{aligned}
\frac{dX}{dT} &= \frac{rX}{1+\gamma_1(S+I)}\left[1-\frac{X}{k}\right] - \big(\alpha_1+\alpha_2(S+I)\big)(S+I)X \\
\frac{dS}{dT} &= e\big(\alpha_1+\alpha_2(S+I)\big)(S+I)X - \beta SI + \frac{\mu I}{\sigma+I} - d_1 S, \\
\frac{dI}{dt} &= \beta SI - \frac{\mu I}{\sigma+I} - d_2 I
\end{aligned}
\tag{1}
$$

where $X(0) = X_0 \geq 0$, $S(0) = S_0 \geq 0$, and $I(0) = I_0 \geq 0$ represent the initial condition of the system (1), and all parameters are assumed nonnegative and can be described in Table 1.

Table 1: The parameters description

Parameters	**Description**
$r > 0$	The prey's intrinsic growth rate
$k > 0$	The environment-carrying capacity
$\gamma_1 \geq 0$	The level of fear that reduces the growth of the prey
$\sigma \geq 0$	The depletion rate of the treatment
$\beta > 0$	The infection rate
$\mu > 0$	The treatment rate
$\alpha_1 > 0$	The attack rate of the predator on the prey
$\alpha_2 > 0$	The predator hunting cooperation level
$d_1 > 0$	The susceptible predator natural death rate
$d_2 > 0$	The infected predator death rate combined natural and disease death rates
$e \in (0,1]$	The conversion rate of prey biomass to predator biomass

The following transformations were applied to remove all units from the system (1).

$$rt = T, \frac{x}{k} = x_1, \frac{\alpha_2}{\alpha_1} S = x_2, \frac{\alpha_2}{\alpha_1} I = x_3.$$

Then system (1) reduces to the following dimensionless form

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{x_1(1-x_1)}{1+w_1(x_2+x_3)} - w_2(1+x_2+x_3)(x_2+x_3)x_1 = x_1 f_1(x_1,x_2,x_3),\\
\frac{dx_2}{dt} &= w_3(1+x_2+x_3)(x_2+x_3)x_1 - w_4 x_2 x_3 + \frac{w_5 x_3}{w_6+x_3} - w_7 x_2 = x_2 f_2(x_1,x_2,x_3), \qquad (2)\\
\frac{dx_3}{dt} &= w_4 x_2 x_3 - \frac{w_5 x_3}{w_6+x_3} - w_8 x_3 = x_3 f_3(x_1,x_2,x_3),
\end{aligned}$$

where $w_1 = \gamma_1 \frac{\alpha_1}{\alpha_2}, w_2 = \frac{{\alpha_1}^2}{r\alpha_2}, w_3 = e\frac{k\alpha_1}{r}, w_4 = \frac{\beta\alpha_1}{r\alpha_2}, w_5 = \frac{\alpha_2\mu}{\alpha_1 r}, w_6 = \frac{\alpha_2\sigma}{\alpha_1}, w_7 = \frac{d_1}{r}, w_8 = \frac{d_2}{r}$.

It is clear from the system (2) that, the functions $x_i f_i(x_1,x_2,x_3); i = 1,2,3$ in the system (2), are continuous and have continuous partial derivatives on the domain $\mathbb{R}_+^3 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$. Therefore, they are locally Lipschitz functions in $\mathbb{R}_+^3$. Consequently, with the use of the fundamental existence and uniqueness theorem, it is obtained that system (2) with any non-negative initial condition $x_1(0) \geq 0$, $x_2(0) \geq 0$, and $x_3(0) \geq 0$ has a unique solution defined in $\mathbb{R}_+^3$.

3. PROPERTIES OF THE SOLUTION

This section discusses the characteristics of the system (2)'s solution, such as positivity and boundedness, as provided in the following theorems.

Theorem 1: All system (2)'s solutions with the initial conditions belonging to $int.\mathbb{R}_+^3$ are positively invariant.

Proof. It is derived from the system's first equation (2):

$$\frac{dx_1}{x_1} = f_1(x_1,x_2,x_3)dt$$

Integrating the above equation within the range [0, t] yields:

$$x_1(t) = x_1(0)e^{\int_0^t f_1(x_1(s),x_2(s),x_3(s))ds} > 0; \forall\, t$$

Similarly, for the second and third equations, it is obtained

$$x_2(t) = x_2(0)e^{\int_0^t f_2(x_1(s),x_2(s),x_3(s))ds} > 0; \forall\, t$$

$$x_3(t) = x_3(0)e^{\int_0^t f_3(x_1(s),x_2(s),x_3(s))ds} > 0; \forall\, t$$

This completes the proof.

Theorem 2: All system (2)'s solutions with the initial conditions belonging to $\mathbb{R}_+^3$ are uniformly bounded

Proof. From the first equation of system (2), it is easy to verify that

$$\frac{dx_1}{dt} \leq x_1(1-x_1)$$

Then according to the lemma (2.2) [54], it is obtained that

$$x_1(t) \leq \left[1 + \left(\frac{1}{x_1(0)} - 1\right) e^{-t}\right]^{-1}$$

Hence for $t \to \infty$, it is obtained that $x_1(t) \leq 1$. Let $W = \frac{w_3}{w_2} x_1 + x_2 + x_3$, then by using system (2) equations, derive W with respect to t gives

$$\frac{dW}{dt} \leq 2 \frac{w_3}{w_2} x_1 - MW,$$

where $M = \min\{1, w_7, w_8\}$. Hence, simple manipulation yields

$$\frac{dW}{dt} + MW \leq 2 \frac{w_3}{w_2}.$$

So, according to the lemma (2.1) [54], it is obtained that

$$W(t) \leq 2 \frac{w_3}{Mw_2} \left[1 + \left(\frac{Mw_2 W(0)}{3w_3} - 1\right) e^{-Mt}\right]$$

Therefore, for $t \to \infty$, it is obtained that:

$$W(t) \leq 2 \frac{w_3}{Mw_2}.$$

That completes the proof.

4. EQUILIBRIA AND STABILITY ANALYSIS

This section determines the stability analysis of each probable equilibrium point. The following equilibrium points (EPs) exist in System (2):

1. The total extinction equilibrium point (TEEP) that is denoted $p_1 = (0,0,0)$ and the axial equilibrium point (AEP) that denoted $p_2 = (1,0,0)$ always exist.
2. The disease-free equilibrium point (DFEP) that is denoted $p_3 = (\bar{x}_1, \bar{x}_2, 0) = (m_1, m_2, 0)$, where

   $$m_1 = 1 - w_2 m_2 (1 + m_2)(1 + w_1 m_2), \quad (3)$$

   while m_2 is a positive root of the following polynomial equation

   $$\begin{aligned} w_1 w_2 w_3 m_2^4 + (w_2 w_3 + 2 w_1 w_2 w_3) m_2^3 + (2 w_2 w_3 + w_1 w_2 w_3) m_2^2 \\ -(w_3 - w_2 w_3) m_2 - w_3 + w_7 = 0 \end{aligned}.$$

   Hence, the conditions of having a unique DFEP are given by

   $$w_7 < w_3. \quad (4)$$

   $$w_2 m_2 (1 + m_2)(1 + w_1 m_2) < 1. \quad (5)$$
3. The positive equilibrium point (PEP) that is denoted $p_4 = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (q_1, q_2, q_3)$, where

   $$\left.\begin{aligned} q_1 &= \left[-1 - w_1 \left(x_3 + \frac{w_5 + w_6 w_8 + w_8 q_3}{w_4 (w_6 + q_3)}\right)\right] \left[w_2 \left(q_3 + \frac{w_5 + w_6 w_8 + w_8 q_3}{w_4 (w_6 + q_3)}\right)\right. \\ &\qquad \left.\left(1 + q_3 + \frac{w_5 + w_6 w_8 + w_8 q_3}{w_4 (w_6 + q_3)}\right) - \frac{1}{1 + w_1 \left(q_3 + \frac{w_5 + w_6 w_8 + w_8 q_3}{w_4 (w_6 + q_3)}\right)}\right] \\ q_2 &= \frac{w_5 + w_6 w_8 + w_8 q_3}{w_4 (w_6 + q_3)} \end{aligned}\right\}, \quad (6)$$

while q_3 represents a positive root of the following tenth-order polynomial equation.

$$\begin{gathered}A_{10}x_3^{10}+A_9x_3^9+A_8x_3^8+A_7x_3^7+A_6x_3^6+A_5x_3^5+A_4x_3^4\\+A_3x_3^3+A_2x_3^2+A_1x_3+A_0=0\end{gathered}, \tag{7}$$

where

$$\begin{gathered}A_0=w_1w_2w_3\left(w_5^5+w_6^5w_8^5\right)+w_2w_3w_4w_6(w_5+w_6w_8)^4(1+2w_1)\\+w_2w_3w_4^2w_6^2(w_5+w_6w_8)^3(2+w_1)+w_3w_4^3w_6^3(w_5+w_6w_8)^2(w_2-1)\\+w_4^4w_6^4(w_5+w_6w_8)(w_7-w_3)+5w_1w_2w_3w_5w_6w_8(w_5^3+w_6^3w_8^3)\\+10w_1w_2w_3w_5^2w_6^2w_8^2(w_5+w_6w_8)\end{gathered}.$$

$$A_{10}=w_1w_2w_3w_4^5.$$

While the other coefficients $A_i; i=1,2,\dots,9$ were computed using Mathematica software and have complicated forms, so it's omitted. Consequently, equation (7) has at least one positive root provided that the following condition is met.

$$A_0<0. \tag{8}$$

Moreover, the PEP exists in the interior of $\mathbb{R}_+^3$, if in addition to condition (8) the following condition holds.

$$\frac{1}{1+w_1\left(q_3+\frac{w_5+w_6w_8+w_8q_3}{w_4(w_6+q_3)}\right)}>w_2\left(q_3+\frac{w_5+w_6w_8+w_8q_3}{w_4(w_6+q_3)}\right)\left(1+q_3+\frac{w_5+w_6w_8+w_8q_3}{w_4(w_6+q_3)}\right). \tag{9}$$

The following calculated Jacobian matrix (JM) can be used to study the local stability analysis of the aforementioned EPs.

$$J=\left[a_{ij}\right]_{3\times3}, \tag{10}$$

where

$$a_{11}=-\frac{x_1}{1+w_1(x_2+x_3)}+\frac{1-x_1}{1+w_1(x_2+x_3)}-w_2(x_2+x_3)(1+x_2+x_3)=x_1\frac{\partial f_1}{\partial x_1}+f_1.$$

$$a_{12}=a_{13}=-x_1\left[w_2(x_2+x_3)+w_2(1+x_2+x_3)+\frac{w_1(1-x_1)}{(1+w_1(x_2+x_3))^2}\right].$$

$$a_{21}=w_3(x_2+x_3)(1+x_2+x_3).$$

$$a_{22}=-w_7-w_4x_3+w_3x_1(x_2+x_3)+w_3x_1(1+x_2+x_3).$$

$$a_{23}=-w_4x_2-\frac{w_5x_3}{(w_6+x_3)^2}+\frac{w_5}{w_6+x_3}+w_3x_1(x_2+x_3)+w_3x_1(1+x_2+x_3).$$

$$a_{31}=0.$$

$$a_{32}=w_4x_3.$$

$$a_{33}=\frac{w_5x_3}{(w_6+x_3)^2}-w_8+w_4x_2-\frac{w_5}{w_6+x_3}=x_3\frac{\partial f_3}{\partial x_3}+f_3.$$

The Jacobian at the TEEP can be written as:

$$J(p_1)=\begin{bmatrix}1&0&0\\0&-w_7&\frac{w_5}{w_6}\\0&0&-\frac{w_5}{w_6}-w_8\end{bmatrix} \tag{11}$$

Therefore, the eigenvalues of $J(p_1)$ are given by

$$\lambda_{11}=1>0\,,\lambda_{12}=-w_7<0\,,\lambda_{13}=-\left(\frac{w_5}{w_6}+w_8\right)<0. \tag{12}$$

As one of the eigenvalues is positive and the others are negative, hence, p_1 is a saddle point.

The Jacobian at the AEP can be written as:

$$J(p_2) = \begin{bmatrix} -1 & -w_2 & -w_2 \\ 0 & w_3 - w_7 & w_3 + \frac{w_5}{w_6} \\ 0 & 0 & -\frac{w_5}{w_6} - w_8 \end{bmatrix}. \quad (13)$$

Therefore, the eigenvalues of $J(p_2)$ are given by

$$\lambda_{21} = -1 < 0\,, \lambda_{22} = w_3 - w_7, \lambda_{23} = -\left(\frac{w_5}{w_6} + w_8\right) < 0. \quad (14)$$

Hence, the AEP is locally asymptotically stable (LAS), nonhyperbolic point, and saddle point provided that the following conditions are met respectively.

$$w_3 < w_7. \quad (15)$$

$$w_3 = w_7. \quad (16)$$

$$w_3 > w_7. \quad (17)$$

The Jacobian at the DFEP can be written as:

$$J(p_3) = \left[c_{ij}\right]_{3\times 3}, \quad (18)$$

where

$$c_{11} = \frac{1-m_1}{1+m_2w_1} - \frac{m_1}{1+m_2w_1} - w_2m_2(1+m_2),$$

$$c_{12} = c_{13} = -m_1\left[\frac{(1-m_1)w_1}{(1+m_2w_1)^2} + m_2w_2 + (1+m_2)w_2\right],$$

$$c_{21} = m_2(1+m_2)w_3,$$

$$c_{22} = m_1m_2w_3 + m_1(1+m_2)w_3 - w_7,$$

$$c_{23} = m_1m_2w_3 + m_1(1+m_2)w_3 - m_2w_4 + \frac{w_5}{w_6},$$

$$c_{31} = c_{32} = 0,$$

$$c_{33} = m_2w_4 - \frac{w_5}{w_6} - w_8.$$

Hence, the characteristic equation of $J(p_3)$ can be written as:

$$[\lambda^2 - (c_{11} + c_{22})\lambda + c_{11}c_{22} - c_{12}c_{21}][c_{33} - \lambda] = 0. \quad (19)$$

Direct computation gives the following roots

$$\left.\begin{aligned} \lambda_{31} &= \frac{(c_{11}+c_{22})}{2} + \frac{1}{2}\sqrt{(c_{11}+c_{22})^2 - 4(c_{11}c_{22} - c_{12}c_{21})} \\ \lambda_{32} &= \frac{(c_{11}+c_{22})}{2} - \frac{1}{2}\sqrt{(c_{11}+c_{22})^2 - 4(c_{11}c_{22} - c_{12}c_{21})} \\ \lambda_{33} &= m_2w_4 - \frac{w_5}{w_6} - w_8 \end{aligned}\right\}. \quad (20)$$

Direct computation shows that, the eigenvalues λ_{31} and λ_{32} have negative real parts if the following conditions hold.

$$\frac{1-m_1}{1+m_2w_1} < \frac{m_1}{1+m_2w_1} + m_2(1+m_2)w_2. \quad (21)$$

$$m_1m_2w_3 + m_1(1+m_2)w_3 < w_7. \quad (22)$$

While the third eigenvalue λ_{33} is negative if the following condition is met.

$$m_2w_4 < \frac{w_5}{w_6} + w_8. \quad (23)$$

Therefore, the DFEP is a LAS if the conditions (21)-(23) are satisfied.

Finally, the Jacobian at the PEP can be written as:

$$J(p_4) = \left[b_{ij}\right]_{3\times 3}, \tag{24}$$

where

$$b_{11} = -\frac{q_1}{1+w_1(q_2+q_3)},$$

$$b_{12} = b_{13} = -q_1\left[\frac{(1-q_1)w_1}{(1+(q_2+q_3)w_1)^2} + w_2(q_2+q_3) + w_2(1+q_2+q_3)\right],$$

$$b_{21} = w_3(q_2+q_3)(1+q_2+q_3),$$

$$b_{22} = w_3q_1(q_2+q_3) + w_3q_1(1+q_2+q_3) - w_4q_3 - w_7,$$

$$b_{23} = w_3q_1(q_2+q_3) + w_3q_1(1+q_2+q_3) - w_4q_2 - \frac{w_5q_3}{(q_3+w_6)^2} + \frac{w_5}{q_3+w_6},$$

$$b_{31} = 0,\ b_{32} = q_3w_4,\ b_{33} = \frac{w_5q_3}{(q_3+w_6)^2}.$$

The characteristic equation of $J(p_4)$ can be written as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{25}$$

where

$$A_1 = -(b_{11} + b_{22} + b_{33}),$$

$$A_2 = b_{11}b_{22} - b_{12}b_{21} + b_{11}b_{33} + b_{22}b_{33} - b_{23}b_{32},$$

$$A_3 = -[b_{11}(b_{22}b_{33} - b_{23}b_{32}) + b_{21}(b_{13}b_{32} - b_{12}b_{33})],$$

$$\Delta = A_1A_2 - A_3 = -(b_{11}+b_{22})[b_{11}b_{22} - b_{12}b_{21}] - (b_{22}+b_{33})[b_{22}b_{33} - b_{23}b_{32}] + b_{11}b_{33}(b_{11}+b_{33}) - 2b_{11}b_{22}b_{33} + b_{13}b_{21}b_{32}.$$

The characteristic equation (25), according to the Routh-Hurwitz criterion, has three eigenvalues with negative real portions if the following conditions are met: $A_1 > 0$; $A_3 > 0$, and $\Delta = A_1A_2 - A_3 > 0$. Moreover, the Routh-Hurwitz requirements are satisfied if the conditions given in the following theorem hold.

Theorem 3: The PEP of the system (2) is LAS if and only if the following sufficient conditions are met.

$$w_3q_1(q_2+q_3) + w_3q_1(1+q_2+q_3) < w_4q_3 + w_7. \tag{26}$$

$$w_3q_1(q_2+q_3) + w_3q_1(1+q_2+q_3) + \frac{w_5}{q_3+w_6} < w_4q_2 + \frac{w_5q_3}{(q_3+w_6)^2}. \tag{27}$$

$$\frac{w_5q_3}{(q_3+w_6)^2} < \min\left\{\frac{w_3q_1(q_2+q_3)+w_3q_1(1+q_2+q_3)-w_4q_2-\frac{w_5q_3}{(q_3+w_6)^2}+\frac{w_5}{q_3+w_6}}{w_3q_1(q_2+q_3)+w_3q_1(1+q_2+q_3)-w_4q_3-w_7}q_3w_4, q_3w_4\right\}. \tag{28}$$

$$\frac{w_5q_3}{(q_3+w_6)^2} < \min\left\{\frac{q_1}{1+w_1(q_2+q_3)}, w_4q_3 + w_7 - w_3q_1(q_2+q_3) - w_3q_1(1+q_2+q_3)\right\}. \tag{29}$$

$$b_{11}b_{33}(b_{11}+b_{33}) - 2b_{11}b_{22}b_{33} + b_{13}b_{21}b_{32} > 0. \tag{30}$$

Proof. Direct application of the Routh-Hurwitz criterion with the given condition the proof follows.

5. PERSISTENCE

The persistence and extinction properties of an eco-epidemiological model including fear and hunting cooperation are investigated, in this section. The goal is to examine how fear and hunting cooperation affect the persistence and extinction of system species in a sick prey-predator system. The system's border levels' dynamics must be understood to understand the conditions that assure the continued existence of all species. Clearly, system (2) has a subsystem representing the infected predator's absence, which can be written as:

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{x_1(1-x_1)}{1+w_1x_2} - w_2(1+x_2)x_1x_2 = g_1(x_1,x_2)\\ \frac{dx_2}{dt} &= w_3(1+x_2)x_1x_2 - w_7x_2 = g_2(x_1,x_2)\end{aligned}. \quad (31)$$

It is noted that, subsystem (31) has three equilibrium points given by $p_{11} = (0,0)$, $p_{12} = (1,0)$, and $p_{13} = (m_1, m_2)$, which are coincide with the projection of TEEP, AEP, and DFEP on the x_1x_2 −plane respectively. As a result, they have the exact prerequisites for local stability. The Dulac-Bendixon criterion is now used to evaluate the possibility of not existence of periodic dynamics in the interior of positive quadrants corresponding to subsystems (31).

Theorem 4: There are no periodic dynamics that fall entirely in the interior of a positive quadrant of x_1x_2 −plane provided that

$$\left.\begin{aligned} w_3 &< \frac{1}{x_2(1+w_1x_2)}\\ &OR\\ w_3 &> \frac{1}{x_2(1+w_1x_2)}\end{aligned}\right\}. \quad (32)$$

Proof. Consider the continuous differential function $D(x_1,x_2) = \frac{1}{x_1x_2}$ on a simple connected region of the interior of a positive quadrant of x_1x_2 −plane. Then the expiration

$$\Delta = \frac{\partial(Dg_1)}{\partial x_1} + \frac{\partial(Dg_2)}{\partial x_2} = -\frac{1}{x_2(1+w_1x_2)} + w_3.$$

It's clear that Δ has the same sign and does not equal zero under the condition (32). Therefore, as a result of the Dulac-Bendixon criterion, a subsystem (31) lacks periodic dynamics in the interior of the positive quadrant of the x_1x_2 −plane.

Theorem 5: Assume that there is no periodic dynamics in the boundary planes of $\mathbb{R}_+^3$. Let the conditions (17), (21), and (22) with the next condition are met, then the system (2) is uniformly persistent.

$$\frac{w_5}{w_6} + w_8 < m_2w_4. \quad (33)$$

Proof: Assume v is a point in the interior of $\mathbb{R}_+^3$ and $O(v)$ is the orbit through it. Let $\Omega(v)$ represent the ω–limit set of $O(v)$. It is worth noting that because the system (2) is bounded, so is $\Omega(v)$. To demonstrate that $p_1 \notin \Omega(v)$, the opposite is assumed first. Because p_1 is a saddle point, it cannot be the only point in $\Omega(v)$, and hence there must be at least one other point u such that $u \in \omega^s(p_1) \cap \Omega(v)$, where $\omega^s(p_1)$ is the stable manifold of p_1, see Butler-McGhee lemma [55].

Given that the stable manifold of p_1 is given by the x_2x_3 −plane, it is included in $\Omega(v)$. As a result, if u lies on the boundary axes of the x_2x_3 −plane, then the positive particular axis (containing u) is contained in $\Omega(v)$, which contradicts its boundedness.

Let u now belong to the inside of the x_2x_3 −plane. Because there is no other equilibrium points in the interior of the x_2x_3 −plane, the orbit through u included in $\Omega(v)$ must be infinite. Giving a contradiction also demonstrates that $p_1 \notin \Omega(v)$.

To demonstrate that $p_2 \notin \Omega(v)$, the opposite is also assumed. Because p_2 is saddle under condition (17) with stable manifold given by x_1x_3 −plane, the evidence is the same as in the proof of the first point p_1. As a result, $p_2 \notin \Omega(v)$ is found.

Similarly, when requirements (21)-(22) hold, $p_3 \notin \Omega(v)$ under condition (33), which produces p_3 saddle point, with stable manifold supplied by x_1x_2 −plane.

Because there are no periodic dynamics in the boundary planes of $\mathbb{R}^3_+$, and the above points p_1, p_2, and p_3 are the only potential attractive points for the solutions of system (2), system (2) uniformly persists.

6. GLOBAL STABILITY ANALYSIS

In this section, we study the global asymptotic stability (GAS) of all single-existence equilibrium points using the Lyapanov function whenever it exists as described in the following theorems

Theorem 5: Assume that condition (15) holds, then the AEP is a GAS provided that the following condition holds.

$$w_3\left(1+2\frac{w_3}{w_2M}\right) < \min\{w_7, w_8\} \tag{34}$$

Proof: Consider the following positive definite real-valued function around the AEP

$$V_1 = \alpha_1[x_1 - 1 - \ln x_1] + \alpha_2 x_2 + \alpha_3 x_3$$

Then, we have

$$\frac{dV_1}{dt} = \alpha_1 \frac{(x_1-1)}{x_1}\frac{dx_1}{dt} + \alpha_2\frac{dx_2}{dt} + \alpha_3\frac{dx_3}{dt}$$

$$\begin{aligned}\frac{dV_1}{dt} = {} & \alpha_1(x_1-1)\left[\frac{1-x_1}{1+w_1(x_2+x_3)} - w_2(1+x_2+x_3)(x_2+x_3)\right] \\ & + \alpha_2\left[w_3(1+x_2+x_3)(x_2+x_3)x_1 - w_4x_2x_3 + \frac{w_5x_3}{w_6+x_3} - w_7x_2\right] \\ & + \alpha_3\left[w_4x_2x_3 - \frac{w_5x_3}{w_6+x_3} - w_8x_3\right]\end{aligned}$$

Now, choosing $\alpha_1 = \frac{w_3}{w_2}$, and $\alpha_2 = \alpha_3 = 1$ with use of the theorem (2), it is obtained that

$$\frac{dV_1}{dt} \leq -\frac{w_3(x_1-1)^2}{w_2[1+w_1(x_2+x_3)]} - \left[w_7 - w_3\left(1+2\frac{w_3}{w_2 M}\right)\right]x_2 - \left[w_8 - w_3\left(1+2\frac{w_3}{w_2 M}\right)\right]x_3$$

Clearly, condition (34) gives that $\frac{dV_1}{dt}$ is negative definite. Moreover, since V_1 is radially unbounded function, then AEP is a GAS.

Theorem 6: assume that the DFEP denoted by $p_3 = (\bar{x}_1, \bar{x}_2, 0) = (m_1, m_2, 0)$ is a LAS, then it is a GAS provided that the following sufficient conditions hold.

$$\left(w_3\left(\frac{w_1(1-m_1)}{B_1 B_2} + w_2(1+x_2+m_2)\right) - \frac{w_2 w_3 m_2(1+m_2)}{x_2}\right)^2 < 4\left(\frac{-w_3}{B_1}\right)\left(w_7 - w_3 x_1(1+x_2+m_2)\right). \quad (35)$$

$$w_7 > w_3 x_1(1+x_2+m_2). \quad (36)$$

$$\frac{x_1}{x_2} > m_1. \quad (37)$$

$$w_2 w_8 > \frac{2w_1 w_3 m_1}{B_2} + w_2 w_3 m_1 + 2w_2 w_3 m_1 x_2 + w_2 w_4 m_2. \quad (38)$$

Proof: consider the following positive definite real valued function around p_3

$$V_2 = \alpha_1\left[x_1 - m_1 - m_1 \ln\frac{x_1}{m_1}\right] + \alpha_2\left[x_2 - m_2 - m_2 \ln\frac{x_2}{m_2}\right] + \alpha_3 x_3$$

Hence, we have

$$\frac{dV_2}{dt} = \alpha_1\frac{(x_1-m_1)}{x_1}\frac{dx_1}{dt} + \alpha_2\frac{(x_2-m_2)}{x_2}\frac{dx_2}{dt} + \alpha_3\frac{dx_3}{dt}$$

$$\frac{dV_2}{dt} = \alpha_1(x_1-m_1)\left[\frac{1-x_1}{1+w_1x_2+w_1x_3} - w_2(1+x_2+x_3)(x_2+x_3) - \frac{1-m_1}{1+w_1 m_2} + w_2(1+m_2)m_2\right] + \alpha_2\frac{(x_2-m_2)}{x_2}\left[w_3(1+x_2+x_3)(x_2+x_3)x_1 - w_4 x_2 x_3 + \frac{w_5 x_3}{w_6+x_3} - w_7 x_2 - w_3(1+m_2)m_1 m_2 + w_7 m_2\right] + \alpha_3\left[w_4 x_2 x_3 - \frac{w_5 x_3}{w_6+x_3} - w_8 x_3\right].$$

By using $\alpha_1 = w_3, \alpha_2 = \alpha_3 = w_2$ it is obtained that:

$$\frac{dV_2}{dt} \leq \frac{-w_3}{B_1}(x_1-m_1)^2 - \frac{w_2}{x_2}[w_7 - w_3 x_1(1+x_2+m_2)](x_2-m_2)^2 - w_2 w_3\left[\frac{x_1}{x_2} - m_1\right]{x_3}^2 - \left[w_3\left(\frac{w_1(1-m_1)}{B_1 B_2} + w_2(1+x_2+m_2)\right) - \frac{w_2 w_3 m_2(1+m_2)}{x_2}\right](x_1-m_1)(x_2-m_2) - \left[w_2 w_8 - \frac{2w_1 w_3 m_1}{B_2} - w_2 w_3 m_1 - 2w_2 w_3 m_1 x_2 - w_2 w_4 m_2\right]x_3.$$

Then we have $\frac{dV_2}{dt}$ is negative definite.

Moreover, since V_2 is radially unbounded function the DFEP is a GAS.

7. LOCAL BIFURCATION

Sotomayor's bifurcation theorem [56] was applied to determine the possibility of local bifurcation near the equilibrium points of the system (2) when the parameter passes through a specific value making the equilibrium point a non-hyperbolic point. The condition that the equilibrium point is non-hyperbolic is a necessary but not sufficient condition for a local bifurcation to occur as is known. The study of the bifurcation of the system (2) is vital because the parameters are not constant in actual reality and are constantly changing according to the conditions of the environment containing the organisms of the system. Now, rewrite the system (2) in the vector form as:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X},\mu),\ \mathbf{X} = (x_1,x_2,x_3)^{\mathrm{T}},\ \mathbf{F} = (x_1 f_1(\mathbf{X},\mu), x_2 f_2(\mathbf{X},\mu), x_3 f_3(\mathbf{X},\mu))^{\mathrm{T}}, \tag{39}$$

where $\mu \in \mathbb{R}$ represents a bifurcation parameter. Hence the second and third directional derivatives for (39) can be written respectively as:

$$D^2\mathbf{F}(\boldsymbol{X},\mu)(\mathbf{V},\mathbf{V}) = [n_{i1}]_{3\times 1}, \tag{40}$$

where $\mathbf{V} = (v_1,v_2,v_3)^T$ be any vector with

$$\begin{aligned}
n_{11} = \frac{1}{[1+w_1(x_2+x_3)]^3} 2\{&-v_1^2[1+w_1(x_2+x_3)]^2 \\
&-(v_2+v_3)^2 x_1[w_2+3w_1w_2(x_2+x_3)+w_1^3w_2(x_2+x_3)^3 \\
&+w_1^2(-1+x_1+3w_2(x_2+x_3)^2)] \\
&-v_1(v_2+v_3)[1+w_1(x_2+x_3)][w_2(1+2x_2+2x_3) \\
&+w_1^2w_2(x_2+x_3)^2(1+2x_2+2x_3) \\
&+w_1(1-2x_1+2w_2(x_2+x_3)(1+2x_2+2x_3))]\}
\end{aligned}$$

$$\begin{aligned}
n_{21} = 2\{&v_2^2w_3x_1 - v_2v_3(w_4-2w_3x_1) + v_1(v_2+v_3)w_3(1+2x_2+2x_3) \\
&+v_3^2(w_3x_1 - \tfrac{w_5w_6}{(w_6+x_3)^3})\}
\end{aligned}$$

$$n_{31} = 2v_3(v_2w_4 + \tfrac{v_3w_5w_6}{(w_6+x_3)^3})$$

And

$$D^3\mathbf{F}(\mathbf{X},\mu)(\mathbf{V},\mathbf{V},\mathbf{V}) = [n_{i2}]_{3\times 1}, \tag{41}$$

where

$$\begin{aligned}
n_{12} = 6(v_2+v_3)\Big[&\tfrac{(v_2+v_3)^2w_1^3(-1+x_1)x_1}{(1+w_1(x_2+x_3))^4} + \tfrac{v_1^2w_1}{(1+w_1(x_2+x_3))^2} \\
&+v_1(v_2+v_3)(-w_2+\tfrac{w_1^2(1-2x_1)}{(1+w_1(x_2+x_3))^3})\Big].
\end{aligned}$$

$$n_{22} = 6(v_1(v_2+v_3)^2w_3 + \tfrac{v_3^3w_5w_6}{(w_6+x_3)^4}).$$

$$n_{32} = -\tfrac{6v_3^3w_5w_6}{(w_6+x_3)^4}.$$

Theorem 7: When the parameter w_3 passes through the value $w_3^* = w_7$, the system (2) undergoes a Transcritical bifurcation (TB) at AEP, provided that $w_2 \neq 1$. Otherwise, pitchfork bifurcation (PB) occurs.

Proof. From the equation (13) with $w_3 = w_3^*$ the JM becomes:

$$J_1^* = J(p_2,w_3^*) = \begin{pmatrix} -1 & -w_2 & -w_2 \\ 0 & 0 & w_3+\frac{w_5}{w_6} \\ 0 & 0 & -\frac{w_5}{w_6}-w_8 \end{pmatrix}.$$

Therefore, the eigenvalues of J_1^* are given by

$$\lambda_{21}(w_3^*) = -1 < 0\,, \lambda_{22}(w_3^*) = 0, \lambda_{23}(w_3^*) = -\frac{w_5}{w_6} - w_8 < 0.$$

Thus AEP is a non-hyperbolic point at $w_3 = w_3^*$.

Let $\mathbf{V}_1 = (v_{11}, v_{12}, v_{13})^{\mathrm{T}}$ and $\mathbf{U}_1 = (u_{11}, u_{12}, u_{13})^{\mathrm{T}}$ be the eigenvectors corresponding to the eigenvalue $\lambda_{22}(w_3^*) = 0$ of the matrices J_1^* and J_1^{*T} respectively. Thus, direct computation gives that $\mathbf{V}_1 = (-w_2, 1, 0)^{\mathrm{T}}$ and $\mathbf{U}_1 = (0, 1, \frac{w_3 w_6 + w_5}{w_5 + w_6 w_8})^T$.

Following Sotomayor's theorem, gives that:

$$\frac{\partial}{\partial w_3}\mathbf{F}(\mathbf{X}, w_3) = \begin{pmatrix} 0 \\ (1 + x_2 + x_3)(x_2 + x_3)x_1 \\ 0 \end{pmatrix} \Rightarrow \mathbf{F}_{w_3}(p_2, w_3^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, ${\mathbf{U}_1}^T \mathbf{F}_{w_3}(p_2, w_3^*) = 0$, as a result, the first requirement for TB is met. Moreover, since

$$D\mathbf{F}_{w_3}(\mathbf{X}, w_3) = \begin{bmatrix} 0 & 0 & 0 \\ (1 + x_2 + x_3)(x_2 + x_3) & x_1(1 + 2x_2 + 2x_3) & x_1(1 + 2x_2 + 2x_3) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow D\mathbf{F}_{w_3}(p_2, w_3^*)\mathbf{V}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore:

$${\mathbf{U}_1}^{\mathrm{T}} D\mathbf{F}_{w_3}(p_2, w_3^*)\mathbf{V}_1 = 1 \neq 0.$$

Now using equation (40), gives

$$D^2\mathbf{F}(p_2, w_3^*)(\mathbf{V}_1, \mathbf{V}_1) = \begin{bmatrix} \frac{2(-w_2^2(1+w_1))^2}{(1+w_1)^3} \\ 2w_7(1 - w_2) \\ 0 \end{bmatrix} \Rightarrow {\mathbf{U}_1}^{\mathrm{T}} D^2\mathbf{F}(p_2, w_3^*)(\mathbf{V}_1, \mathbf{V}_1) = 2w_7(1 - w_2) \neq 0.$$

Hence a TB take place near AEP when $w_2 \neq 1$. Otherwise using equation (41) gives

$$D^3\mathbf{F}(p_2, w_3^*)(\mathbf{V}_1, \mathbf{V}_1, \mathbf{V}_1) = [n_{i2}(p_2, w_3^*)],$$

where

$$n_{12}(p_3, w_3^*) = 6(w_1 w_1^2 - 2w_2 - w_1^2)$$
$$n_{22}(p_3, w_3^*) = -6w_2 w_7$$
$$n_{32}(p_3, w_3^*) = 0.$$

Accordingly, obtained

$${\mathbf{U}_1}^T D^3\mathbf{F}(p_2, w_3^*)(\mathbf{V}_1, \mathbf{V}_1, \mathbf{V}_1) = -6w_2 w_7 \neq 0.$$

Therefore, PB takes place near AEP, and the proof is complete.

Theorem 8: Assume that conditions (21)-(22) hold, and the parameter w_4 passes through the value $w_4^* = \frac{w_8}{m_2} + \frac{w_5}{m_2 w_6}$, then system (2) undergoes a TB at DFEP provided that the following condition holds

$$\sigma_2 w_4^* + \frac{w_5}{{w_6}^2} \neq 0, \tag{42}$$

Otherwise, PB takes place, where all the new symbols are defined in the proof.

Proof. From the equation (18) with $w_4 = w_4^*$ the JM becomes

$$J_2^* = J(p_3, w_4^*) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23}^* \\ 0 & 0 & 0 \end{bmatrix}.$$

where $c_{ij}; i = 1,2, j = 1,2,3$ are given in (18), while $c_{23}^* = c_{23}(w_4^*) = m_1 m_2 w_3 + m_1(1 + m_2)w_3 - w_8$. Therefore, the eigenvalues of J_2^* are given by equation (20), so that λ_{31} and λ_{32} have negative real parts under conditions (20)-(21), while the third one $\lambda_{31} = 0$. Thus DFEP is a non-hyperbolic point at $w_4 = w_4^*$.

Let $\mathbf{V}_2 = (v_{21}, v_{22}, v_{23})^{\mathrm{T}}$ and $\mathbf{U}_2 = (u_{21}, u_{22}, u_{23})^{\mathrm{T}}$ be the eigenvectors corresponding to the eigenvalue $\lambda_{33} = 0$ of the metrics J_2^* and J_2^{*T} respectively. Direct computation gives that $\mathbf{V}_2 = (\sigma_1, \sigma_2, 1)^{\mathrm{T}}$ and $\mathbf{U}_2 = (0,0,1)^T$, where $\sigma_1 = \frac{c_{12}c_{23}^* - c_{22}c_{13}}{c_{11}c_{22} - c_{12}c_{21}}$ and $\sigma_2 = -\frac{c_{11}c_{23}^* - c_{21}c_{13}}{c_{11}c_{22} - c_{12}c_{21}}$.

Following Sotomayor's theorem, gives that:

$$\frac{\partial}{\partial w_4}\mathbf{F}(\mathbf{X}, w_4) = \begin{pmatrix} 0 \\ -x_2 x_3 \\ x_2 x_3 \end{pmatrix} \Rightarrow \mathbf{F}_{w_4}(p_3, w_4^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, ${\mathbf{U}_2}^{\mathrm{T}}\mathbf{F}_{w_4}(p_3, w_4^*) = 0$, as a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$D\mathbf{F}_{w_4}(\mathbf{X}, w_4) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -x_3 & -x_2 \\ 0 & x_3 & x_2 \end{bmatrix} \Rightarrow D\mathbf{F}_{w_4}(p_3, w_4^*)\mathbf{V}_2 = \begin{bmatrix} 0 \\ -m_2 \\ m_2 \end{bmatrix}.$$

Therefore

$${\mathbf{U}_2}^{\mathrm{T}} D\mathbf{F}_{w_4}(p_3, w_4^*)\mathbf{V}_2 = m_2 \neq 0.$$

Using equation (40) gives

$$D^2\mathbf{F}(p_3, w_4^*)(\mathbf{V}_2, \mathbf{V}_2) = [n_{i1}(p_3, w_4^*)],$$

where

$$\begin{aligned} n_{11}(p_3, w_4^*) = \frac{1}{[1 + w_1 m_2]^3} 2\{&-\sigma_1^2[1 + w_1 m_2]^2 \\ &-(\sigma_2 + 1)^2 w_2 m_1[1 + 3w_1 m_2 + w_1^3 {m_2}^3 \\ &+w_1^2(-1 + m_1 + 3w_2 {m_2}^2)] \\ &-\sigma_1(\sigma_2 + 1)[1 + w_1 m_2][w_2(1 + 2m_2) \\ &+w_1^2 w_2 {m_2}^2(1 + 2m_2) \\ &+w_1(1 - 2m_1 + 2w_2 m_2(1 + 2m_2))]\} \\ n_{21}(p_3, w_4^*) = 2\{&\sigma_2^2 w_3 m_1 - \sigma_2(w_4^* - 2w_3 m_1) + \sigma_1(\sigma_2 + 1)w_3(1 + 2m_2) \\ &+(w_3 m_1 - \frac{w_5}{{w_6}^2})\} \\ n_{31}(p_3, w_4^*) = 2(&\sigma_2 w_4^* + \frac{w_5}{{w_6}^2}) \end{aligned}$$

Now, when the condition (42) is met, it is obtained that

$${\mathbf{U}_2}^T D^2\mathbf{F}(p_3, w_4^*)(\mathbf{V}_2, \mathbf{V}_2) = 2(\sigma_2 w_4^* + \frac{w_5}{{w_6}^2}) \neq 0.$$

Hence a TB take place near DFEP. Otherwise, if the condition (42) is violated, it is obtained that

$$D^3\mathbf{F}(p_3, w_4^*)(\mathbf{V}_2, \mathbf{V}_2, \mathbf{V}_2) = [n_{i2}(p_3, w_4^*)],$$

where

$$n_{12}(p_3, w_4^*) = 6(\sigma_2 + 1)\left(\frac{(\sigma_2 + 1)^2(m_1 - 1)w_1^3 m_1}{(1 + w_1 m_2)^4}\right) + \frac{\sigma_1^2 w_1}{(1 + w_1 m_2)^2}$$
$$+ \sigma_1(\sigma_2 + 1)\left(-w_2 + \frac{w_1^3(1 - 2m_1)}{(1 + w_1 m_2)^3}\right)$$
$$n_{22}(p_3, w_4^*) = 6\left(\sigma_1(\sigma_2 + 1)^2 w_3 + \frac{w_5}{w_6^2}\right)$$
$$n_{32}(p_3, w_4^*) = -\frac{w_5}{w_6^3}.$$

Accordingly, it is obtained

$$\mathbf{U}_2{}^T D^3\mathbf{F}(p_3, w_4^*)(\mathbf{V}_2, \mathbf{V}_2, \mathbf{V}_2) = -\frac{w_5}{w_6^3} \neq 0.$$

Therefore, PB takes place near DFEP, and the proof is complete.

Theorem 9: Assume that conditions (26), (27), and (29) are met along with the following conditions

$$b_{33} < \frac{b_{23}b_{32}}{b_{11} + b_{22}^*}, \tag{43}$$
$$\sigma_5 n_{11}(p_4, w_7^*) + \sigma_6 n_{21}(p_4, w_7^*) + n_{31}(p_4, w_7^*) \neq 0, \tag{44}$$

where all the new symbols are defined in the proof. Then system (2) undergoes a saddle-node bifurcation (SNB) near PEP when the parameter w_7 passes through the value w_7^*, where

$$w_7^* = [q_1(q_2 + q_3)w_3 + q_1(1 + q_2 + q_3)w_3 - q_3 w_4] + \frac{1}{b_{11}b_{33}}[b_{13}b_{21}b_{32} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33}].$$

Proof. From the equation (24) with $w_7 = w_7^*$ the JM becomes

$$J_4^* = J(p_4, w_7^*) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22}^* & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix}_{3\times 3},$$

where $b_{ij;\, i,j=1,2,3}$ are given in equation (24), with $b_{22}^* = b_{22}(w_7^*)$.

As a result, it is simple to verify that the coefficient $A_3 = 0$ in equation (25) at $w_7 = w_7^*$. Therefore, the characteristic equation (25) becomes

$$(\lambda^2 + A_1^*\lambda + A_2^*)\lambda = 0,$$

where $A_1^* = A_1(w_7^*)$ and $A_2^* = A_2(w_7^*)$ with A_1 and A_2 as given in equation (25). Clearly, $A_1^* > 0$ and $A_2^* > 0$ under the conditions (26), (27), (29) and (43). Therefore, due to Routh-Hurwitz criterion, the quadratic term in the above-obtained characteristic equation has two eigenvalues with negative real parts, while the third one is zero eigenvalue. Thus PEP is a non-hyperbolic point at $w_7 = w_7^*$.

Let $\mathbf{V}_4 = (v_{41}, v_{42}, v_{43})^{\mathrm{T}}$ and $\mathbf{U}_4 = (u_{41}, u_{42}, u_{43})^{\mathrm{T}}$ be the eigenvectors corresponding to the zero eigenvalue of J_4^* and ${J_4^*}^T$ respectively. Then direct computation gives that $\mathbf{V}_4 = (\sigma_3, \sigma_4, 1)^{\mathrm{T}}$ and

$\mathbf{U}_4 = (\sigma_5, \sigma_6, 1)^{\mathrm{T}}$, where $\sigma_3 = \frac{b_{12}b_{23} - b_{13}b_{22}^*}{b_{11}b_{22}^* - b_{12}b_{21}}$, $\sigma_4 = -\frac{b_{11}b_{23} - b_{13}b_{21}}{b_{11}b_{22}^* - b_{12}b_{21}} < 0$, $\sigma_5 = \frac{b_{21}b_{32}}{b_{11}b_{22}^* - b_{12}b_{21}} > 0$, and $\sigma_6 = -\frac{b_{11}b_{32}}{b_{11}b_{22}^* - b_{12}b_{21}} > 0$.

Following Sotomayor's theorem, gives that:

$$\frac{\partial}{\partial w_7}\mathbf{F}(\mathbf{X}, w_7) = \begin{pmatrix} 0 \\ -x_2 \\ 0 \end{pmatrix} \Rightarrow \mathbf{F}_{w_7}(p_4, w_7^*) = \begin{pmatrix} 0 \\ -q_2 \\ 0 \end{pmatrix}.$$

Therefore, it is obtained.

$$\mathbf{U}_4{}^{\mathrm{T}}\mathbf{F}_{w_7}(p_4, w_7^*) = -q_2\sigma_6 \neq 0$$

Now using equation (40) gives that

$$D^2\mathbf{F}(p_4, w_7^*)(\mathbf{V}_4, \mathbf{V}_4) = [n_{i1}(p_4, w_7^*)],$$

where

$$\begin{aligned} n_{11}(p_4, w_7^*) = \frac{1}{[1 + w_1(q_2 + q_3)]^3} 2\{&-\sigma_3^2[1 + w_1(q_2 + q_3)]^2 \\ &-(\sigma_4 + 1)^2 q_1[w_2 + 3w_1w_2(q_2 + q_3) + w_1^3 w_2(q_2 + q_3)^3 \\ &+w_1^2(-1 + q_1 + 3w_2(q_2 + q_3)^2)] \\ &-\sigma_3(\sigma_4 + 1)[1 + w_1(q_2 + q_3)][w_2(1 + 2q_2 + 2q_3) \\ &+w_1^2 w_2(q_2 + q_3)^2(1 + 2q_2 + 2q_3) \\ &+w_1(1 - 2q_1 + 2w_2(q_2 + q_3)(1 + 2q_2 + 2q_3))]\} \\ n_{21}(p_4, w_7^*) = 2\{&\sigma_4^2 w_3 q_1 - \sigma_4(w_4 - 2w_3q_1) + \sigma_3(\sigma_4 + 1)w_3(1 + 2q_2 + 2q_3) \\ &+(w_3q_1 - \tfrac{w_5w_6}{(w_6 + q_3)^3})\} \\ n_{31}(p_4, w_7^*) = 2(&\sigma_4 w_4 + \tfrac{w_5w_6}{(w_6 + q_3)^3}) \end{aligned}$$

Thus, due to condition (44), the following is obtained.

$$\mathbf{U}_4{}^{T}D^2\mathbf{F}(p_4, w_7^*)(\mathbf{V}_4, \mathbf{V}_4) = \sigma_5 n_{11}(p_4, w_7^*) + \sigma_6 n_{21}(p_4, w_7^*) + n_{31}(p_4, w_7^*) \neq 0.$$

Therefore, SNB takes place as the $w_7 = w_7^*$.

8. NUMERICAL SIMULATION

The purpose of this section is to explore the diseased prey-predator interaction contained by the system (2) using numerical approaches. We are particularly interested in the consequences of prey fear, sickness, and predator-hunting cooperation. Unless otherwise specified, parameter values are fixed according to the biologically feasible set listed below.

$$w_1 = 0.2, w_2 = 0.3, w_3 = 0.25, w_4 = 0.2, w_5 = 0.1, w_6 = 2, w_7 = 0.1, w_8 = 0.1. \quad (45)$$

In this case, the system (2) approaches PEP asymptotically, as shown in Figure (1).

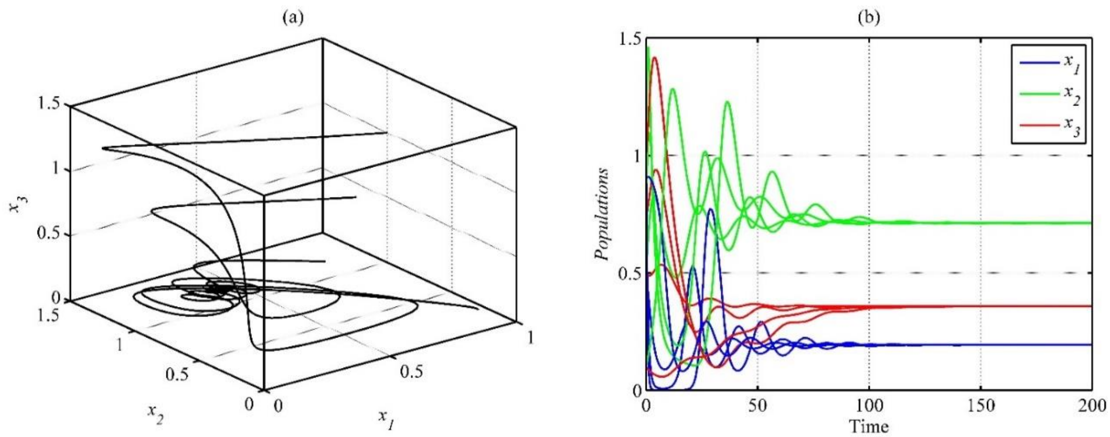


Fig. 1: The solutions of system (2) approach to $p_4 = (0.19,0.71,0.35)$ using the data set (45) and starting from different initial values. (a) 3D phase portrait. (b) Time series.

Figure (1) shows that system (2) has a distinct PEP that is asymptotically stable and dependent on the data collection (45). To investigate the effect of modifying the parameters on the dynamic of the system (2), the numerical solution is calculated using data set (45) by varying one parameter at a time, and the results are then displayed in the form of phase portraits and their time series. It is observed that as $w_1 \in [0,1.3)$ and $w_1 \geq 1.3$ the solution of system (2) approaches to PEP and DFEP respectively, see Figure (1) for the first case and Figure (2) for a selected value of w_1 in the second case. Similar behavior is obtained as that of w_1 when the parameters w_5 and w_8 varying.

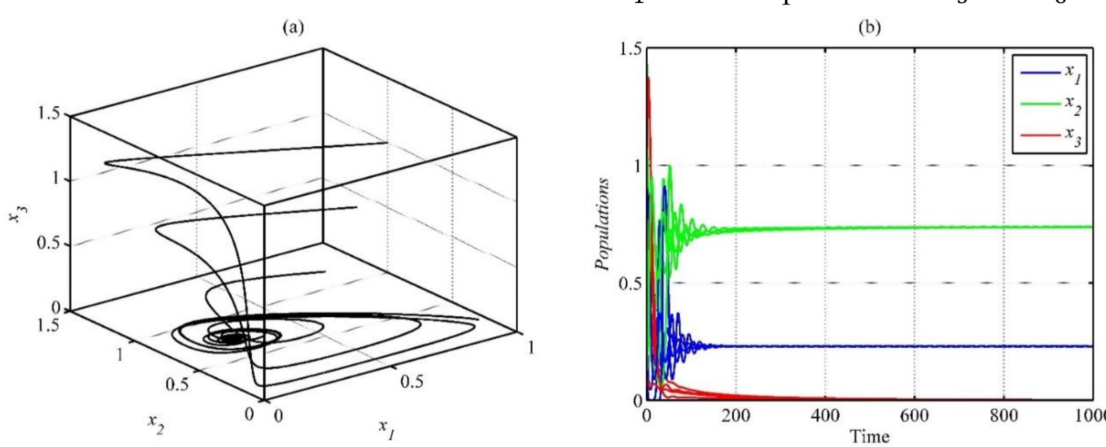


Fig. 2: The solutions of system (2) approach to $p_3 = (0.22,0.73,0)$ using the data set (45) with $w_1 = 1.35$ and starting from different initial values. (a) 3D phase portrait. (b) Time series.

For the ranges $w_2 \in (0,0.06]$, $w_2 \in (0.06,0.51]$, and $w_2 > 0.51$ the solution of system (2) goes asymptotically to 3D periodic dynamic, p_4 see Figure (1), and p_3 respectively. Figure (3) is drawn to explain the obtained results at selected values of w_2.

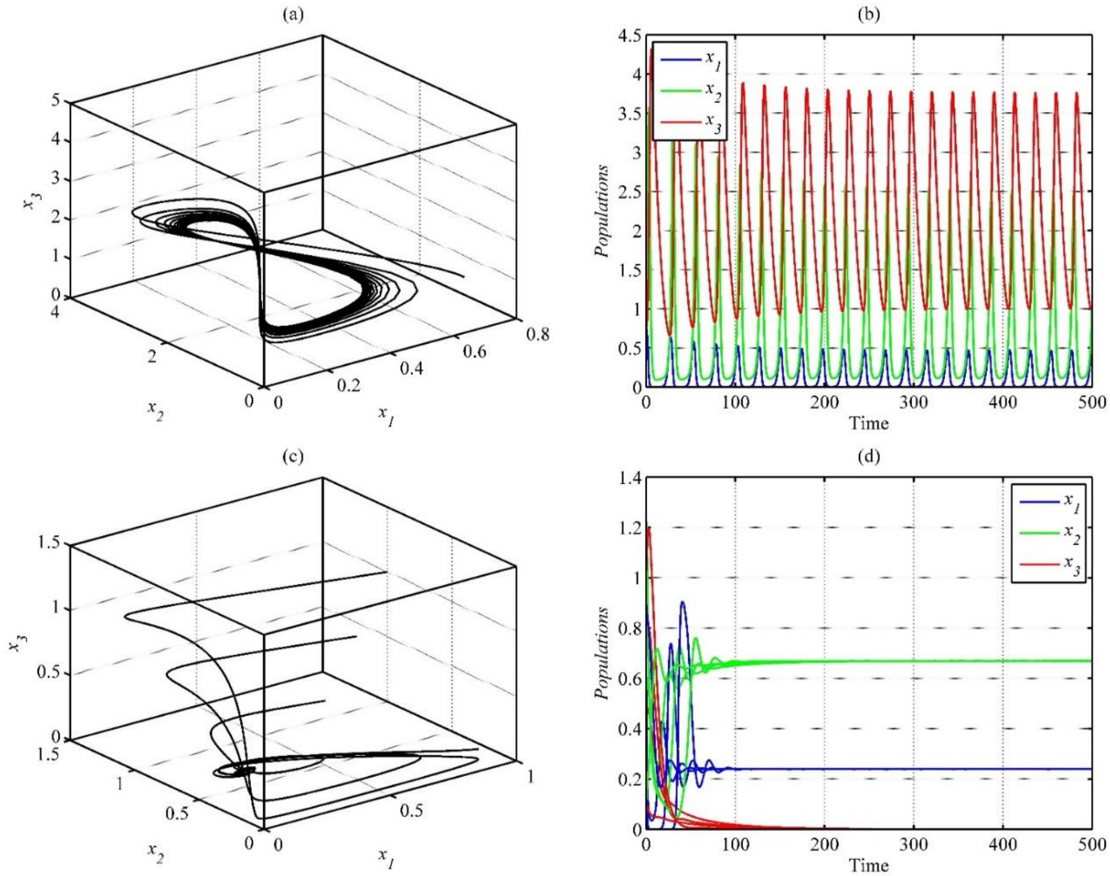
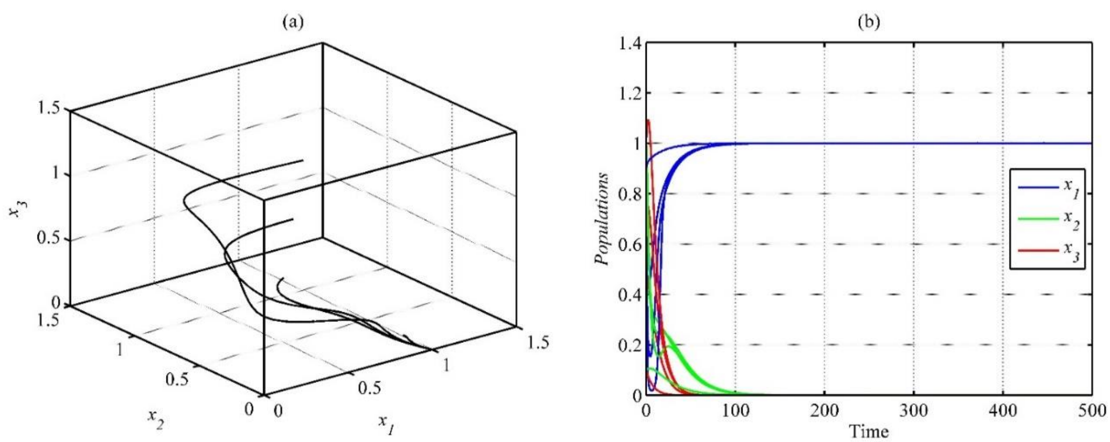


Fig. 3: The solutions of system (2) using the data set (45) go asymptotically to (a) 3D periodic attractor when $w_2 = 0.05$. (b) Time series of the periodic case. (c) $p_3 = (0.22,0.73,0)$ when $w_2 = 0.6$ starting from different initial points. (d) Time series when $w_2 = 0.6$.

On the other hand, it is obtained that system (2) approaches asymptotically to p_2, bistable between p_2 and p_3, p_4, and 3D periodic attractor when $w_3 \in (0,0.06]$, $w_3 \in (0.06,0.1]$, $w_3 \in (0.1,0.66)$ see Figure (1), and $w_3 \geq 0.66$ respectively. Figure (4) shows the behavior of the system (3) at selected values of w_3.



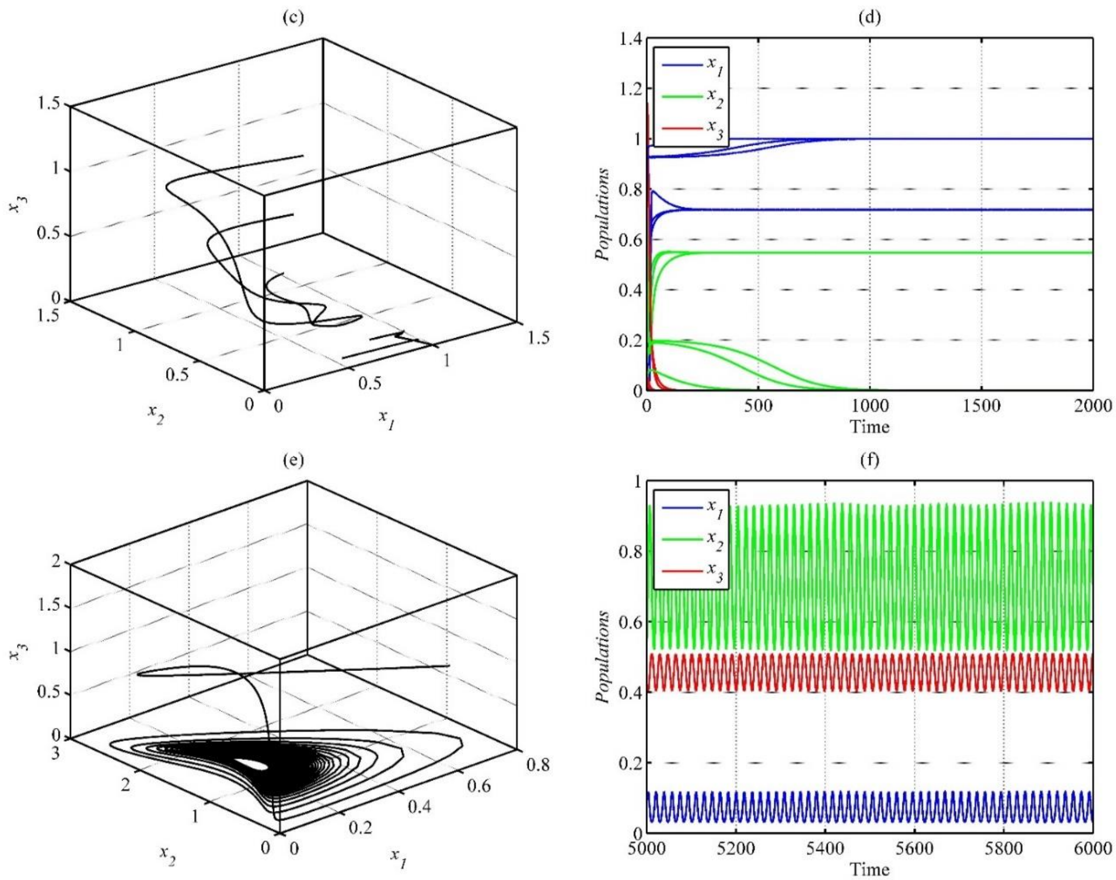
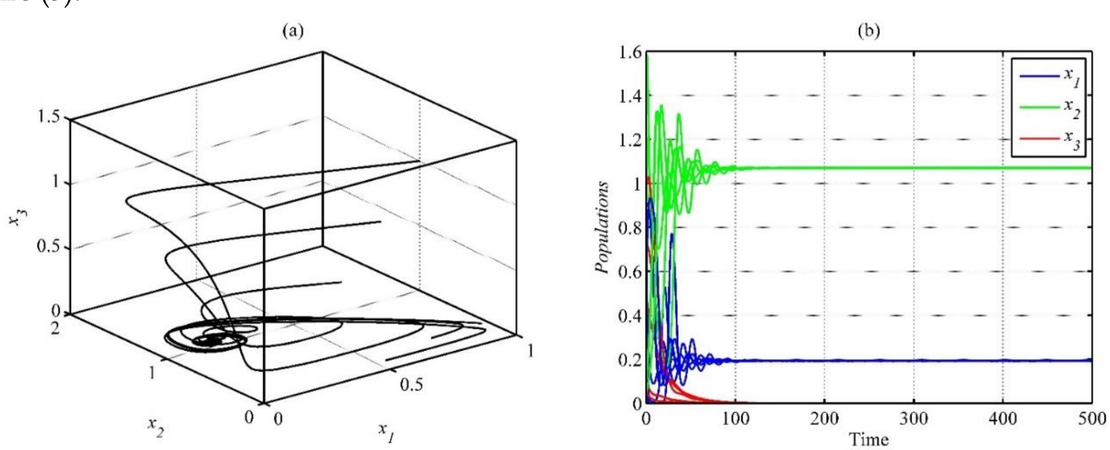


Fig. 4: The solutions of system (2) using the data set (45) goes asymptotically to: (a) $p_2 = (1,0,0)$ when $w_3 = 0.05$ starting from different initial points. (b) Time series when $w_3 = 0.05$. (c) Bistable case between p_2 abd $p_3 = (0.71,0.54,0)$ when $w_3 = 0.09$. (d) Time series when $w_3 = 0.09$. (e) 3D periodic attractor when $w_3 = 0.7$. (f) Time series of the periodic case.

Now, when $w_4 \in (0,0.14]$, and $w_4 > 0.14$, it is observed that the solution of system (2) goes asymptotically to p_3 and p_4 respectively. The obtained results at selected values are shown in Figure (5).



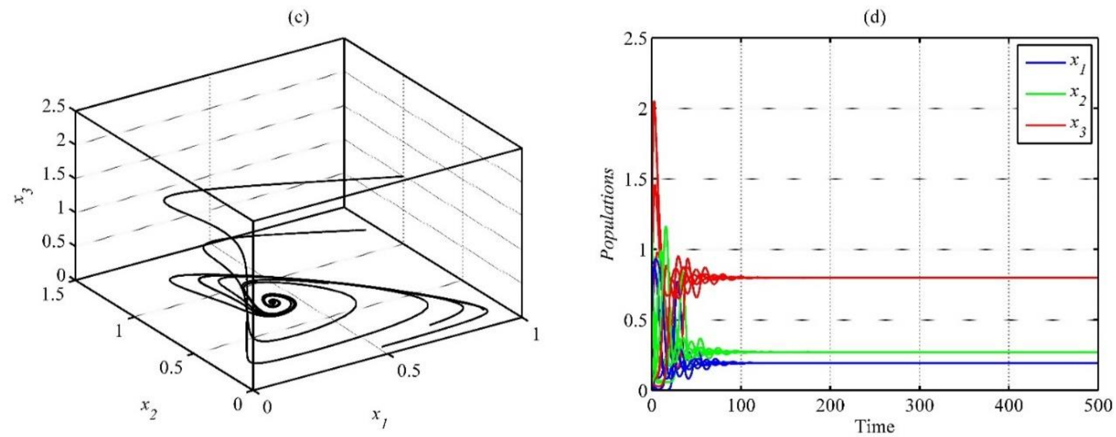
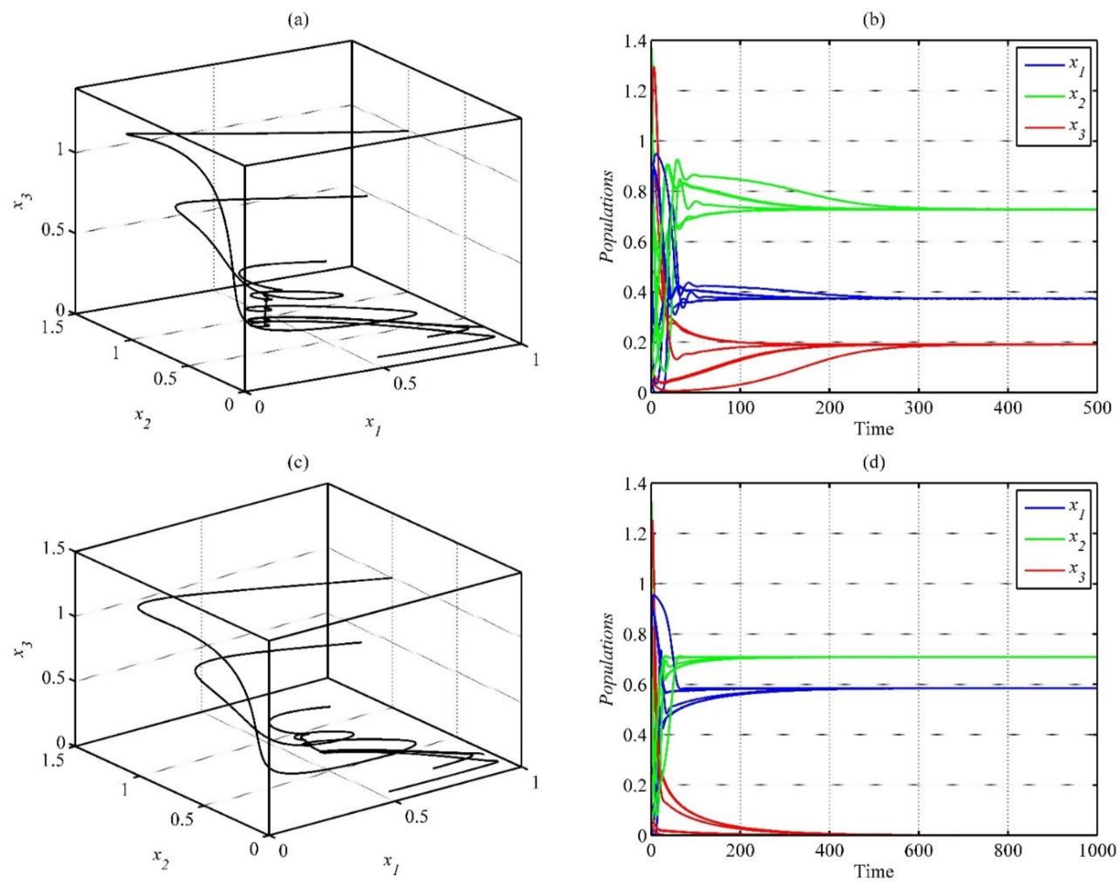


Fig. 5: The solutions of system (2) using the data set (45) and starting from different initial values goes asymptotically to: (a) $p_3 = (0.19,1.07,0)$ when $w_4 = 0.1$. (b) Time series when $w_4 = 0.1$. (c) $p_4 = (0.19,0.27,0.79)$ when $w_4 = 0.5$. (b) Time series when $w_4 = 0.5$.

Similar behavior as that shown in case of varying w_4 is obtained when w_6 is varied with different bifurcating point. Finally, for the ranges $w_7 \in (0,0.24)$, $w_7 \in [0.24,0.29)$, and $w_7 \geq 0.29$ the solution of system (2) goes asymptotically to p_4, p_3, and p_2 respectively as explain in Figure (6) as selected values of w_7.



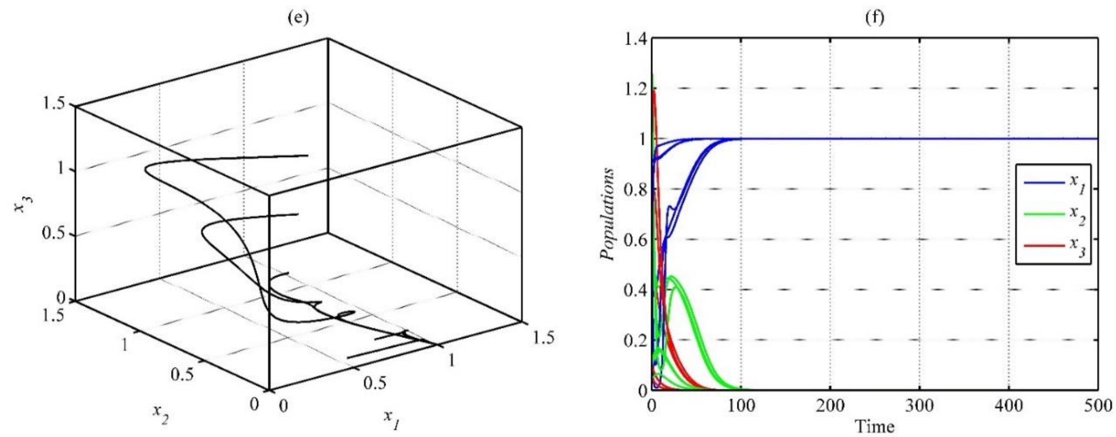


Fig. 6: The solutions of system (2) using the data set (45) and starting from different initial values goes asymptotically to: (a) $p_4 = (0.37,0.72,0.19)$ when $w_7 = 0.2$. (b) Time series when $w_7 = 0.2$. (c) $p_3 = (0.58,0.7,0)$ when $w_7 = 0.25$. (b) Time series when $w_7 = 0.25$. (e) $p_2 = (1,0,0)$ when $w_7 = 0.35$. (b) Time series when $w_7 = 0.35$.

9. CONCLUSIONS

This research involved the study of the impact of both fear and hunting cooperation on the eco-epidemiological system of the prey-predator. The mathematical model was first formulated and then the proposed model was studied theoretically using known mathematical methods and it was noted that the model contains at most four equilibrium points. The proposed system's stability, uniform persistence, and local bifurcation analysis were performed, and all of the prerequisites required to acquire these concepts were determined. To confirm the analytical findings and understand the impact of parameters on the dynamic of the system (2), numerical simulation was used. Figure (2) shows that fear has an extinction effect on the infected predator. Concerning the recovery rate and infected predator mortality rate, similar results have been observed with fear levels. While, as shown in Figure (3), the hunting cooperation level first has a stabilizing influence on the system's (2) dynamic behavior and later, at a critical threshold, becomes an extinction factor for the infected predator. The conversion rate has a beneficial effect on the overall coexistence of the system since it is a stabilizing component at the positive equilibrium point at the start, but when it exceeds a certain point, it has an instability effect and the system switches to cyclic dynamics, see Figure (4).The infection rate (similarly, treatment rate depletion) has a stabilizing influence on the system dynamics at the PEP, see Figure (5). Finally, as shown in Figure (6), healthy predator death rate works as a extinction factor for the predator species.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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