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Grüss Type k-Fractional Integral Operator Inequalities and Allied Results

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Abstract. This paper aims to derive fractional Grüss type integral inequalities for generalized k-fractional integral operators with Mittag-Leffler function in the kernel. Many new results can be deduced for several integral operators by giving specific values to the parameters involved in Mittag-Leffler function. Moreover, the results of this paper reproduce a lot of already published results.

1. Introduction

In 1935, Grüss derived the following inequality which is well-known as the Grüss inequality [7]:

Theorem 1.1. *Let f and g be two integrable functions on $[\omega_1, \omega_2]$. Also, let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{N}_1$ and $\mathcal{N}_2$ be four real numbers satisfying the following conditions:*

$$\mathcal{M}_1 \leq f(x) \leq \mathcal{M}_2 \quad \text{and} \quad \mathcal{N}_1 \leq g(x) \leq \mathcal{N}_2 \tag{1.1}$$

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for all $x \in [\omega_1, \omega_2]$. *Then the following inequality holds:*

$$\left|\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} f(x)g(x)dx - \left(\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} f(x)dx\right)\left(\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} g(x)dx\right)\right| \\ \leq \frac{1}{4}(\mathcal{M}_2-\mathcal{M}_1)(\mathcal{N}_2-\mathcal{N}_1). \tag{1.2}$$

Grüss inequality (1.2) estimates the integral mean of the product of two functions to the product of their integral means. In recent years, many authors have introduced generalizations and extensions of inequality (1.2) for different integral operators. For example, in [29], Tariboon et al. derived the Grüss type integral inequalities for Riemann-Liouville integral operator. In [11], Kacar et al. obtained the Grüss type integral inequalities for generalized Riemann-Liouville integral operator. In [22], Rashid et al. established the Grüss type integral inequalities for generalized k-Riemann-Liouville integral operator. In [17], Mubeen and Iqbal present the Grüss type integral inequalities for generalized Riemann-Liouville (k,r)-integral operator. In [24], Rahman et al. proved the Grüss type integral inequalities for the conformable integral operator. In [8], Habib et al. gave the Grüss type integral inequalities for generalized (k,s)-conformable integral operator. In [5], Farid et al. derived the Grüss type integral inequalities for generalized integral operator containing Mittag-Leffler function. For more detail related to the Grüss inequality (1.2), see [19].

Inspired by the above-defined works, this paper aims to derive the Grüss type integral inequalities for k-fractional integral operators containing Mittag-Leffler function in their kernel. Several new Grüss type integral inequalities can be deduced from the presented results.

Fractional integral operators involving the Mittag-Leffler function are very useful in mathematical inequalities. A large number of integral inequalities involving the Mittag-Leffler function exist in literature. For example, in [20], Purohit et al. derived the Chebyshev type inequalities for fractional integral operator containing multi-index Mittag-Leffler function. In [23], Rashid et al. gave the Hadamard type inequalities for exponentially m-convex functions via an extended generalized Mittag-Leffler function. In [6], Farid gave the bounds of fractional integral operators involving the Mittag-Leffler function. In [2], Andrić et al. derived Chebyshev and Pólya-Szegö types inequalities for generalized Mittag-Leffler function. For further details related to the fractional integral inequalities involving Mittag-Leffler function, see [3]. For further such results one can see [32–34].

Next, we give the definitions of generalized integral operators containing the Mittag-Leffler function.

Definition 1.1 ([1]). *Let* $f : [\omega_1, \omega_2] \to \mathbb{R}$, $0 < \omega_1 < \omega_2$ *be a integrable function. Also, let* $\xi, \alpha, \theta, \rho, \epsilon, \sigma \in \mathbb{C}, \Re(\alpha), \Re(\theta), \Re(\rho) > 0, \Re(\sigma) > \Re(\epsilon) > 0$ *with* $q \geq 0, \nu > 0$ *and* $0 < \gamma \leq \nu + \Re(\alpha)$. *Then for* $x \in [\omega_1, \omega_2]$, *the generalized integral operators are defined by:*

$$\left(\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+} f\right)(x;q) = \int_{\omega_1}^{x} (x-\psi)^{\theta-1} E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho}(\xi(x-\psi)^{\alpha};q) f(\psi)d\psi \tag{1.3}$$

and

$$\left(\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_2^-}f\right)(x;q)=\int_x^{\omega_2}(\psi-x)^{\theta-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho}(\xi(\psi-x)^{\alpha};q)f(\psi)d\psi, \tag{1.4}$$

where $E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho}(\psi;q)$ *is the generalized Mittag-Leffler function defined by:*

$$E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho}(\psi;q)=\sum_{n=0}^{\infty}\frac{\mathrm{B}_q(\epsilon+n\gamma,\sigma-\epsilon)}{\mathrm{B}(\epsilon,\sigma-\epsilon)}\frac{(\sigma)_{n\gamma}}{\Gamma(\alpha n+\theta)}\frac{\psi^n}{(\rho)_{n\nu}}$$

$$\mathrm{B}_q(x,y)=\int_0^1\psi^{x-1}(1-\psi)^{y-1}e^{-\frac{q}{\psi(1-\psi)}}d\psi$$

and $(\sigma)_{n\gamma}=\frac{\Gamma(\sigma+n\gamma)}{\Gamma(\sigma)}$.

Recently, in [30], Zhang et al. introduced the generalized k-integral operators involving Mittag-Leffler function as follows:

Definition 1.2. *Let* $f,\zeta:[\omega_1,\omega_2]\to\mathbb{R}$, $0<\omega_1<\omega_2$ *be the functions such that* f *be a integrable and positive and* ζ *be a strictly increasing and differentiable. Also, let* $\xi,\theta,\rho,\epsilon,\sigma\in\mathbb{C}$, $\Re(\theta),\Re(\rho)>0$, $\Re(\sigma)>\Re(\epsilon)>0$ *with* $q\geq 0,\alpha,\nu>0$, $0<\gamma\leq\nu+\alpha$ *and* $k>0$. *Then for* $x\in[\omega_1,\omega_2]$ *the integral operators are defined by:*

$$\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)=\int_{\omega_1}^{x}(\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\psi))^{\frac{\alpha}{k}};q)f(\psi)d(\zeta(\psi)) \tag{1.5}$$

and

$$\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_2^-}f\right)(x;q)=\int_{x}^{\omega_2}(\zeta(\psi)-\zeta(x))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(\psi)-\zeta(x))^{\frac{\alpha}{k}};q)f(\psi)d(\zeta(\psi)), \tag{1.6}$$

where $E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\psi;q)$ *is the modified Mittag-Leffler function defined by:*

$$E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\psi;q)=\sum_{n=0}^{\infty}\frac{\mathrm{B}_q(\epsilon+n\gamma,\sigma-\epsilon)}{\mathrm{B}(\epsilon,\sigma-\epsilon)}\frac{(\sigma)_{n\gamma}}{k\Gamma_k(\alpha n+\theta)}\frac{\psi^n}{(\rho)_{n\nu}}.$$

Remark 1.1. *The integral operators* (1.5) *and* (1.6) *reproduce several well-known integral operators exist in literature. For example, for* $k=1$, *the integral operators defined in [16] are reproduced, for* $k=1$ *and* $\zeta(x)=x$, *the integral operators defined in* (1.3) *and* (1.4) *are reproduced, for* $k=1,\zeta(x)=x$ *and* $q=0$, *the integral operators defined in [26] are reproduced, for* $k=1,\zeta(x)=x$ *and* $\nu=\rho=1$, *the integral operators defined in [25] are reproduced, for* $k=1,\zeta(x)=x,q=0$ *and* $\nu=\rho=1$, *the integral operators defined in [27] are reproduced, for* $k=1,\zeta(x)=x,q=0$ *and* $\gamma=\nu=\rho=1$, *the integral operators defined in [21] are reproduced, for* $k=1,\zeta(x)=\frac{x^\tau}{\tau},\tau>0$ *and* $\xi=q=0$, *the integral operators defined in [4] are reproduced, for* $k=1,\zeta(x)=\ln x$ *and* $\xi=q=0$, *the integral operators defined in [14] are reproduced, for* $\zeta(x)=\frac{x^{\tau+1}}{\tau+1}$ *and* $\xi=q=0$, *the integral operators defined in [28] are reproduced, for* $k=1,\zeta(x)=\frac{(x)^{\tau+\vartheta}}{\tau+\vartheta}$ *and* $\xi=q=0$, *the integral operators defined in [13] are reproduced, for* $\zeta(x)=\frac{(x-a)^\tau}{\tau},\tau>0$ *in* (1.5) *and* $\zeta(x)=-\frac{(b-x)^\tau}{\tau},\tau>0$ *in* (1.6) *with* $\xi=q=0$, *the integral operators defined in [9] are reproduced, for* $\zeta(x)=\frac{(x-a)^\tau}{\tau},\tau>0$ *in* (1.5) *and* $\zeta(x)=-\frac{(b-x)^\tau}{\tau},\tau>0$ *in* (1.6) *with* $k=1$ *and* $\xi=q=0$, *the integral operators* *defined in [10] are reproduced, for $\xi = q = 0$, the integral operators defined in [15] are reproduced, for $\xi = q = 0$ and $k = 1$, the integral operators defined in [14] are reproduced, for $\xi = q = 0$ and $\zeta(x) = x$, the integral operators defined in [18] are reproduced, for $\xi = q = 0, \zeta(x) = x$ and $k = 1$, the classical Riemann-Liouville integral operators are reproduced.*

In [31], Zhang et al. proved the following formulas for constant function, which we will use in our results:

$$\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}1\right)(x;q) = k(\zeta(x)-\zeta(\omega_1))^{\frac{\theta}{k}} E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta+k,\rho,k}(\xi(\zeta(x)-\zeta(\omega_1))^{\frac{\alpha}{k}};q) := \mathcal{F}^{\theta}_{\omega_1^+}(x;q) \tag{1.7}$$

and

$$\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_2^-}1\right)(x;q) = k(\zeta(\omega_2)-\zeta(x))^{\frac{\theta}{k}} E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta+k,\rho,k}(\xi(\zeta(\omega_2)-\zeta(x))^{\frac{\alpha}{k}};q) := \mathcal{F}^{\theta}_{\omega_2^-}(x;q). \tag{1.8}$$

In the upcoming section, we give Grüss type integral inequalities for generalized k-integral operators containing the Mittag-Leffler function. Also, we have given generalizations and extensions of Grüss type inequalities for different integral operators proved in [5, 8, 11, 12, 17, 22, 24, 29]. Moreover, some new fractional versions of Grüss type inequalities can be deduced for integral operators given in [31, Remark 1].

2. Main Results

First we prove the following inequality by utilizing the k-integral operator (1.5).

Theorem 2.1. *Let f, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be positive and integrable functions on $[0,\infty)$, such that*

$$\mathcal{Q}_1(x) \le f(x) \le \mathcal{Q}_2(x). \tag{2.1}$$

Then for k-integral operator (1.5), we have

$$\begin{aligned} &\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}f\right)(x;q) + \left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\right)(x;q) \\ &\ge \left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\right)(x;q) + \left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}f\right)(x;q). \end{aligned} \tag{2.2}$$

Proof. From inequality (2.1), for all $\psi, \phi \ge 0$, we can write:

$$(\mathcal{Q}_2(\psi) - f(\psi))(f(\phi) - \mathcal{Q}_1(\phi)) \ge 0. \tag{2.3}$$

Therefore, from (2.3), we have

$$\mathcal{Q}_2(\psi)f(\phi) + \mathcal{Q}_1(\phi)f(\psi) \ge \mathcal{Q}_1(\phi)\mathcal{Q}_2(\psi) + f(\psi)f(\phi). \tag{2.4}$$

Multiplying both sides of (2.4) with the following expression:

$$(\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1} E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi), \tag{2.5}$$

then integrating with respect to ψ on $[\omega_1, x]$, we have

$$\begin{aligned}
&\int_{\omega_1}^{x} (\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1} E_{\alpha,\theta,\rho,k}^{\epsilon,\nu,\gamma,\sigma}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)\mathcal{Q}_2(\psi)f(\phi)d\psi \\
&\quad + \int_{\omega_1}^{x} (\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1} E_{\alpha,\theta,\rho,k}^{\epsilon,\nu,\gamma,\sigma}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)\mathcal{Q}_1(\phi)f(\psi)d\psi \\
&\geq \int_{\omega_1}^{x} (\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1} E_{\alpha,\theta,\rho,k}^{\epsilon,\nu,\gamma,\sigma}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)\mathcal{Q}_1(\phi)\mathcal{Q}_2(\psi)d\psi \\
&\quad + \int_{\omega_1}^{x} (\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1} E_{\alpha,\theta,\rho,k}^{\epsilon,\nu,\gamma,\sigma}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)f(\psi)f(\phi)d\psi. \qquad (2.6)
\end{aligned}$$

By utilizing k-integral operator (1.5), we achieve

$$\begin{aligned}
&f(\phi)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_2\right)(x;q) + \mathcal{Q}_1(\phi)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q) \\
&\geq \mathcal{Q}_1(\phi)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_2\right)(x;q) + f(\phi)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q). \qquad (2.7)
\end{aligned}$$

Now multiplying both sides of (2.7) with following expression:

$$(\zeta(x)-\zeta(\phi))^{\frac{\vartheta}{k}-1} E_{\alpha,\vartheta,\rho,k}^{\epsilon,\nu,\gamma,\sigma}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi), \qquad (2.8)$$

then integrating with respect to ϕ on $[\omega_1, x]$, we have

$$\begin{aligned}
&\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_2\right)(x;q)\int_{\omega_1}^{x} (\zeta(x)-\zeta(\phi))^{\frac{\vartheta}{k}-1} E_{\alpha,\vartheta,\rho,k}^{\epsilon,\nu,\gamma,\sigma}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)f(\phi)d\phi \\
&\quad + \left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q)\int_{\omega_1}^{x} (\zeta(x)-\zeta(\phi))^{\frac{\vartheta}{k}-1} E_{\alpha,\vartheta,\rho,k}^{\epsilon,\nu,\gamma,\sigma}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)\mathcal{Q}_1(\phi)d\phi \\
&\geq \left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_2\right)(x;q)\int_{\omega_1}^{x} (\zeta(x)-\zeta(\phi))^{\frac{\vartheta}{k}-1} E_{\alpha,\vartheta,\rho,k}^{\epsilon,\nu,\gamma,\sigma}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)\mathcal{Q}_1(\phi)d\phi \\
&\quad + \left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q)\int_{\omega_1}^{x} (\zeta(x)-\zeta(\phi))^{\frac{\vartheta}{k}-1} E_{\alpha,\vartheta,\rho,k}^{\epsilon,\nu,\gamma,\sigma}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)f(\phi)d\phi. \qquad (2.9)
\end{aligned}$$

Again, by utilizing k-integral operator (1.5), the inequality (2.2) is achieved. □

Corollary 2.1. *Let f be a positive and integrable function on $[0,\infty)$. Also, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two real numbers such that*

$$\mathcal{M}_1 \leq f(x) \leq \mathcal{M}_2 \qquad (2.10)$$

for all $x \in [0,\infty)$. Then we have

$$\begin{aligned}
&\mathcal{M}_2\mathcal{F}_{\omega_1^+}^{\theta}(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q) + \mathcal{M}_1\mathcal{F}_{\omega_1^+}^{\vartheta}(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q) \\
&\geq \mathcal{M}_1\mathcal{M}_2\mathcal{F}_{\omega_1^+}^{\theta}(x;q)\mathcal{F}_{\omega_1^+}^{\vartheta}(x;q) + \left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q). \qquad (2.11)
\end{aligned}$$

Remark 2.1. *In Theorem 2.1, by using substitution of parameters several results are reproduced for different integral operators. For example, for $k=1$ and $\zeta(x)=x$, we achieve [5, Theorem 2.1], for $\xi=q=0$ and $\zeta(x)=\frac{(x-\omega_1)^{\tau}}{\tau}$, we achieve [8, Theorem 2.1], for $\xi=q=\omega_1=0$ and $k=1$, we achieve [11, Theorem 2.11], for $\xi=q=\omega_1=0, k=1$ and $\zeta(x)=\frac{x^{\tau+1}}{\tau+1}$, we achieve [12, Theorem 5], for $\xi=q=0$ and $\zeta(x)=\frac{x^{\tau+1}}{\tau+1}$, we achieve [17, Theorem 2.1], for $\xi=q=\omega_1=0$, we* *achieve [22, Theorem 2.1], for* $\xi = q = \omega_1 = 0, k = 1$ *and* $\zeta(x) = \frac{x^\tau}{\tau}$, *we achieve [24, Theorem 2.1], for* $\xi = q = \omega_1 = 0, k = 1$ *and* $\zeta(x) = x$, *we achieve [29, Theorem 2].*

Theorem 2.2. *Let* f, $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *be positive and integrable functions on* $[0, \infty)$ *and satisfying* (2.1). *Also, let* g, $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *be positive and integrable functions on* $[0, \infty)$, *such that*

$$\mathcal{R}_1(x) \leq g(x) \leq \mathcal{R}_2(x). \tag{2.12}$$

Then for k-integral operator (1.5), *we have*

(i)

$$\begin{aligned}&\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{R}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\\&\geq\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{R}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}g\right)(x;q);\end{aligned} \tag{2.13}$$

(ii)

$$\begin{aligned}&\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{R}_2\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\\&\geq\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_2\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}f\right)(x;q);\end{aligned} \tag{2.14}$$

(iii)

$$\begin{aligned}&\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_2\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\\&\geq\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_2\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}f\right)(x;q);\end{aligned} \tag{2.15}$$

(iv)

$$\begin{aligned}&\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_1\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\\&\geq\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}f\right)(x;q).\end{aligned} \tag{2.16}$$

Proof. (i) From inequalities (2.1) and (2.12), we can write:

$$(\mathcal{Q}_2(\psi) - f(\psi))(g(\phi) - \mathcal{R}_1(\phi)) \geq 0. \tag{2.17}$$

Therefore, from (2.17), we have

$$\mathcal{Q}_2(\psi)g(\phi) + \mathcal{R}_1(\phi)f(\psi) \geq \mathcal{Q}_1(\phi)\mathcal{Q}_2(\psi) + f(\psi)g(\phi). \tag{2.18}$$

Multiplying both sides of (2.18) with (2.5) and (2.8), then integrating with respect to ψ and ϕ on $[\omega_1, x]$, after this by utilizing k-integral operator (1.5), the inequality (i) of (2.13) is achieved.

To prove the inequalities $(ii) - (iv)$ of (2.13), we utilize the following inequalities, respectively:

(ii) $(\mathcal{R}_2(\psi) - g(\psi))(f(\phi) - \mathcal{Q}_1(\phi)) \geq 0$;
(iii) $(\mathcal{Q}_2(\psi) - f(\psi))(g(\phi) - \mathcal{R}_2(\phi)) \leq 0$;
(iv) $(\mathcal{Q}_1(\psi) - f(\psi))(g(\phi) - \mathcal{R}_1(\phi)) \leq 0$.

□

Corollary 2.2. *Let f and g be two positive and integrable functions on $[0,\infty)$. Also, let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1$ and $\mathcal{N}_2$ be four real numbers satisfying (2.10) and the following inequalities:*

$$\mathcal{N}_1 \leq g(x) \leq \mathcal{N}_2 \tag{2.19}$$

for all $x \in [0,\infty)$. Then we have

(i)

$$\begin{aligned}&\mathcal{N}_1 \mathcal{F}^{\vartheta}_{\omega_1^+}(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q) + \mathcal{M}_2\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}g\right)(x;q)\\ &\geq \mathcal{N}_1\mathcal{M}_2\mathcal{F}^{\vartheta}_{\omega_1^+}(x;q)\mathcal{F}^{\theta}_{\omega_1^+}(x;q) + \left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}g\right)(x;q);\end{aligned} \tag{2.20}$$

(ii)

$$\begin{aligned}&\mathcal{M}_1 \mathcal{F}^{\vartheta}_{\omega_1^+}(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q) + \mathcal{N}_2\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}f\right)(x;q)\\ &\geq \mathcal{M}_1\mathcal{N}_2\mathcal{F}^{\vartheta}_{\omega_1^+}(x;q)\mathcal{F}^{\theta}_{\omega_1^+}(x;q) + \left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q);\end{aligned} \tag{2.21}$$

(iii)

$$\begin{aligned}&\mathcal{M}_2 \mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}g\right)(x;q) + \mathcal{N}_2\mathcal{F}^{\vartheta}_{\omega_1^+}(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\\ &\geq \mathcal{M}_2\mathcal{N}_2\mathcal{F}^{\vartheta}_{\omega_1^+}(x;q)\mathcal{F}^{\theta}_{\omega_1^+}(x;q) + \left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}g\right)(x;q);\end{aligned} \tag{2.22}$$

(iv)

$$\begin{aligned}&\mathcal{M}_1 \mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}g\right)(x;q) + \mathcal{N}_1\mathcal{F}^{\vartheta}_{\omega_1^+}(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\\ &\geq \mathcal{M}_1\mathcal{N}_1\mathcal{F}^{\vartheta}_{\omega_1^+}(x;q)\mathcal{F}^{\theta}_{\omega_1^+}(x;q) + \left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\vartheta,\rho,\xi,\omega_1^+}g\right)(x;q).\end{aligned} \tag{2.23}$$

Remark 2.2. *In Theorem 2.2, by using substitution of parameters several results are reproduced for different integral operators. For example, for $k=1$ and $\zeta(x)=x$, we achieve [5, Theorem 2.2], for $\xi=q=0$ and $\zeta(x)=\frac{(x-\omega_1)^\tau}{\tau}$, we achieve [8, Theorem 2.5], for $\xi=q=\omega_1=0$ and $k=1$, we achieve [11, Theorem 2.15], for $\xi=q=\omega_1=0, k=1$ and $\zeta(x)=\frac{x^{\tau+1}}{\tau+1}$, we achieve [12, Theorem 6], for $\xi=q=0$ and $\zeta(x)=\frac{x^{\tau+1}}{\tau+1}$, we achieve [17, Theorem 2.5], for $\xi=q=\omega_1=0$, we achieve [22, Theorem 2.5], for $\xi=q=\omega_1=0, k=1$ and $\zeta(x)=\frac{x^\tau}{\tau}$, we achieve [24, Theorem 2.2], for $\xi=q=\omega_1=0, k=1$ and $\zeta(x)=x$, we achieve [29, Theorem 5].*

Theorem 2.3. *Let f, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be positive and integrable functions on $[0,\infty)$, satisfying (2.1). Then for k-integral operator (1.5), we have*

$$\begin{aligned}&\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f^2\right)(x;q) - \left[\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\right]^2\\ &= \left[\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q) - \left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\right]\\ &\quad\times\left[\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q) - \left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\right)(x;q)\right]\\ &\quad - \mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left[\left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q) - \left({}^k_\zeta\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\right]\end{aligned}$$

$$\times\left[\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q)-\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_1\right)(x;q)\right]$$
$$+\mathcal{F}_{\omega_1^+}^{\theta}(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_1f\right)(x;q)-\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_1\right)(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q)$$
$$+\mathcal{F}_{\omega_1^+}^{\theta}(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_2f\right)(x;q)-\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_2\right)(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q)$$
$$-\mathcal{F}_{\omega_1^+}^{\theta}(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_1\mathcal{Q}_2\right)(x;q)+\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_1\right)(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{Q}_2\right)(x;q). \quad (2.24)$$

Proof. For any $\psi,\phi>0$, we have

$$\begin{aligned}&(\mathcal{Q}_2(\phi)-f(\phi))(f(\psi)-\mathcal{Q}_1(\psi))+(\mathcal{Q}_2(\psi)-f(\psi))(f(\phi)-\mathcal{Q}_1(\phi))\\&\quad-(\mathcal{Q}_2(\psi)-f(\psi))(f(\psi)-\mathcal{Q}_1(\psi))-(\mathcal{Q}_2(\phi)-f(\phi))(f(\phi)-\mathcal{Q}_1(\phi))\\&=f^2(\psi)+f^2(\phi)-2f(\psi)f(\phi)+\mathcal{Q}_2(\phi)f(\psi)+\mathcal{Q}_1(\psi)f(\phi)\\&\quad-\mathcal{Q}_1(\psi)\mathcal{Q}_2(\phi)+\mathcal{Q}_2(\psi)f(\phi)+\mathcal{Q}_1(\phi)f(\psi)-\mathcal{Q}_1(\phi)\mathcal{Q}_2(\phi)\\&\quad-\mathcal{Q}_2(\psi)f(\psi)+\mathcal{Q}_1(\psi)\mathcal{Q}_2(\psi)-\mathcal{Q}_1(\psi)f(\psi)-\mathcal{Q}_2(\phi)f(\phi)\\&\quad+\mathcal{Q}_1(\phi)\mathcal{Q}_2(\phi)-\mathcal{Q}_1(\phi)f(\phi).\end{aligned} \quad (2.25)$$

Multiplying both sides of (2.25) with (2.5) and the following expression:

$$(\zeta(x)-\zeta(\phi))^{\frac{\theta}{k}-1}E_{\alpha,\theta,\rho,k}^{\epsilon,\nu,\gamma,\sigma}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi), \quad (2.26)$$

then integrating with respect to ψ and ϕ on $[\omega_1,x]$, after this by utilizing k-integral operator (1.5), the required identity (2.24) is achieved. □

Remark 2.3. *In Theorem 2.3, by using substitution of parameters several results are reproduced for different integral operators. For example, for $k=1$ and $\zeta(x)=x$, we achieve [5, Theorem 2.3], for $\xi=q=0$ and $\zeta(x)=\frac{(x-\omega_1)^\tau}{\tau}$, we achieve [8, Theorem 2.7], for $\xi=q=\omega_1=0$ and $k=1$, we achieve [11, Lemma 2.19], for $\xi=q=\omega_1=0, k=1$ and $\zeta(x)=\frac{x^{\tau+1}}{\tau+1}$, we achieve [12, Lemma 2], for $\xi=q=0$ and $\zeta(x)=\frac{x^{\tau+1}}{\tau+1}$, we achieve [17, Theorem 2.7], for $\xi=q=\omega_1=0$, we achieve [22, Lemma 2.9], for $\xi=q=\omega_1=0, k=1$ and $\zeta(x)=\frac{x^\tau}{\tau}$, we achieve [24, Theorem 2.3], for $\xi=q=\omega_1=0, k=1$ and $\zeta(x)=x$, we achieve [29, Lemma 7].*

Theorem 2.4. *Let $f,g,\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{R}_1$ and $\mathcal{R}_2$ be positive and integrable functions on $[0,\infty)$ satisfying (2.1) and (2.12). Then for k-integral operator (1.5), we have*

$$\begin{aligned}&\left|\mathcal{F}_{\omega_1^+}^{\theta}(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}fg\right)(x;q)-\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}f\right)(x;q)\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}g\right)(x;q)\right|\\&\leq\sqrt{\mathcal{H}(f,\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{H}(g,\mathcal{R}_1,\mathcal{R}_2)},\end{aligned} \quad (2.27)$$

where

$$\begin{aligned}\mathcal{H}(\mathcal{U},\mathcal{V},\mathcal{W})&=\left[\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{W}\right)(x;q)-\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{U}\right)(x;q)\right]\\&\times\left[\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{U}\right)(x;q)-\left({}_{\zeta}^{k}\mathcal{J}_{\alpha,\theta,\rho,\xi,\omega_1^+}^{\epsilon,\nu,\gamma,\sigma}\mathcal{V}\right)(x;q)\right]\end{aligned}$$

$$+\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{U}\mathcal{V}\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{U}\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{V}\right)(x;q)$$
$$+\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{U}\mathcal{W}\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{U}\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{W}\right)(x;q)$$
$$-\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{V}\mathcal{W}\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{V}\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{W}\right)(x;q).$$

Proof. As we know f and g are two integrable functions on $[0,\infty)$ and satisfying (2.1) and (2.12). Therefore, we can write

$$[f(\psi)-f(\phi)]\,[g(\psi)-g(\phi)]=f(\psi)g(\psi)+f(\phi)g(\phi)-f(\psi)g(\phi)-f(\phi)g(\psi). \tag{2.28}$$

Multiplying both sides of (2.28) with $\frac{1}{2}$, (2.5) and (2.26), then integrating with respect to ψ and ϕ on $[\omega_1,x]$, we have

$$\begin{aligned}&\Bigg(\frac{1}{2}\int_{\omega_1}^{x}\int_{\omega_1}^{x}(\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)\,(\zeta(x)-\zeta(\phi))^{\frac{\theta}{k}-1}\\&\quad\times E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)\,[f(\psi)-f(\phi)]\,[g(\psi)-g(\phi)]\,d\psi d\phi\Bigg)\\&=\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}fg\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q).\end{aligned} \tag{2.29}$$

Now by utilizing Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\Bigg(\frac{1}{2}\int_{\omega_1}^{x}\int_{\omega_1}^{x}(\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)\,(\zeta(x)-\zeta(\phi))^{\frac{\theta}{k}-1}\\&\quad\times E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)\,[f(\psi)-f(\phi)]\,[g(\psi)-g(\phi)]\,d\psi d\phi\Bigg)^2\\&\leq\frac{1}{2}\int_{\omega_1}^{x}\int_{\omega_1}^{x}(\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)\\&\quad\times(\zeta(x)-\zeta(\phi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)\,[f(\psi)-f(\phi)]^2\,d\psi d\phi\\&\quad\times\frac{1}{2}\int_{\omega_1}^{x}\int_{\omega_1}^{x}(\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)\\&\quad\times(\zeta(x)-\zeta(\phi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)\,[g(\psi)-g(\phi)]^2\,d\psi d\phi.\end{aligned} \tag{2.30}$$

From (2.30), by utilizing k-integral operator (1.5), we have

$$\begin{aligned}&\frac{1}{2}\int_{\omega_1}^{x}\int_{\omega_1}^{x}(\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)\\&\quad\times(\zeta(x)-\zeta(\phi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)\,[f(\psi)-f(\phi)]^2\,d\psi d\phi\\&=\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f^2\right)(x;q)-\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\right]^2.\end{aligned} \tag{2.31}$$

Similarly, we have

$$\frac{1}{2}\int_{\omega_1}^{x}\int_{\omega_1}^{x}(\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)$$
$$\times(\zeta(x)-\zeta(\phi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)\left[g(\psi)-g(\phi)\right]^2 d\psi d\phi$$
$$=\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g^2\right)(x;q)-\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\right]^2. \tag{2.32}$$

Utilizing identities (2.31) and (2.32) in (2.30), we achieve

$$\left(\frac{1}{2}\int_{\omega_1}^{x}\int_{\omega_1}^{x}(\zeta(x)-\zeta(\psi))^{\frac{\theta}{k}-1}E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\psi))^{\alpha};q)\zeta'(\psi)(\zeta(x)-\zeta(\phi))^{\frac{\theta}{k}-1}\right.$$
$$\left.\times E^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,k}(\xi(\zeta(x)-\zeta(\phi))^{\alpha};q)\zeta'(\phi)\times[f(\psi)-f(\phi)]\,[g(\psi)-g(\phi)]\,d\psi d\phi\right)^2$$
$$\le\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f^2\right)(x;q)-\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\right]^2$$
$$\times\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g^2\right)(x;q)-\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\right]^2. \tag{2.33}$$

From identity (2.29) together with the inequality (2.33), we achieve

$$\left(\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}fg\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\right)^2$$
$$\le\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f^2\right)(x;q)-\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\right]^2$$
$$\times\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g^2\right)(x;q)-\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\right]^2. \tag{2.34}$$

As we know $(\mathcal{Q}_2(x)-f(x))(f(x)-\mathcal{Q}_1(x))\ge 0$ and $(\mathcal{R}_2(x)-g(x))(g(x)-\mathcal{R}_1(x))\ge 0$ holds for $x\in[0,\infty)$. Therefore, we have the following inequalities:

$$\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}(\mathcal{Q}_2-f)(f-\mathcal{Q}_1)\right)(x;q)\ge 0$$

and

$$\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}(\mathcal{R}_2-g)(g-\mathcal{R}_1)\right)(x;q)\ge 0.$$

By utilizing Theorem 2.3, we have

$$\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f^2\right)(x;q)-\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\right]^2$$
$$\le\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)\right]$$
$$\times\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\right)(x;q)\right]$$
$$+\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_1f\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)$$
$$+\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_2f\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}f\right)(x;q)$$
$$-\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\mathcal{Q}_2\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{Q}_2\right)(x;q)$$
$$=\mathcal{H}(f,\mathcal{Q}_1,\mathcal{Q}_2). \tag{2.35}$$

Similarly, we have

$$\begin{aligned}
&\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g^2\right)(x;q)-\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\right]^2\\
&\le\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_2\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\right]\\
&\quad\times\left[\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_1\right)(x;q)\right]\\
&\quad+\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_1 f\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\\
&\quad+\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_2 g\right)(x;q)-\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_2\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}g\right)(x;q)\\
&\quad-\mathcal{F}^{\theta}_{\omega_1^+}(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_1\mathcal{R}_2\right)(x;q)+\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_1\right)(x;q)\left({}^{k}_{\zeta}\mathcal{J}^{\epsilon,\nu,\gamma,\sigma}_{\alpha,\theta,\rho,\xi,\omega_1^+}\mathcal{R}_2\right)(x;q)\\
&=\mathcal{H}(g,\mathcal{R}_1,\mathcal{R}_2).
\end{aligned}\tag{2.36}$$

From identities (2.35) and (2.36) together with inequality (2.34), the inequality (2.27) is achieved. □

Remark 2.4. *In Theorem 2.4, by using substitution of parameters several results are reproduced for different integral operators. For example, for $k=1$ and $\zeta(x)=x$, we achieve [5, Theorem 2.4], for $\xi=q=0$ and $\zeta(x)=\frac{(x-\omega_1)^{\tau}}{\tau}$, we achieve [8, Theorem 2.10], for $\xi=q=\omega_1=0$ and $k=1$, we achieve [11, Theorem 2.23], for $\xi=q=\omega_1=0$, $k=1$ and $\zeta(x)=\frac{x^{\tau+1}}{\tau+1}$, we achieve [12, Theorem 7], for $\xi=q=0$ and $\zeta(x)=\frac{x^{\tau+1}}{\tau+1}$, we achieve [17, Theorem 2.10], for $\xi=q=\omega_1=0$, we achieve [22, Theorem 2.13], for $\xi=q=\omega_1=0, k=1$ and $\zeta(x)=\frac{x^{\tau}}{\tau}$, we achieve [24, Theorem 2.4], for $\xi=q=\omega_1=0$, $k=1$ and $\zeta(x)=x$, we achieve [29, Theorem 9].*

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