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Analysis of the Economic Cost of Coxian-2 Service with Encouraged Arrival and Balking

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Abstract. The queuing model is widely used in the production, inventory, and service industries. In order to improve the performance of a queuing model, it is crucial to characterize the practical queuing characteristics. The purpose of this work is to examine an analysis of the economic cost of Coxian-2 service with encouraged arrival and balking in a queuing system. In particular, we discussed Coxian-2 service-encouraged arrival queuing system and an accelerated distribution. According to our presumption, units (customers) enter the system one at a time in an encouraged arrival procedure, and the server offers Coxian-2 service one at a time according to the first in first out (FIFO) rule. As probability-generating functions, the typical customer count, and the typical customer wait time in the system and queue, respectively. We also derive steady-state probabilities and performance measures for the proposed model. Finally, the economic analysis of the model is performed by introducing cost model with an empirical example is given to show the effectiveness of the proposed model. The created formula also fulfills Little's formula.

1. Introduction

The queuing theory deals with situations where some kind of customers are gathered at a service facility to receive service, some are waiting for service, and others are leaving. Service facilities develop queues or waiting lines when they cannot handle the number of units requiring service. We all have to wait in a line or queue in our daily lives, whether it is at the food court, the clinic, or the bank. The research on queuing systems with server vacation received more prominence and importance in the literature on queuing theory. To effectively use idle time, servers take vacations. Production, financial services, communications networks, internet technologies, etc. are just a few of the areas where vacation queuing models have been effectively applied. Many academics have an interest in studying queueing models with different vacation guidelines, such as single

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and multiple vacation regulations. In terms of managing operations, planning, executing, and growing services for customers and other areas, a range of organizations can greatly benefit from the encouraged arrival policy.

The M/D/1 queue is widely found in queue with optional server vacations Al-Jararha and Madan [1]. The average number of customers, their average waiting time in the system, and the steady-state queue size distributions in terms of these variables are found in Al-Rawi and Al Shboul [2]. The first three moments are used to study phase-type distributions that approximate generic distributions with known squared coefficients of variation, and the phase-type distributions are also discussed as being used to simulate processes as well as fit observable data sets and approximate broad distributions for simulation and analytical models are investigated in [3], [8] and [18]. Bounkhel [4] investigated how system-size steady-state probabilities are determined using the probability generating function and linear operator techniques with measures of effectiveness. Since the late 1970s, queueing systems with server vacations (servers doing non-queueing tasks) have been the subject of much research. The early 1980s saw the completion of a sizable number of works in this area and described about the balking (they can decide not to join the queue) in vacation queuing models which were examined by Doshi and Haight [5] and [7] in 1986. In order to arrive at the decision that the complementary waiting-time distribution function in the Gl/G/I queue is the sum of two exponential functions when the service time has a Coxian-2 distribution, are disscused in [6]. Basics of queuing theory be found in [9] and [19]. Zhang [10], have investigated the vacation queues with different vacation policies with single or multiple server vacations. Due to an additional optional service, breakdowns, and repeated vacations with dissatisfied customers,for a batch encouraged arrival of Markovian queuing model has been studied in [11] and [12]. The Markovian model's quality control methodology provides an iterative approach to the nth customer in the system are studied in [13]. Stationary distribution of coxian-2 service with vaction times and deterministic service were studied in [14], [16] and [17]. Performance of load-balancing measures in steady-state conditions in conditions of high traffic, with the system's normalised load being studied in [15]. Single server finite capacity of maarkovian service with encouraged arrival are discussed in [20]. The Coxian service time transition matrices of industrial lines and to obtain the extact solution of a sparse linear systems are explained in [23] and [24]. Yücel and Bulut [25] and [26] are explained the measures for steady-state distribution and performance, including throughput, the average number of items in the buffers, and the average system cost, which includes holding, production, and shortage costs in coxian-2 distrubution.

In this work, we investigate an analysis of the economic costs of coxian-2 service with encouraged arrival and balking with an accelerated distribution. When a customer uses a service, his service time is a random variable distributed as Coxian-2 service. Furthermore, we suppose that after each service, the server may take a vacation of arbitrary length with probability p or resume the next service with probability (1-p). When the server takes a vacation, his vacation time is dispersed accelerated distribution. We obtained the steady-state probabilities with a queue size distribution. Additionally, for the steady state, we find the mean queue size, the mean system size, and the mean customer waiting time.

Maximisation of system size and optimisation of the cost analysis of Coxian-2 service by encouraging arrival and balking are proposed in this work. An introduction is described in Section 1. The coxian-2 service with encouraged arrival for model elaboration is described in Section 2. The governing system of equations is derived in Section 3. Steady-state solutions and performance measures are also described in Sections 4 and 5. Numerical illustrations are provided in Section 6. To optimise the cost analysis, the economic cost was explained with an example in Section 7. Results and discussion are given in Section 8. Section 9 contains the conclusion.
The mathematical model of our study is briefly described by the following assumptions:

2. Model Elaboration

In this work we assume that

(1) Customers arrive to the system in a Encouraged arrival rate and balking $(\lambda(1+\zeta)\vartheta)$, where, ζ represents the percentage increase in number of customers calculated from past or observed data and ϑ represents the customers are impatient, they can decide not to join the queue (balking).
(2) Coxian-2 service is accelerated with kth phase mean service time $\frac{1}{\mu_k}$, k=1,2.
(3) The server's vacation period has an accelerated distribution with mean vacation time $\frac{1}{\beta}$
(4) The system's random variables, such as customer service times, customer service intervals, and server vacation intervals, are all independent of one another.

Also we define.

(1) $Prob_n^k(t)$: Probability that a time t there are n customers in the queue excluding one unit in phase-k service, k=1,2; n=0,1,2...
(2) Q(t) : Probability that at time t there is no customer in the queue and the server is idle.
(3) $M_n(t)$: Probability that at time t there is no customer in the queue and the server is on vacation.

Then we have the following sets of equations:

$$\begin{aligned} Prob_n^1(t+\Delta t) =& Prob_n^1(t)[1-(\lambda(1+\zeta)\vartheta)\Delta t](1-\mu_1\Delta t) \\ &+ Prob_{n-1}^1(t)[(\lambda(1+\zeta)\vartheta)\Delta t]((1-\mu_1\Delta t) \\ &+ Prob_{n+1}^1(t)(1-(\lambda(1+\zeta)\vartheta)\Delta t)(1-b)(\mu_1\Delta t)(1-p) \\ &+ Prob_{n+1}^2(t)(1-(\lambda(1+\zeta)\vartheta)\Delta t)(\mu_2\Delta t)(1-p) \\ &+ M_{n+1}(t)(1-(\lambda(1+\zeta)\vartheta)\Delta t)\beta\Delta t \end{aligned} \tag{2.1}$$

$$\begin{aligned} Prob_0^1(t+\Delta t) =& Prob_0^1(t)[1-(\lambda(1+\zeta)\vartheta)\Delta t](1-\mu_1\Delta t) \\ &+ Prob_1^1(t)(1-(\lambda(1+\zeta)\vartheta)\Delta t)(1-b)(\mu_1\Delta t)(1-p) \end{aligned}$$

$$
\begin{aligned}
&+ Prob_1^2(t)(1-(\lambda(1+\zeta)\vartheta)\Delta t)(\mu_2\Delta t)(1-p) \\
&+ M_1(t)(1-(\lambda(1+\zeta)\vartheta)\Delta t)\beta\Delta t + Q(t)(\lambda(1+\zeta)\vartheta)\Delta t \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
Prob_n^2(t+\Delta t) =& Prob_n^2(t)[1-\lambda(1+\zeta)\vartheta\Delta t](1-\mu_2\Delta t) \\
&+ Prob_{n-1}^2(t)[\lambda(1+\zeta)\vartheta\Delta t](1-\mu_2\Delta t) \\
&+ Prob_n^1(1-\lambda(1+\zeta)\vartheta\Delta t)(\mu_1\Delta t)b, n\geq 1 \qquad (2.3)
\end{aligned}
$$

$$
\begin{aligned}
Prob_0^2(t+\Delta t) =& Prob_0^2(t)[1-\lambda(1+\zeta)\vartheta\Delta t](1-\mu_2\Delta t) \\
&+ Prob_0^1(1-\lambda(1+\zeta)\vartheta\Delta t)(\mu_1\Delta t)b \qquad (2.4)
\end{aligned}
$$

$$
\begin{aligned}
M_n(t+\Delta t) =& M_n(t)[1-\lambda(1+\zeta)\vartheta\Delta t](1-\beta\Delta + M_{n-1}(t)[1-\lambda(1+\zeta)\vartheta\Delta t](1-\beta\Delta t) \\
&+ Prob_n^2(t)[1-\lambda(1+\zeta)\vartheta\Delta t](\mu_2\Delta t)p \\
&+ Prob_n^1(t)[1-\lambda(1+\zeta)\vartheta\Delta t](\mu_1\Delta t)(1-b)p, n\geq 1 \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
M_0(t+\Delta t) =& M_0(t)[1-\lambda(1+\zeta)\vartheta\Delta t](1-\beta\Delta t + Prob_0^2(t)[1-\lambda(1+\zeta)\vartheta\Delta t](\mu_2\Delta t)p \\
&+ Prob_0^1(t)[1-\lambda(1+\zeta)\vartheta\Delta t](\mu_1\Delta t)(1-b)p \qquad (2.6)
\end{aligned}
$$

$$
\begin{aligned}
Q(t+\Delta t) =& Q(t)[1-\lambda(1+\zeta)\vartheta\Delta t] + Prob_0^1(t)[1-\lambda(1+\zeta)\vartheta\Delta t](\mu_1\Delta t)(1-b)(1-p) \\
&+ Prob_0^2(t)[1-\lambda(1+\zeta)\vartheta\Delta t](\mu_2\Delta t)(1-p) + M_0(t)[1-\lambda(1+\zeta)\vartheta\Delta t]\beta\Delta t \qquad (2.7)
\end{aligned}
$$

The probabilities of the system at time t and those at time t+Δt can be connected by considering $Prob_n 1(t+\Delta t)$, which represents the probability that there are n customers at time t, excluding one unit in phase 1. The following scenarios are mutually exclusive and included:

- It is assumed that at time t, there are n customers in the queue, which excludes one customer in phase 1 service. There are no encouraged arrivals, no balking and no service completions during (t,t+Δt] and there are (n-1) customers in the queue, which excludes one customer in phase 1 service. There are one encouraged arrival, one balking and no service completions during (t,t+Δt]. These case has a joint probability $Prob_n^1(t)[1-(\lambda(1+\zeta)\vartheta)\Delta t](1-\mu_1\Delta t)$ and $Prob_{n-1}^1(t)[(\lambda(1+\zeta)\vartheta)\Delta t]((1-\mu_1\Delta t)$
- There are (n+1) customers in the queue at time t, with the excluding of one customer in phase 1 and phase 2 service, and there are no encouraged arrival, no balking and one service completion during (t,t+Δt], and the customer prefers not to use phase 2 and phase 1 of service. Then the server also doesn't take a vacation with probability (1-p). In this situation joint probability exits in $Prob_{n+1}^1(t)(1-(\lambda(1+\zeta)\vartheta)\Delta t)(1-b)(\mu_1\Delta t)(1-p)$ and $Prob_{n+1}^2(t)(1-(\lambda(1+\zeta)\vartheta)\Delta t)(\mu_2\Delta t)(1-p)$. When there are there are (n+1) customers in the queue and the server is on vacation, and no encouraged arrival, no balking , but one vacation completed during (t,t+Δt]. These case has a joint probability $M_{n+1}(t)(1-(\lambda(1+\zeta)\vartheta)\Delta t)\beta\Delta t$.

3. The Governing System of Equation

The following set of difference-differential equations is obtained by rearranging the terms of the equations and let $\Delta t \to 0$,

$$\begin{aligned}\frac{d}{dt}Prob_n^1(t) =& -[(\lambda(1+\zeta)\vartheta)+\mu_1]Prob_n^1(t)+(\lambda(1+\zeta)\vartheta)Prob_{n-1}^1(t)\\&+(1-b)(1-p)\mu_1Prob_{n+1}^1(t)+(1-p)\mu_2Prob_{n+1}^2(t)+\beta V_{n+1}(t)\end{aligned}\tag{3.1}$$

$$\begin{aligned}\frac{d}{dt}Prob_0^1(t) =& -[(\lambda(1+\zeta)\vartheta)+\mu_1]Prob_0^1(t)+(1-b)(1-p)\mu_1Prob_1^1(t)\\&+(1-p)\mu_2Prob_1^2(t)+\beta V_1(t)+(\lambda(1+\zeta)\vartheta)Q(t)\end{aligned}\tag{3.2}$$

$$\frac{d}{dt}Prob_n^2(t) = -[(\lambda(1+\zeta)\vartheta)+\mu_2]Prob_n^2(t)+(\lambda(1+\zeta)\vartheta)Prob_{n-1}^2(t)+b\mu_1Prob_n^1(t)\tag{3.3}$$

$$\frac{d}{dt}Prob_0^2(t) = -[(\lambda(1+\zeta)\vartheta)+\mu_2]Prob_0^2(t)+b\mu_1Prob_0^1(t)\tag{3.4}$$

$$\begin{aligned}\frac{d}{dt}M_n(t) =& -[(\lambda(1+\zeta)\vartheta)+\beta]M_n(t)+(\lambda(1+\zeta)\vartheta)M_{n-1}(t)+p\mu_2Prob_n^2(t)\\&+(1-b)p\mu_1Prob_n^1(t)\end{aligned}\tag{3.5}$$

$$\frac{d}{dt}M_0(t) = -[(\lambda(1+\zeta)\vartheta)+\beta]M_0(t)+p\mu_2Prob_0^2(t)+(1-b)p\mu_1Prob_0^1(t)\tag{3.6}$$

$$\begin{aligned}\frac{d}{dt}Q(t) =& -[\lambda(1+\zeta)\vartheta]Q(t)+(1-b)(1-p)\mu_1Prob_0^1(t)\\&+(1-p)\mu_2Prob_0^2(t)+\beta M_0(t)\end{aligned}\tag{3.7}$$

Let us assume that initially there are no customers in the system and the server is idle. We have the following initial conditions:

$$Prob_n^k(0)=0, k=1,2, M_n(0)=0, \forall n\geq 0, Q(0)=1\tag{3.8}$$

Now, by applying the Laplace transformation to equations (3.1) through (3.7), and by using (3.8), we obtain the following:

$$\begin{aligned}cProb_n^{*1}(c)-Prob_n^1(0) =& -[(\lambda(1+\zeta)\vartheta)+\mu_1]Prob_n^{*1}(c)+(\lambda(1+\zeta)\vartheta)Prob_{n-1}^{*1}(c)\\&+(1-b)(1-p)\mu_1Prob_{n+1}^{*1}(c)\\&+(1-p)\mu_2Prob_{n+1}^{*2}(c)+\beta M_{n+1}^*(c)\end{aligned}$$

$$\begin{aligned}[c+(\lambda(1+\zeta)\vartheta)+\mu_1]Prob_n^{*1}(c) =& (\lambda(1+\zeta)\vartheta)Prob_{n-1}^{*1}(c)+(1-b)(1-p)\mu_1Prob_{n+1}^{*1}(c)\\&+(1-p)\mu_2Prob_{n+1}^{*2}(c)+\beta M_{n+1}^*(c), n\geq 1\end{aligned}\tag{3.9}$$

$$\begin{aligned}[c+(\lambda(1+\zeta)\vartheta)+\mu_1]Prob_0^{*1}(c) =& (1-b)(1-p)\mu_1Prob_1^{*1}(c)+(1-p)\mu_2Prob_1^{*2}(c)\\&+\beta M_1^*(c)+(\lambda(1+\zeta)\vartheta)Q^*(c)\end{aligned}\tag{3.10}$$

$$[c+(\lambda(1+\zeta)\vartheta)+\mu_2]Prob_n^{*2}(c) = (\lambda(1+\zeta)\vartheta)Prob_{n-1}^{*2}(c)+b\mu_1Prob_n^{*1}(c)\tag{3.11}$$

$$[c+(\lambda(1+\zeta)\vartheta)+\mu_2]Prob_0^{*2}(c) = b\mu_1Prob_0^{*1}(c)\tag{3.12}$$

$$[c+(\lambda(1+\zeta)\vartheta)+\beta]M_n^*(c) = (\lambda(1+\zeta)\vartheta)M_{n-1}^*(c)+p\mu_2Prob_n^{*2}(c)$$

$$+ (1-b)p\mu_1 Prob_n^{*1}(c) \tag{3.13}$$

$$[c + (\lambda(1+\zeta)\vartheta) + \beta]M_n^*(c) = p\mu_2 Prob_n^{*2}(c) + (1-b)p\mu_1 Prob_n^{*1}(c) \tag{3.14}$$

$$[c + \lambda(1+\zeta)\vartheta]Q^*(c) = (1-b)(1-p)\mu_1 Prob_0^{*1}(c)$$

$$+ (1-p)\mu_2 Prob_0^{*2}(c) + \beta M_0^*(c) + 1 \tag{3.15}$$

To define the probability-generating functions listed below in terms of their Laplace transforms are:

$$Prob^{*k}(y,c) = \sum_{n=0}^{\infty} Prob_n^{*k}(c)y^n, k = 1,2 \tag{3.16}$$

$$M^*(y,c) = \sum_{n=0}^{\infty} P_n^*(c)y^n \tag{3.17}$$

Multiply the equation (3.9) by y^{n+1} and sum over n=1 to ∞, and multiply eqaution (3.10) by y, then add together get,

$$[c + (\lambda(1+\zeta)\vartheta) + \mu_1]\sum_{n=0}^{\infty} Prob_n^{*1}(c)y^{n+1} = (\lambda(1+\zeta)\vartheta)y^2 \sum_{n=0}^{\infty} Prob_n^{*1}(c)y^n + (1-b)(1-p)\mu_1$$

$$\sum_{n=1}^{\infty} Prob_n^{*1}(c)y^n + (1-p)\mu_2 \sum_{n=1}^{\infty} Prob_n^{*2}(c)y^n + \beta \sum_{n=1}^{\infty} M_n^*(c)y^n + Q^*(c)y$$

And by using the terms define by equations (3.16) and (3.17), we obtain,

$$yProb^{*1}(y,c)[c + (\lambda(1+\zeta)\vartheta) + \mu_1 - (\lambda(1+\zeta)\vartheta)y] = (1-b)(1-p)\mu_1 P^{*1}(y,c)$$

$$+(1-p)\mu_2 P^{*2}(y,c) + \beta M^*(y,c) + (\lambda(1+\zeta)\vartheta)yQ^*(c)$$

$$-[(1-b)(1-p)\mu_1 Prob_0^{*1}(c) + (1-p)\mu_2 Prob_0^{*2}(c) + \beta M_0^*(c)] \tag{3.18}$$

Now using equations (3.15), (3.18) can be written as,

$$Prob^{*1}(y,c)[y(c + (\lambda(1+\zeta)\vartheta) + \mu_1 - (\lambda(1+\zeta)\vartheta))y - (1-b)(1-p)\mu_1]$$

$$= (1-p)\mu_2 Prob^{*2}(y,c) + \beta V^*(y,c) + Q^*(c)[(\lambda(1+\zeta)\vartheta)y - c - (\lambda(1+\zeta)\vartheta)] + 1 \tag{3.19}$$

Next multiply equation (3.11) by y^n and sum over n=1 to ∞, equation (3.12) to the result. Thus we have,

$$Prob^{*2}(y,c)[c + (\lambda(1+\zeta)\vartheta) + \mu_2 - (\lambda(1+\zeta)\vartheta)y] = b\mu_1 Prob^{*1}(y,c) \tag{3.20}$$

Then multiplying equation (3.13) by y^n and summing over n=1 to ∞ then, adding the result to equation (3.14) we give,

$$M^*(y,c)[c + (\lambda(1+\zeta)\vartheta) + \beta - (\lambda(1+\zeta)\vartheta)y] = p\mu_2 Prob^{*2}(y,c) + p(1-b)\mu_1 Prob^{*1}(y,c) \tag{3.21}$$

Now, on solving equations (3.19) - (3.21) using Cramerś rule we get;

$$Prob^{*1}(y,c) = \frac{[Q^*(c)((\lambda(1+\zeta)\vartheta)y - (\lambda(1+\zeta)\vartheta) - c) + 1][c + \mu_2 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y][c + \beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)z]}{E(c,y)} \tag{3.22}$$

$$Prob^{*2}(y,c) = \frac{b\mu_1[Q^*(c)((\lambda(1+\zeta)\vartheta)y - (\lambda(1+\zeta)\vartheta) - c) + 1][c + \beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y]}{E(c,y)} \tag{3.23}$$

$$M^*(y,c) = \frac{[Q^*(c)((\lambda(1+\zeta)\vartheta)y - (\lambda(1+\zeta)\vartheta) - c) + 1][bp\mu_1\mu_2 + p(1-b)\mu_1][c + \mu_2 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y]}{E(c,y)} \tag{3.24}$$

where,

$$\begin{aligned} E(c,y) =& z(c + \mu_1 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y)(c + \mu_2 + (\lambda(1+\zeta)\vartheta) \\ &- (\lambda(1+\zeta)\vartheta)y)(c + \beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y) \\ &- (1-b)(1-p)\mu_1(c + \mu_2 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y)(c + \beta + (\lambda(1+\zeta)\vartheta) \\ &- (\lambda(1+\zeta)\vartheta)z) - (1-p)b\mu_1\mu_2(c + \beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y) \\ &- \beta p(1-b)\mu_1(c + \mu_1 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y) - \beta pb\mu_1\mu_2 \end{aligned} \tag{3.25}$$

4. Steady State Solution

Using the well known property of Laplace transform(L.T) $\lim_{c\to 0} cQ^*(c) = Q = \lim_{t\to\infty} Q(t)$ We obtain from equation (3.25)

$$\begin{aligned} E(y) = \lim_{c\to 0} E(c,y) =& z(\mu_1 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y)(\mu_2 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y) \\ &(\beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y) \\ &- (1-b)(1-p)\mu_1(\mu_2 + (\lambda(1+\zeta)\vartheta) \\ &- (\lambda(1+\zeta)\vartheta)y)(\beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y) \\ &- (1-p)b\mu_1\mu_2(\beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y) \\ &- \beta p(1-b)\mu_1(\mu_1 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y) - \beta pb\mu_1\mu_2 \end{aligned} \tag{4.1}$$

then, for the steady state we have,

$$Prob^1(y) = \lim_{c\to 0} sProb^{*1}(y,c)$$

$$Prob^1(y) = \frac{\lim_{c\to 0} c[Q^*(c)((\lambda(1+\zeta)\vartheta)c - (\lambda(1+\zeta)\vartheta) - c) + 1][c + \mu_2 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y][c + \beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)z]}{E(c,y)}$$

$$Prob^1(y) = \frac{[Q(\lambda(1+\zeta)\vartheta)y - (\lambda(1+\zeta)\vartheta)][\mu_2 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y][\beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y]}{E(y)} \tag{4.2}$$

$$\begin{aligned} Prob^1(y) &= \lim_{c\to 0} sProb^{*2}(y,c) \\ &= \frac{b\mu_1 Q[(\lambda(1+\zeta)\vartheta)y - (\lambda(1+\zeta)\vartheta)][\beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y]}{E(y)} \end{aligned} \tag{4.3}$$

$$\begin{aligned} M(y) &= \lim_{c\to 0} cM^*(y,c) \\ &= \frac{Q[(\lambda(1+\zeta)\vartheta)y - (\lambda(1+\zeta)\vartheta)][bp\mu_1\mu_2 + p(1-b)\mu_1][\mu_2 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y]}{E(y)} \end{aligned} \tag{4.4}$$

Where E(y) is given in equation (4.1), In order to find the only unknown probability Q, we shall use the normalizing condition.

$$Q + Prob^1(1) + Prob^2(1) + M(1) = 1 \tag{4.5}$$

Now, since each of $Prob^1(1), Prob^2(1) \& M(1)$ in equations (4.2) - (4.4) is indeterminate of the $\frac{0}{0}$ form at z=1, we use L'hospital rule and obtain,

$$Prob^1(1) = \lim_{y\to 1} Prob^1(y) = \frac{(\lambda(1+\zeta)\vartheta)\mu_2\beta Q}{\mu_1\mu_2\beta - (\lambda(1+\zeta)\vartheta)\mu_2\beta - (\lambda(1+\zeta)\vartheta)b\mu_1\beta - (\lambda(1+\zeta)\vartheta)p\mu_1\mu_2} \tag{4.6}$$

$$Prob^2(1) = \lim_{y\to 1} Prob^2(y) = \frac{(\lambda(1+\zeta)\vartheta)b\mu_1\beta Q}{\mu_1\mu_2\beta - (\lambda(1+\zeta)\vartheta)\mu_2\beta - (\lambda(1+\zeta)\vartheta)b\mu_1\beta - (\lambda(1+\zeta)\vartheta)p\mu_1\mu_2} \tag{4.7}$$

$$M(1) = \lim_{y\to 1} M(y) = \frac{(\lambda(1+\zeta)\vartheta)p\mu_1\mu_2 Q}{\mu_1\mu_2\beta - (\lambda(1+\zeta)\vartheta)\mu_2\beta - (\lambda(1+\zeta)\vartheta)b\mu_1\beta - (\lambda(1+\zeta)\vartheta)p\mu_1\mu_2} \tag{4.8}$$

Using equations (4.6) - (4.8) in equation (4.5) and simplifying we obtain

$$Q = \frac{\mu_1\mu_2\beta - (\lambda(1+\zeta)\vartheta)\mu_2\beta - (\lambda(1+\zeta)\vartheta)b\mu_1\beta - (\lambda(1+\zeta)\vartheta)p\mu_1\mu_2}{\mu_1\mu_2\beta}$$

Which on simplifying gives,

$$\begin{aligned} Q =& 1 - \frac{\lambda(1+\zeta)\vartheta}{\mu_1} - \frac{(\lambda(1+\zeta)\vartheta)b}{\mu_2} - \frac{(\lambda(1+\zeta)\vartheta)p}{\beta} \\ =& 1 - [\frac{\lambda(1+\zeta)\vartheta}{\mu_1} + \frac{(\lambda(1+\zeta)\vartheta)b}{\mu_2} + \frac{(\lambda(1+\zeta)\vartheta)p}{\beta}] \\ =& 1 - \rho \end{aligned} \tag{4.9}$$

$$and\ so,\ \rho = [\frac{\lambda(1+\zeta)\vartheta}{\mu_1} + \frac{(\lambda(1+\zeta)\vartheta)b}{\mu_2} + \frac{(\lambda(1+\zeta)\vartheta)p}{\beta}] \tag{4.10}$$

4.1. **Some special Cases:**

(1) We may note that when p=0, (no vacation),

$$Q = 1 - [\frac{\lambda(1+\zeta)\vartheta}{\mu_1} + \frac{(\lambda(1+\zeta)\vartheta)b}{\mu_2}]$$

(2) Further, when p=0,b=1 (no vacation,two phase service),

$$Q = 1 - [\frac{\lambda(1+\zeta)\vartheta}{\mu_1} + \frac{\lambda(1+\zeta)\vartheta}{\mu_2}]$$

(3) And when p=0,b=0 (no vacation, no second phase service)

$$Q = 1 - \frac{\lambda(1+\zeta)\vartheta}{\mu_1} \tag{4.11}$$

This equation (4.11) is known as the result of M/M/1 Queue.

5. Performance Measures

In this section, our aim is to determine the average number of customers in the system and their mean waiting time. We define:
L = The average number in the system
L_q = The average number in the queue (mean queue length)
W = The mean waiting time in the system
W_q = The mean waiting time in the queue.

Let $Prob(z) = Prob^1(z) + Prob^2(z) + M(z)$ define the p.g.f of the number of units present in the queue without regard to the state of the server.Then we write,
$Prob(z) = \frac{S(y)}{E(y)}$ Where,

$$\begin{aligned} S(y) =& Q(\lambda(1+\zeta)\vartheta)[(y-1)(\mu_2 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y)(\beta + (\lambda(1+\zeta)\vartheta) \\ & - (\lambda(1+\zeta)\vartheta)y) + b\mu_1(y-1)(\beta + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y) \\ & + (y-1)[bp\mu_1\mu_2 + p(1-b)\mu_1(\mu_2 + (\lambda(1+\zeta)\vartheta) - (\lambda(1+\zeta)\vartheta)y)]] \end{aligned}$$

and E(y) is given by equation (4.1)
Now since, $L_q = \frac{d}{dz}Prob(y)_{|y|=1} = \frac{0}{0}$,Then, we use L'Hosptial's rule twice to get,

$$Ł_q = \frac{E^{'}(1)S^{''}(1) - S^{'}(1)E^{''}(1)}{2[E^{'}(1)]}$$

$$\begin{aligned} S^{'}(1) =& Q(\lambda(1+\zeta)\vartheta)[\mu_2\beta + b\mu_1\beta + p\mu_1\mu_2] \\ S^{''}(1) =& Q(\lambda(1+\zeta)\vartheta)[2(\lambda(1+\zeta)\vartheta)bp\mu_1 - 2(\lambda(1+\zeta)\vartheta)\beta - 2(\lambda(1+\zeta)\vartheta)p\mu_1 \\ & - 2(\lambda(1+\zeta)\vartheta)b\mu_1 - 2(\lambda(1+\zeta)\vartheta)\mu_2] \\ E^{'}(1) =& \mu_1\mu_2\beta - (\lambda(1+\zeta)\vartheta)b\mu_1\beta - (\lambda(1+\zeta)\vartheta)p\mu_1\mu_2 - (\lambda(1+\zeta)\vartheta)b\mu_2\beta \\ E^{''}(1) =& 2(\lambda(1+\zeta)\vartheta)^2\beta - 2(\lambda(1+\zeta)\vartheta)\mu_1\mu_2 - 2(\lambda(1+\zeta)\vartheta)\mu_1\beta + 2(\lambda(1+\zeta)\vartheta)^2\mu_2 \\ & - 2(\lambda(1+\zeta)\vartheta)\mu_2\beta + 2p(\lambda(1+\zeta)\vartheta)^2\mu_1 + 2b(\lambda(1+\zeta)\vartheta)^2\mu_1 - 2bp(\lambda(1+\zeta)\vartheta)^2\mu_1 \end{aligned}$$

Further, $L = L_q + \rho$
where, $\rho = [\frac{\lambda(1+\zeta)\vartheta}{\mu_1} + \frac{(\lambda(1+\zeta)\vartheta)b}{\mu_2} + \frac{(\lambda(1+\zeta)\vartheta)p}{\beta}]$, $W = \frac{L}{\lambda(1+\zeta)\vartheta}$ and so $W_q = \frac{L_q}{\lambda(1+\zeta)\vartheta}$.

6. Numerical Illustrations

In order to see the effective of various parameters on server's idle time Q and various other queue characteristics such as ρ, L, W, Lq and Wq, are the basic numerical examples on our main results. For this purpose, we choose arbitrary values of the parameters $(\lambda(1+\zeta)\vartheta)$, β, p, b, μ_1 and μ_2 such that the steady state condition is always satisfied. We have assumed $(\lambda(1+\zeta)\vartheta)$ = 0.0220, β= 0.1, μ_1 = 6, μ_2 = 8,ζ = 0.1, ϑ =0.01 with varying values of p from 0.0 to 1.0 at the intervals of 0.1 and varying values of b = 0, b = 0.5 and b = 1. Using MATLAB, we obtain the following tables: Table 1 presents an encouraged arrival for 10% discounts with a Coxian-2 service b=0, Table 2 presents an encouraged arrival for 10% discounts with a Coxian-2 service b=0.5 and Table 3 presents an encouraged arrival for 10% discounts with a Coxian-2 service b=1.0

b	p	Q	ρ	L	L_q	W	W_q
	0.0	0.9963	0.0037	0.0037	0.0000	0.1673	0.0006
	0.1	0.9743	0.0257	0.0307	0.0051	1.3969	0.2302
	0.2	0.9523	0.0477	0.0580	0.0103	2.6370	0.4704
	0.3	0.9303	0.0697	0.0855	0.0159	3.8886	0.7219
	0.4	0.9083	0.0917	0.1134	0.0217	5.1523	0.9856
0.0	0.5	0.8863	0.1137	0.1414	0.0278	6.4291	1.2624
	0.6	0.8643	0.1357	0.1698	0.0342	7.7200	1.5533
	0.7	0.8423	0.1577	0.1986	0.0409	9.0261	1.8595
	0.8	0.8203	0.1797	0.2277	0.0480	10.3486	2.1820
	0.9	0.7983	0.2017	0.2572	0.0555	11.6889	2.5223
	1.0	0.7763	0.2237	0.2871	0.0634	13.0485	2.8819

Table 1. Encouraged arrival for 10% discounts with a Coxian-2 service b=0.

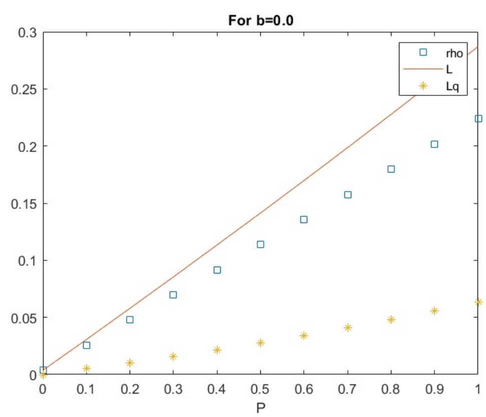

Figure 1. For b=0 the Coxian-2 service with 10% encouraged arrival and balking for the length in system and queue, as well as waiting time in system and queue

For b=0, the Coxian-2 service with 10% encouraged arrival and balking for the length in systems and queues are increasing as shows in the Figure 1.

b	p	Q	ρ	L	L_q	W	W_q
	0.0	0.9950	0.0050	0.0051	0.0000	0.2302	0.0010
	0.1	0.9730	0.0270	0.0322	0.0051	1.4615	0.2323
	0.2	0.9510	0.0490	0.0595	0.0104	2.7035	0.4744
	0.3	0.9290	0.0710	0.0871	0.0160	3.9570	0.7278
	0.4	0.9070	0.0930	0.1149	0.0219	5.2228	1.2726
0.5	0.5	0.8850	0.1150	0.1430	0.0280	6.5018	1.2726
	0.6	0.8630	0.1370	0.1715	0.0344	7.7950	1.5658
	0.7	0.8410	0.1590	0.2003	0.0412	9.1036	1.8744
	0.8	0.8190	0.1810	0.2294	0.0484	10.4287	2.1996
	0.9	0.7970	0.2030	0.2590	0.0559	11.7718	2.5427
	1.0	0.7750	0.2250	0.2890	0.0639	13.1344	2.90552

TABLE 2. Encouraged arrival for 10% discounts with a Coxian-2 service b=0.5

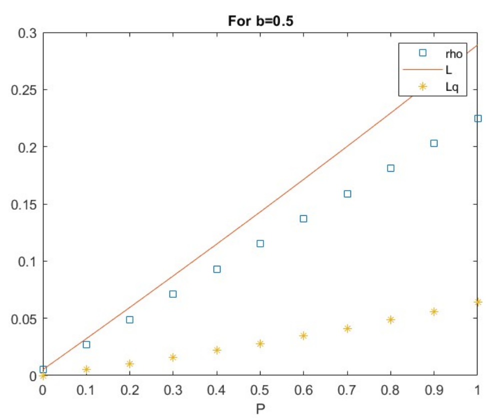

FIGURE 2. For b=0.5 the Coxian-2 service with 10% encouraged arrival and balking for the length in system and queue, as well as waiting time in system and queue

The Coxian-2 service with 10% encouraged arrival and balking for the length in systems and queues are increasing in b=0.5 as shown in the Figure 2.

b	p	Q	ρ	L	L_q	W	W_q
	0.0	0.9936	0.0064	0.0064	0.0000	0.2931	0.0014
	0.1	0.9716	0.0284	0.0336	0.0052	1.5262	0.2345
	0.2	0.9496	0.0505	0.0609	0.0105	2.7700	0.4784
	0.3	0.9276	0.0724	0.0886	0.0161	4.0255	0.7338
	0.4	0.9056	0.0944	0.1165	0.0220	5.2933	1.0017
1.0	0.5	0.8836	0.1164	0.1446	0.0282	6.5745	1.2828
	0.6	0.8616	0.1384	0.1731	0.0347	7.8701	1.5784
	0.7	0.8396	0.1604	0.2020	0.0416	9.1811	1.8894
	0.8	0.8176	0.1824	0.2312	0.0488	10.5089	2.2172
	0.9	0.7956	0.2044	0.2608	0.0564	11.8548	2.5631
	1.0	0.7736	0.2264	0.2908	0.0644	13.2203	2.9287

TABLE 3. Encouraged arrival for 10% discounts with a Coxian-2 service b=1.0

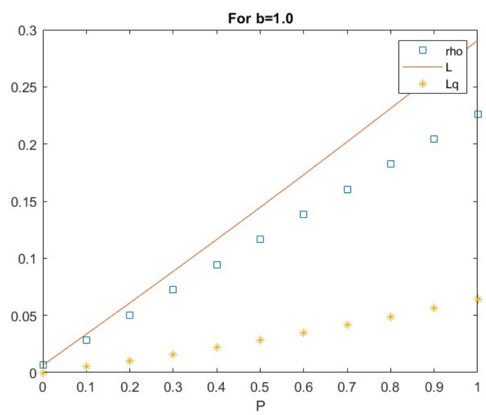

FIGURE 3. For b=1.0 the Coxian-2 service with 10% encouraged arrival and balking for the length in system and queue, as well as waiting time in system and queue

When b=1.0 the system size and the queues are increased in coxian-2 service with encouraged arrival and with balking as shown in Figure 3.

7. Economic Cost Analysis

In order to optimise the system's operating costs, we established a cost function. In the system, c_h represents the unit holding cost per customer, c_o represents the operating cost per unit of time, c_a represents the startup cost per unit of time for the server setup, and c_s represents the setup cost per busy cycle. Then, the total expected cost per unit of time is

$$TC(c) = c_h L + c_o \frac{exp[B]}{exp[C]} + c_a \frac{exp[I]}{exp[C]} + c_s \frac{1}{exp[C]}, \tag{7.1}$$

Where the expected idle period, the expected busy period, and the expected busy cycle are respectively given by

$$exp[I] = \frac{1}{\lambda(1+\zeta)\vartheta}, exp[B] = \frac{1-P_o}{P_o} exp[I], exp[C] = exp[I] + exp[B], \tag{7.2}$$

Thus,

$$TC(c) = c_h L + c_o + ((\lambda(1+\zeta)\vartheta)c_s + c_a - c_o)P_o \tag{7.3}$$

Let us considered the unit costs are c_h = 10, c_o = 20, c_a = 50, c_s = 500 and the system of linear equations yields $Prob_0 = 0.7522 * (\lambda(1+\zeta)\vartheta)$.

p	TC at b=0	TC at b=0.5	TC at b=1
0.0	20.712	20.726	20.739
0.1	20.982	20.997	21.011
0.2	21.255	21.270	21.284
0.3	21.530	21.546	21.561
0.4	21.809	21.824	21.840
0.5	22.089	22.105	22.121
0.6	22.373	22.390	22.406
0.7	22.661	22.678	22.695
0.8	22.952	22.969	22.987
0.9	23.247	23.265	23.283
1.0	23.546	23.565	23.583

Table 4. Expected total cost for b=0, 0.5, 1 with encouraged arrival and balking

8. Results and Discussion

- The above table(1, 2, 3) clearly shows that as p increases for a fixed value of b, or b increases for a fixed value of p. Although the idle time on servers decreases, the utilisation factor, the average size of the system, and the average queue size are all increasing.
- Further note that values of the above queues characteristics for p = 0 corresponds to the case, when the server does not take any vacations.
- From the values of b = 0 were corresponds to the vacation periods with a encouraged arrival and balking of accelerate phase, same as the values of b = 1 are corresponds to vacation periods with two accelerate phases. Table 4 shows that the expected total cost increases for each b=0, 0.5, 1 when p increases for a fixed value of b. Further verification in the tables, it shows that the following graphs are increased.

9. Conclusion

We have investigated the Economic cost of Coxian-2 service with encouraged arrival and balking with accelerated distribution in queuing model. Additionally, the system utilization factor, the average number of customers in queue, and the average wait time for each customer have all been gathered in closed form. The system size and the queue size are increased. A numerical example is given that illustrates distinct and significant patterns caused by the impacts of specific factors, and the graphs further support these trends.The above calculated values are found with varying values of p from 0.0 to 1.0 at the intervals of 0.1 and varying values of b = 0, 0.5 & 1 and to improve the standard of service to be more effective and efficient. The economic cost anlaysis are also more efficiency to optimize the cost analysis.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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