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On Frames in Hilbert Modules Over Locally C^*-Algebras

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Abstract. Frame is a fundamental notion in the study of vector spaces; they offer redundancy and flexibility, which favor their application in various fields of mathematics. This article aims to collect important results of frames in Hilbert pro-C^*-modules: Frame, $*$-frame, $*$-K-frame, g-frame, $*$-g-frame, $*$-K-g-frame, operator frame, $*$-operator frame, $*$-K-operator frames. We also prove some new notions.

1. Introduction

It is known that the bases in vector spaces represent every item uniquely and conveniently. However, the linear independence property of a basis, which allows each vector to be represented only as a linear combination, is very limited for practical problems. A frame is more general than a basis: one can represent each element of the vector space by a frame, but this representation may not necessarily be unique.

Although Duffin and Schaefer introduced the concept of Frames [3] in 1952, it was only in the last ten years that the theory of frames was developed extensively by Grossman and Meyer [6]. n the 1970s and 1980s, the study of frames was extended to the context of Hilbert spaces, where it was used to understand the representation and analysis of signals and images; this led to the development of the theory of frames, a well-established area of mathematics with a rich body of research and numerous applications in signal processing, image analysis, and numerical analysis.

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In recent years, the study of frames has continued to evolve and expand, and it has become an essential tool for understanding the representation and analysis of signals, images, and functions in a wide range of areas, including mathematics, physics, engineering, and computer science.

In 2008, Joita [11] extended the theory of frame in Hilbert modules over pro-C^*-algebras, which led to the initiation of new notions of frames in Hilbert pro-C^*-modules.

In this work, we gather all the crucial results concerning frames in Hilbert pro-C^*-modules and prove some new results.

2. Preliminaries

Pro-C^*-algebra is considered as a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous C^*-seminorms in the sens that a net $\{a_\alpha\}$ converges to 0 if and only if $p_\alpha(a_\alpha)$ converges to 0 for all continuous C^*-seminorm p_α on $\mathcal{A}$, and we have:

1) $p_\alpha(\gamma\beta) \leq p_\alpha(\gamma)p_\alpha(\beta)$
2) $p_\alpha(\gamma^*\gamma) = p_\alpha(\gamma)^2$

for all $\gamma, \beta \in \mathcal{A}$ $sptr(\gamma)$ denotes the spectrum of γ such that: $sptr(\gamma) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - \gamma$ is not invertible $\}$ for all $\gamma \in \mathcal{A}$. Where $\mathcal{A}$ is unital pro-C^*-algebra with unite $1_{\mathcal{A}}$.

We denote by $Se(\mathcal{A})$ the set of all continunous C^*-seminorms on $\mathcal{A}$. $\mathcal{A}^+$ designates the set of all positive elements of $\mathcal{A}$, and it is a closed convex C^*-seminorms on $\mathcal{A}$.

$\mathcal{H}_{\mathcal{A}}$ denotes the set of all sequences $(\gamma_n)_n$ with $\gamma_n \in \mathcal{A}$ such that $\sum_n \gamma_n^*\gamma_n$ converges in $\mathcal{A}$.

Example 2.1. *Every C^*-algebra is a locally C^*-algebra.*

Definition 2.1. *[15] A pre-Hilbert module over locally C^*-algebra $\mathcal{A}$, is a complex vector space $\mathcal{X}$ which is also a left $\mathcal{A}$-module compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle$ $\mathcal{X} \times \mathcal{X} \to \mathcal{A}$ which is $\mathbb{C}$-and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:*

1) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ *for every* $\xi, \eta \in \mathcal{X}$
2) $\langle \xi, \xi \rangle \geq 0$ *for every* $\xi \in \mathcal{X}$
3) $\langle \xi, \xi \rangle = 0$ *if and only if* $\xi = 0$

for every $\xi, \eta \in \mathcal{X}$. We say $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module (or Hilbert pro-C^*-module over $\mathcal{A}$) if it is complete with respect to the topology determined by the family of seminorms

$$\bar{p}_{\mathcal{X}}(\xi) = \sqrt{p_\alpha(\langle \xi, \xi \rangle)} \quad \xi \in \mathcal{X}, p \in S(\mathcal{A})$$

In all the rest $\mathcal{A}$ is a pro-C^*-algebra, $\mathcal{X}$ and $\mathcal{Y}$ are two Hilbert $\mathcal{A}$-modules and I and J are countable index sets.

We call an operator from $\mathcal{X}$ to $\mathcal{Y}$ every bounded $\mathcal{A}$-module map from $\mathcal{X}$ to $\mathcal{Y}$. The set of all operator from $\mathcal{X}$ to $\mathcal{Y}$ by is denoted by $Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$.

Proposition 2.1. *[9] Let $\mathcal{A}$ be a unital pro-C^*-algebra with an identity $1_{\mathcal{A}}$. Then for any $p_\alpha \in Se(\mathcal{A})$, we have:*

(1) $p_\alpha(\gamma) = p_\alpha(\gamma^*)$ *for all* $\gamma \in A$
(2) $p_\alpha(1_{\mathcal{A}}) = 1$
(3) *If* $1_{\mathcal{A}} \leq \beta$, *then* β *is invertible and* $\beta^{-1} \leq 1_{\mathcal{A}}$
(4) *If* $\gamma, \beta \in \mathcal{A}^+$ *are invertible and* $0 \leq \gamma \leq \beta$, *then* $0 \leq \beta^{-1} \leq \gamma^{-1}$
(5) *If* $\gamma, \beta \in \mathcal{A}^+$ *and* $\gamma^2 \leq \beta^2$, *then* $0 \leq \gamma \leq \beta$

Proposition 2.2. *[2]. Let $\mathcal{X}$ be a Hilbert module over pro-C^*-algebra $\mathcal{A}$ and T be an invertible element in $Hom^*_{\mathcal{A}}(\mathcal{X})$ such that both are uniformly bounded. Then for each $\xi \in \mathcal{X}$,*

$$\left\|T^{-1}\right\|_\infty^{-2} \langle \xi, \xi\rangle \leq \langle T\xi, T\xi\rangle \leq \|T\|_\infty^2 \langle \xi, \xi\rangle.$$

3. Frames

Definition 3.1. *[11] A sequence $\{\xi_i\}_i$ in $M(\mathcal{X})$ is a standard frame of multipliers in $\mathcal{X}$ if for each $\xi \in \mathcal{X}, \sum_i \langle \xi, \xi_i\rangle_{M(\mathcal{X})} \langle \xi_i, \xi\rangle_{M(\mathcal{X})}$ converges in $\mathcal{A}$, and there are two positive constants C and D such that*

$$C\langle \xi, \xi\rangle_{\mathcal{X}} \leqslant \sum_i \langle \xi, \xi_i\rangle_{M(\mathcal{X})} \langle \xi_i, \xi\rangle_{M(\mathcal{X})} \leqslant D\langle \xi, \xi\rangle_{\mathcal{X}}$$

for all $\xi \in \mathcal{X}$. If $D = C = 1$ we say that $\{\xi_i\}_i$ is a standard normalized frame of multipliers.

Particularly if the right inequality

$$\sum_i \langle \xi, \xi_i\rangle_{M(\mathcal{X})} \langle \xi_i, \xi\rangle_{M(\mathcal{X})} \leqslant D\langle \xi, \xi\rangle_{\mathcal{X}} \quad \forall \xi \in \mathcal{X}$$

holds true, we call $\{\xi_i\}_{i\in I}$ a Bessel sequence.

Remark 3.1. *[11] Let $\{\eta_i\}_i$ be a sequence in $M(\mathcal{X})$. Then $\{\eta_i\}_i$ is a standard normalized frame of multipliers in $\mathcal{X}$ if and only if $\left\{\left(\pi_p^{\mathcal{A},\mathcal{X}}\right)_*(\eta_i)\right\}_i$ is a standard normalized frame of multipliers in $\mathcal{X}_p$ for each $p_\alpha \in Se(\mathcal{A})$.*

Example 3.1. *[11] For any pro-C^*-algebra, $\{u_i\}_i$ is a standard normalized frame of multipliers in $\mathcal{H}_{\mathcal{A}}$. Indeed, if $(\alpha_i)_i \in \mathcal{H}_{\mathcal{A}}$ then, since*

$$\left\langle (\alpha_j)_j, u_i\right\rangle_{M(\mathcal{H}_{\mathcal{A}})} \left\langle u_i, (\alpha_j)_j\right\rangle_{M(\mathcal{H}_{\mathcal{A}})} = \alpha_i^* \alpha_i$$

for each positive integer i, we have

$$\sum_i \left\langle (\alpha_j)_j, u_i\right\rangle_{M(\mathcal{H}_{\mathcal{A}})} \left\langle u_i, (\alpha_j)_j\right\rangle_{M(\mathcal{H}_{\mathcal{A}})} = \sum_i \alpha_i^* \alpha_i = \left\langle (\alpha_j)_j, (\alpha_j)_j\right\rangle_{\mathcal{H}_{\mathcal{A}}}$$

and so $\{u_i\}_i$ is a standard normalized frame of multipliers in $\mathcal{H}_{\mathcal{A}}$.

Proposition 3.1. *[11] Any countably generated Hilbert $\mathcal{A}$-module $\mathcal{X}$ in $M(\mathcal{X})$ admits a standard normalized frame of multipliers.*

Theorem 3.1. *[11][The reconstruction formula]*

Let $\mathcal{X}$ be a countably generated Hilbert $\mathcal{A}$-module in $M(\mathcal{X})$ and let $\{\eta_i\}_i$ be a sequence in $M(\mathcal{X})$. Then $\{\eta_i\}_i$ is a standard normalized frame of multipliers if and only if for all $\xi \in \mathcal{X}$, $\sum_i \eta_i \cdot \langle \eta_i, \xi \rangle_{M(\mathcal{X})}$ converges in $\mathcal{X}$ and moreover,

$$\xi = \sum_i \eta_i \cdot \langle \eta_i, \xi \rangle_{M(\mathcal{X})}.$$

Proof. By Remark 3.1 and [13, Theorem 3.4], $\{h_n\}_n$ is a standard normalized frame of multipliers in $\mathcal{X}$ if and only if $\sum_n \left(\pi_p^{A,\mathcal{X}}\right)_* (h_n) \cdot \left\langle \left(\pi_p^{A,\mathcal{X}}\right)_* (h_n), \sigma_p^{\mathcal{X}}(\xi) \right\rangle_{M(\mathcal{X})}$ converges in $\mathcal{X}_p$ for all $\xi \in \mathcal{X}$ and for each $p \in S(A)$, and moreover,

$$\sigma_p^{\mathcal{X}}(\xi) = \sum_n \left(\pi_p^{A,\mathcal{X}}\right)_* (h_n) \cdot \left\langle \left(\pi_p^{A,\mathcal{X}}\right)_* (h_n), \sigma_p^{\mathcal{X}}(\xi) \right\rangle_{M(\mathcal{X})}$$

From this fact and taking into account that

$$\begin{aligned}\bar{p}_{\mathcal{X}}\left(\xi - \sum_{k=1}^{n} h_k \cdot \langle h_k, \xi \rangle_{M(\mathcal{X})}\right) &= \left\| \sigma_p^{\mathcal{X}}(\xi) - \sigma_p^{\mathcal{X}}\left(\sum_{k=1}^{n} h_k \cdot \langle h_k, \xi \rangle_{M(\mathcal{X})}\right) \right\|_{\mathcal{X}_p} \\ &= \left\| \sigma_p^{\mathcal{X}}(\xi) - \sum_{k=1}^{n} \left(\pi_p^{A,\mathcal{X}}\right)_* (h_k) \cdot \left\langle \left(\pi_p^{A,\mathcal{X}}\right)_* (h_k), \sigma_p^{\mathcal{X}}(\xi) \right\rangle_{M(\mathcal{X})} \right\|_{\mathcal{X}_p}\end{aligned}$$

for all $\xi \in \mathcal{X}$, for all $p \in S(A)$ and for all positive integers n, we deduce that $\{h_n\}_n$ is a standard normalized frame of multipliers in $\mathcal{X}$ if and only if $\sum_n h_n \cdot \langle h_n, \xi \rangle_{M(\mathcal{X})}$ converges in $\mathcal{X}$ for all $\xi \in \mathcal{X}$, and moreover, $\xi = \sum_n h_n \cdot \langle h_n, \xi \rangle_{M(\mathcal{X})}$ for all $\xi \in \mathcal{X}$. □

Remark 3.2. *[11] If $\{h_n\}_n$ is a standard normalized frame of multipliers in $\mathcal{X}$, then $\sum_n (h_n \circ h_n^*)(\xi) = \xi$ for all $\xi \in E$, since $(h_n \circ h_n^*)(\xi) = h_n \cdot \langle h_n, \xi \rangle_{M(\mathcal{X})}$ for each positive integer n. Therefore, $\{h_n\}_n$ is a standard normalized frame of multipliers in $\mathcal{X}$ if and only if $\sum_n (h_n \circ h_n^*)(\xi)$ converges in $\mathcal{X}$ for each $\xi \in \mathcal{X}$ and moreover, $\sum_n (h_n \circ h_n^*)(\xi) = \xi$.*

Corollary 3.1. *[11] If The bounded part of $\mathcal{X}$ noted $b(\mathcal{X})$ admits a standard normalized frame of multipliers, then $\mathcal{X}$ admits a standard normalized frame of multipliers.*

Proof. We will show that if $\{h_n\}_n$ is a standard normalized frame of multipliers in $b(\mathcal{X})$, then $\left\{\tilde{h}_n\right\}_n$, where $\tilde{h}_n$ is the extension of h_n to an element in $M(\mathcal{X})$, is a standard normalized frame of multipliers in $\mathcal{X}$.

Let $\xi \in \mathcal{X}, p \in Se(\mathcal{A})$ and $\varepsilon > 0$. Since $b(\mathcal{X})$ is dense in $\mathcal{X}$, there is $\xi_0 \in b(\mathcal{X})$ such that $\bar{p}_{\mathcal{X}}(\xi - \xi_0) \leqslant \varepsilon/3$. Since $\{h_n\}_n$ is a standard normalized frame of multipliers in $b(\mathcal{X})$, there is n_0 such that

$$\left\| \xi_0 - \sum_{k=1}^{n} (h_k \circ h_k^*)(\xi_0) \right\|_\infty \leqslant \varepsilon/3$$

for all n with $n \geqslant n_0$. Then

$$\begin{aligned}
&\bar{p}_{\mathcal{X}}\left(\xi-\sum_{k=1}^{n}\left(\tilde{h}_k \circ \tilde{h}_k^*\right)(\xi)\right) \\
&\leqslant \bar{p}_{\mathcal{X}}\left(\xi-\xi_0\right)+\bar{p}_{\mathcal{X}}\left(\xi_0-\sum_{k=1}^{n}\left(h_k \circ h_k^*\right)\left(\xi_0\right)\right)+\bar{p}_{\mathcal{X}}\left(\sum_{k=1}^{n}\left(\tilde{h}_k \circ \tilde{h}_k^*\right)\left(\xi-\xi_0\right)\right) \\
&\leqslant \varepsilon/3+\left\|\xi_0-\sum_{k=1}^{n}\left(h_k \circ h_k^*\right)\left(\xi_0\right)\right\|_{\infty}+\tilde{p}_{L(\mathcal{X})}\left(\sum_{k=1}^{n} \tilde{h}_k \circ \tilde{h}_k^*\right) \bar{p}_{\mathcal{X}}\left(\xi-\xi_0\right) \\
&\leqslant \varepsilon/3\left(2+\left\|\sum_{k=1}^{n} \tilde{h}_k \circ \tilde{h}_k^*\right\|_{\infty}\right) \\
&=\varepsilon/3\left(2+\left\|\sum_{k=1}^{n} h_k \circ h_k^*\right\|\right) \\
&\leqslant \varepsilon
\end{aligned}$$

for all n with $n \geqslant n_0$. This shows that $\sum_n \left(\tilde{h}_n \circ \tilde{h}_n^*\right)(\xi)$ converges to ξ in $\mathcal{X}$ for each $\xi \in \mathcal{X}$ and so $\left\{\tilde{h_n}\right\}_n$ is a standard normalized frame of multipliers in $\mathcal{X}$

If $\{h_n\}_n$ is a standard frame of multipliers in $\mathcal{X}$, then $\sum_n \langle \xi, h_n\rangle_{M(\mathcal{X})} \langle h_n, \xi\rangle_{M(\mathcal{X})}$ converges in $\mathcal{X}$ for all $\xi \in \mathcal{X}$. From this fact and taking into account that $\langle h_n, \xi\rangle_{M(\mathcal{X})} \in A$ for all positive integers n, we conclude that $\left(\langle h_n, \xi\rangle_{M(\mathcal{X})}\right)_n \in H_A$. Thus we can define a linear map $\theta : \mathcal{X} \to H_A$ by

$$\theta(\xi)=\left(\langle h_n, \xi\rangle_{M(\mathcal{X})}\right)_n$$

Moreover, θ is a continuous module morphism, since

$$\theta(\xi a)=\left(\langle h_{\mathrm{n}}, \xi a\rangle_{M(\mathcal{X})}\right)_{\mathrm{n}}=\left(\langle h_{\mathrm{n}}, \xi\rangle_{M(\mathcal{X})} a\right)_{\mathrm{n}}=\theta(\xi) a$$

for all $\xi \in \mathcal{X}$ and for all $a \in A$ and

$$\bar{p}_{H_A}(\theta(\xi))^2=p\left(\sum_n \langle \xi, h_n\rangle_{M(\mathcal{X})} \langle h_n, \xi\rangle_{M(\mathcal{X})}\right) \leqslant C \bar{p}_{\mathcal{X}}(\xi)^2$$

for all $\xi \in \mathcal{X}$ and for all $p \in S(A)$. □

3.1. *g*-frame.

Definition 3.2. *[7] A sequence $\Gamma=\left\{\Gamma_i \in Hom_{\mathcal{A}}^*\left(\mathcal{X}, \mathcal{Y}_i\right)\right\}_{i \in I}$ is called a g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ if there are two positive constants C and D such that for every $\xi \in \mathcal{X}$,*

$$C\langle \xi, \xi\rangle \leq \sum_{i \in I}\langle \Gamma_i \xi, \Gamma_i \xi\rangle \leq D\langle \xi, \xi\rangle .$$

The constants C and D are called g-frame bounds for Γ. The g-frame is called tight if $C = D$ and called a Parseval if $C = D = 1$. If in the above we only need to have the upper bound, then Γ is called a g-Bessel sequence. Also if for each $i \in I, \mathcal{Y}_i = \mathcal{Y}$, we call it a g-frame for $\mathcal{X}$ with respect to $\mathcal{Y}$.

Example 3.2. *Let $\{\xi_i\}_{i\in I}$ be a frame for $\mathcal{X}$ with bounds, C and D. Then by definition for each $\xi \in \mathcal{X}$,*

$$C\langle \xi, \xi\rangle \leq \sum_{i\in I} \langle \xi, \xi_i\rangle \langle \xi_i, \xi\rangle \leq D\langle \xi, \xi\rangle.$$

Now for $i \in I$ define the operator Γ_{ξ_i} as follows:

$$\Gamma_{\xi_i} : \mathcal{X} \to \mathcal{A} \quad , \quad \Gamma_{\xi_i}(\xi) = \langle \xi, \xi_i\rangle .$$

Clearly Γ_{ξ_i} is a bounded operator in $Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{A})$ and has adjoint as follows:

$$\Gamma^*_{\xi_i} : \mathcal{A} \to \mathcal{X} \quad , \quad \Gamma^*_{\xi_i}(a) = a\xi_i.$$

Hence $\Gamma_{\xi_i} \in Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{A}), i \in I$. Also for each $\xi \in \mathcal{X}$,

$$C\langle \xi, \xi\rangle \leq \sum_{i\in I} \langle \xi, \xi_i\rangle \langle \xi_i, \xi\rangle = \sum_{i\in I} \langle \Gamma_i\xi, \Gamma_i\xi\rangle \leq D\langle \xi, \xi\rangle.$$

Therefore $\Gamma = \left\{\Gamma_{\xi_i}\right\}_{i\in I}$ is a g-frame for $\mathcal{X}$ with respect to $\mathcal{A}$. Let $\Gamma = \left\{\Gamma_i \in Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}_i)\right\}_{i\in I}$ be a g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ and bounds C, D. We define the corresponding g-frame transform as follows:

$$T_\Gamma : \mathcal{X} \to \bigoplus_{i\in I} \mathcal{Y}_i \quad , \quad T_\Gamma(\xi) = \{\Gamma_i\xi\}_{i\in I}$$

Since Γ is a g-frame, hence for each $\xi \in \mathcal{X}$ we have:

$$C\langle \xi, \xi\rangle \leq \sum_{i\in I} \langle \Gamma_i\xi, \Gamma_i\xi\rangle \leq D\langle \xi, \xi\rangle.$$

So T_Γ is well-defined. Also for any $p \in S(A)$ and $\xi \in \mathcal{X}$ the following inequality is obtained:

$$\sqrt{C}\bar{p}_{\mathcal{X}}(\xi) \leq \bar{p}_{\oplus_{i\in I}\mathcal{Y}_i}(T_\Gamma\xi) \leq \sqrt{D}\bar{p}_{\mathcal{X}}(\xi).$$

Frome the above, it follows that the g-frame transform is an uniformly bounded below operator in $Hom_{\mathcal{A}}\left(\mathcal{X}, \bigoplus_{i\in I} \mathcal{Y}_i\right)$. Thus by Proposition 2.13, T_Γ is closed and injective.

Also, we define the synthesis operator for g-frame Γ as follows:

$$T^*_\Gamma : \bigoplus_{i\in I} \mathcal{Y}_i \to \mathcal{X} \quad , \quad T^*_\Gamma(\{\eta_i\}_i) = \sum_{i\in I} \Gamma^*_i(\eta_i) . \tag{3.1}$$

Where Γ^*_i is the adjoint operator of Γ_i.

Proposition 3.2. *[7] The synthesis operator defined by 3.1 is well-defined, uniformly bounded and adjoint of the transform operator. Since $\Gamma = \{\Gamma_i : i \in I\}$ is a g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$, there exist positive constants C and D such that for any $\xi \in \mathcal{X}$,*

$$C\langle \xi, \xi\rangle \leq \sum_{i\in I} \langle \Gamma_i\xi, \Gamma_i\xi\rangle \leq D\langle \xi, \xi\rangle.$$

Let J be an arbitrary finite subset of I. Using Cauchy-Bunyakovskii inequality for any $p \in Se(\mathcal{A})$ and $(\eta_i)_i \in \oplus_{i\in I}\mathcal{Y}_i$ we have:

$$\begin{aligned}
\bar{p}_{\mathcal{X}}\left(\sum_{i\in J}\Gamma_i^*(\eta_i)\right) &= \sup\left\{p\left\langle\sum_{i\in J}\Gamma_i^*(\eta_i),\xi\right\rangle : \xi\in\mathcal{X}, \bar{p}_{\mathcal{X}}(\xi)\le 1\right\}\\
&= \sup\left\{p\left(\sum_{i\in J}\langle\eta_i,\Gamma_i\xi\rangle\right) : \xi\in\mathcal{X}, \bar{p}_{\mathcal{X}}(\xi)\le 1\right\}\\
&\le \sup_{\bar{p}_{\mathcal{X}}(\xi)\le 1}\left(p\left(\sum_{i\in J}\langle\eta_i,\eta_i\rangle\right)\right)^{1/2}\left(p\left(\sum_{i\in J}\langle\Gamma_i\xi,\Gamma_i\xi\rangle\right)\right)^{1/2}\\
&\le \sup_{\bar{p}_{\mathcal{X}}(\xi)\le 1}\left(\sqrt{D}\bar{p}_{\mathcal{X}}(\xi)\left(p\sum_{i\in J}\langle\eta_i,\eta_i\rangle\right)^{1/2}\right)\\
&\le \sqrt{D}\left(p\left(\sum_{i\in J}\langle\eta_i,\eta_i\rangle\right)\right)^{1/2}
\end{aligned}$$

Now, since the series $\sum_{i\in I}\langle\eta_i,\eta_i\rangle$ converges in A, the above inequality shows that $\sum_{i\in I}\Gamma_i^*(\eta_i)$ is convergent. Hence T_Γ^* is well-defined. On the other hand for any $\xi\in X$ and $(\eta_i)_i\in\oplus_{i\in I}\mathcal{Y}_i$, we have:

$$\begin{aligned}
\langle T_\Gamma(\xi),(\eta_i)_i\rangle &= \langle(\Gamma_i x)_i,(\eta_i)_i\rangle\\
&= \sum_{i\in I}\langle\Gamma_i\xi,\eta_i\rangle\\
&= \sum_{i\in I}\langle\xi,\Gamma_i^*\eta_i\rangle\\
&= \left\langle\xi,\sum_{i\in I}\Gamma_i^*\eta_i\right\rangle\\
&= \langle\xi,T_\Gamma^*(\eta_i)_i\rangle.
\end{aligned}$$

Therefore by Proposition 2.11 it follows that the synthesis operator is adjoint of the transform operator. Also, for any $p\in S(A)$ we have:

$$\bar{p}_X\left(T_\Gamma^*(\eta)\right)\le\sqrt{D}\bar{p}_{\oplus i\in I}Y_i(\eta),\quad \eta=(\eta_i)_i\in\oplus_{i\in I}\mathcal{Y}_i$$

Hence the synthesis operator is uniformly bounded.

Theorem 3.2. *[7] Let $\Gamma=\{\Gamma_i\}_{i\in I}$ be a g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ and with bounds C, D. Then S_Γ is invertible positive operator. Also it is a self-adjoint operator such that:*

$$CI_{\mathcal{X}}\le S_\Gamma\le DI_{\mathcal{X}} \tag{3.2}$$

Here $I_{\mathcal{X}}$ is the identity function on $\mathcal{X}$.

Proof. According to the definition of the transform operator, for any $\xi \in \mathcal{X}$ we can write:

$$\langle T_\Gamma(\xi), T_\Gamma(\xi)\rangle = \left\langle \{\Gamma_i x\}_{i\in I}, \{\Gamma_i \xi\}_{i\in I}\right\rangle = \sum_{i\in I} \langle \Gamma_i\xi, \Gamma_i\xi\rangle .$$

Since Γ is a g-frame for $\mathcal{X}$ with bounds C and D, for each $\xi \in \mathcal{X}$ it follows that:

$$C\langle \xi, \xi\rangle \le \langle T_\Gamma(\xi), T_\Gamma(\xi)\rangle \le D\langle \xi, \xi\rangle.$$

On the other hand,

$$\langle S_\Gamma(\xi), \xi\rangle = \langle T_\Gamma^* T_\Gamma(\xi), \xi\rangle = \langle T_\Gamma(\xi), T_\Gamma(\xi)\rangle = \langle \xi, T_\Gamma^* T_\Gamma(x)\rangle = \langle \xi, S_\Gamma(\xi)\rangle .$$

Consequently, S_Γ is a self-adjoint operator. Also for any $\xi \in \mathcal{X}$, we obtain:

$$C\langle \xi, \xi\rangle \le \langle S_\Gamma(\xi), \xi\rangle \le D\langle \xi, \xi\rangle.$$

From the above, it follows that the g-frame operator is positive and 3.2 is obtained too. Moreover by Proposition 2.1 it follows that S_Γ is invertible. By previous discussions, we have the following useful result. □

3.2. $*$**-frame.**

Definition 3.3. *[13] Let $\mathcal{X}$ be a Hilbert pro-C^*-module. The sequence $\{T_n\}_n$ in $M(\mathcal{X})$ we call a standard $*$-frame of multipliers for $\mathcal{X}$ if for each $\xi \in \mathcal{X}$, the series $\sum_n \langle \xi, T_n\rangle_{M(\mathcal{X})} \langle T_n, \xi\rangle_{M(\mathcal{X})}$ is convergent in $\mathcal{A}$ and there exist two strictly nonzero elements C and D in $\mathcal{A}$ such that*

$$C\langle \xi, \xi\rangle_E C^* \leqslant \sum_n \langle \xi, T_n\rangle_{M(\mathcal{X})} \langle T_n, \xi\rangle_{M(\mathcal{X})} \leqslant D\langle \xi, \xi\rangle_{\mathcal{X}} D^*$$

*for all $\xi \in \mathcal{X}$. If $\lambda = C = D$, then standard $*$-frame $\{T_i\}_{i\in I}$ of multipliers is called a standard λ-tight *-frame. If $\{T_i\}_{i\in I}$ possesses an upper $*$-frame bound, but not necessarily a lower $*$-frame bound, we call it standard $*$-Bessel sequence of multipliers for $\mathcal{X}$.*

Remark 3.3. *[13] Every standard frame of multipliers in $\mathcal{X}$ with bounds C and D is a standard *-frame of multipliers in $\mathcal{X}$ with $\mathcal{A}$-valued *-frame bounds $(\sqrt{C})1_{\mathcal{A}}$ and $(\sqrt{D})1_{\mathcal{A}}$.*

Example 3.3. *Let $\mathcal{H}_{\mathcal{A}}$ be a Hilbert $\mathcal{A}$-module with the following operations:*

$$\xi\eta := \{\xi_i\eta_i\}_{i\in\mathbb{N}}, \qquad \xi^* := \left\{\bar{\xi}_i\right\}_{i\in\mathbb{N}}, \qquad \langle \{\xi_i\}, \{\eta_i\}\rangle := \sum_{i\in\mathbb{N}} \xi_i^*\eta_i,$$

$$\bar{\rho}_{\mathcal{H}_{\mathcal{A}}}(\xi) := \left(\rho\left(\langle \xi, \xi\rangle_{\mathcal{H}_{\mathcal{A}}}\right)\right)^{\frac{1}{2}}, \qquad \forall \xi = \{\xi_i\}_{i\in\mathbb{N}}, \quad \eta = \{\eta_i\}_{i\in\mathbb{N}} .$$

Let $J = \mathbb{N}$ and define $\{\phi_j\}_{j\in J} \in \mathcal{H}_{\mathcal{A}}$ by $\phi_j = \left\{\phi_i^j\right\}_{i\in\mathbb{N}}$ such that

$$\phi_i^j = \begin{cases} 1_{\mathcal{A}} i = j \\ 0 \quad i \neq j \end{cases}, \quad \forall j \in \mathbb{N}.$$

We observe that

$$\left\langle \{\xi_i\}, \phi_j \right\rangle_{\mathcal{H}_\mathcal{A}} \left\langle \phi_j, \{\xi_i\} \right\rangle_{\mathcal{H}_\mathcal{A}} = \overline{\xi_j} 1_\mathcal{A} \overline{1_\mathcal{A}} \xi_j = \overline{\xi_j}\xi_j.$$

Also,

$$\sum_{j\in J} \left\langle \xi, \phi_j \right\rangle_{\mathcal{H}_\mathcal{A}} \left\langle \phi_j, \xi \right\rangle_{\mathcal{H}_\mathcal{A}} = \sum_{j\in J} \overline{\xi_j}\xi_j = \langle \xi, \xi \rangle_{\mathcal{H}_\mathcal{A}}.$$

So $\left\{\phi_j\right\}_{j\in J} \in \mathcal{H}_\mathcal{A}$ *is a standard normalized* $*$*-frame.*

Example 3.4. *[13] Let* $\mathcal{H}_\mathcal{A}$ *be a Hilbert* $\mathcal{A}$*-module. Then* $Hom^*_\mathcal{A}(\mathcal{A}, \mathcal{H}_\mathcal{A})$ *is* $Hom(\mathcal{A})$*-module with the following operations:*

$$\xi\eta := \{\xi_i\eta_i\}_{i\in\mathbb{N}}, \quad \xi^* := \left\{\bar{\xi}_i\right\}_{i\in\mathbb{N}}, \quad \langle \{\xi_i\}, \{\eta_i\} \rangle := \sum_{i\in\mathbb{N}} \xi_i \eta_i^*,$$

$$\bar{p}_{\mathcal{H}_\mathcal{A}}(\xi) = \left(p\left(\langle \xi, \xi \rangle_{\mathcal{H}_\mathcal{A}}\right)\right)^{\frac{1}{2}}, \quad \forall \xi = \{\xi_i\}_{i\in\mathbb{N}}, \quad \eta = \{\eta_i\}_{i\in\mathbb{N}}.$$

Let $J = \mathbb{N}$ *and define* $\phi_j \in L(\mathcal{A}, H_\mathcal{A})$ *by* $\phi_j = \left\{\phi_i^j\right\}_{i\in\mathbb{N}}$ *such that*

$$\phi_i^j(\alpha) = \begin{cases} \langle \alpha, K1_\mathcal{A} \rangle & i = j \\ 0 & i \neq j \end{cases}, \quad \forall j \in \mathbb{N},$$

where K *is constant.*

$$p\left(\sum_j \phi_i^j(\alpha)\overline{\phi_i^j(\alpha)}\right) = p\left(\phi_i^j(\alpha)\overline{\phi_i^j(\alpha)}\right) = p\left(\langle \alpha, K1_\mathcal{A}\rangle \overline{\langle \alpha, K1_\mathcal{A}\rangle}\right) = p\left(\langle \alpha, K1_\mathcal{A}\rangle\right)^2 < \infty,$$

which implies that ϕ_j *is well-defined and adjointable.* $\phi_j^* \in Hom^*_\mathcal{A}(\mathcal{H}_\mathcal{A}, \mathcal{A})$ *is obtained by* $\phi_j^* = \left\{\phi_i^{j^*}\right\}_{i\in\mathbb{N}}$*, as* $\phi_i^{j^*}(\{x_i\}) = K1_\mathcal{A}x_j$*, and we have*

$$\left\langle \{x_i\}, \phi_j \right\rangle_{M(H_\mathcal{A})} \left\langle \phi_j, \{x_i\} \right\rangle_{M(\mathcal{H}_\mathcal{A})} = \overline{\phi_j^*(\{x_i\})}\phi_j^*(\{x_i\}) = \overline{K1_\mathcal{A}}\bar{x}_j K1_\mathcal{A} x_j.$$

So,

$$\begin{aligned} \sum_{j\in J} \left\langle x, \phi_j \right\rangle_{M(\mathcal{H}_\mathcal{A})} \left\langle \phi_j, x \right\rangle_{M(\mathcal{H}_\mathcal{A})} &= \sum_{j\in J} \left\langle \{x_i\}_{i\in\mathbb{N}}, \phi_j \right\rangle_{M(\mathcal{H}_\mathcal{A})} \left\langle \phi_j, \{x_i\}_{i\in\mathbb{N}} \right\rangle_{M(\mathcal{H}_\mathcal{A})} \\ &= \sum_{j\in J} \overline{K1_\mathcal{A}}\bar{x}_j x_j K1_\mathcal{A} \\ &= K1_\mathcal{A} \sum_{j\in J} \bar{x}_j x_j K1_\mathcal{A} = K1_\mathcal{A} \langle x, x \rangle_{\mathcal{H}_\mathcal{A}} K1_\mathcal{A}. \end{aligned}$$

Consequently, $\left\{\phi_j\right\}_{j\in I}$ *in* $M(\mathcal{H}_\mathcal{A})$ *is a standard* $K1_\mathcal{A}$*-tight* $*$*-frame of multipliers in* $\mathcal{H}_\mathcal{A}$*.*

Proposition 3.3. *[13] Let the sequence* $\{T_i\}_{i\in I}$ *be a standard* $*$*-frame of multipliers in* $\mathcal{X}$*. Then* $\left\{\langle T_i, \xi \rangle_{M(\mathcal{X})}\right\}_{i\in I} \in \mathcal{H}_\mathcal{A}$*.*

Definition 3.4. *[13] Let* $\{T_i\}_{i\in I}$ *be a standard* $*$*-frame of multipliers in* $\mathcal{X}$*, thus we can define a linear map* $T : E \to H_\mathcal{A}$ *by* $T(x) = \left\{\langle T_i, x \rangle_{M(E)}\right\}_{i\in I}$ *is called the pre-*$*$*-frame operator or* $*$*-frame transform for* $\{T_i\}_{i\in I}$*.*

Theorem 3.3. *Let $\{\phi_i\}_{i\in I}$ be a standard $*$-frame of multiplier s in E with lower and upper $*$-frame bounds C and D, respectively. Then the pre-$*$-frame operator T is invertible and $\hat{p}_{\mathcal{X},H_{\mathcal{A}}}(T) \leqslant p(D)$.*

Proof. Let J be an arbitrary finite subset of I. Using the Cauchy-Schwarz inequality, for any $p \in Se(\mathcal{A})$ and $\{\eta_i\}_{i\in I} \in H_{\mathcal{A}}$, we have

$$\begin{aligned}
\left(\bar{p}_{\mathcal{H}_{\mathcal{A}}}(T\xi)\right)^2 &= \left(\bar{p}_{\mathcal{H}_{\mathcal{A}}}\left(\left\{\langle \phi_i, \xi\rangle_{M(\mathcal{X})}\right\}_{i\in J}\right)\right)^2 \\
&= \sup\left\{\left(p\left(\left\langle\left\{\langle h_i, \xi\rangle_{M(\mathcal{X})}\right\}_i, \{\eta_i\}_i\right\rangle\right)\right)^2 : \bar{p}_{H_{\mathcal{A}}}\left(\{\eta_i\}_i\right) \leqslant 1\right\} \\
&\leqslant \sup_{\bar{p}_{\mathcal{H}_{\mathcal{A}}}(\{\eta_i\}_i)\leqslant 1} p\left(\left\langle\left\{\langle \phi_i, \xi\rangle_{M(\mathcal{X})}\right\}_i, \left\{\langle \phi_i, \xi\rangle_{M(\mathcal{X})}\right\}_i\right\rangle\right) \times \sup_{\bar{p}_{H_{\mathcal{A}}}(\{\eta_i\}_i)\leqslant 1} p\left(\langle\{\eta_i\}, \{\eta_i\}\rangle\right) \\
&\leqslant p\left(\sum_{i\in J} \overline{\langle \phi_i, \xi\rangle_{M(\mathcal{X})}} \langle \phi_i, \xi\rangle_{M(\mathcal{X})}\right) \\
&\leqslant p(D)p\left(\langle \xi, \xi\rangle_{\mathcal{X}}\right) p\left(D^*\right) \\
&= (p(D))^2 \left(\bar{p}_{\mathcal{X}}(\xi)\right)^2.
\end{aligned}$$

This shows $\bar{p}_{H_{\mathcal{A}}}(T\xi) \leqslant p(D)\bar{p}_{\mathcal{X}}(\xi)$ for all $\xi \in \mathcal{X}$, so T is well-defined, bounded and $\hat{p}_{\mathcal{X},H_{\mathcal{A}}}(T) \leqslant p(D)$. Since $\langle T\xi, T\xi\rangle_{\mathcal{X}} = \sum_n \langle \xi, \phi_n\rangle_{M(\mathcal{X})} \langle \phi_n, \xi\rangle_{M(\mathcal{X})}$ for each $\xi \in \mathcal{X}$, we observe that

$$C\langle \xi, \xi\rangle_{\mathcal{X}} C^* \leqslant \langle T\xi, T\xi\rangle_{\mathcal{X}} \leqslant D\langle \xi, \xi\rangle_{\mathcal{X}} D^*.$$

Suppose that $\xi \in \mathcal{X}$ and $T\xi = 0$. Thus, $\xi = 0$ and T is invertible. □

Proposition 3.4. *[13] Let $\{\phi_i\}_{i\in I}$ be a sequence in $M(\mathcal{X})$. Suppose that $P : \xi \to \left\{\langle \phi_i, \xi\rangle_{M(\mathcal{X})}\right\}_{i\in I}$ is an invertible element in $b\left(Hom^*_{\mathcal{A}}\left(\mathcal{X}, \mathcal{X}_{\mathcal{A}}\right)\right)$. Then $\{\phi_i\}_{i\in I}$ is a standard $*$-frame of multipliers in $\mathcal{X}$.*

Proof. Let the sequence $\{\alpha_i\}_{i\in I}$ be in $\mathcal{H}_{\mathcal{A}}$. We can write

$$\begin{aligned}
\left\langle \{\alpha_i\}_{i\in I}, P(\xi)\right\rangle_{\mathcal{X}_{\mathcal{A}}} &= \left\langle \{\alpha_i\}_{i\in I}, \left\{\langle \phi_i, \xi\rangle_{M(\mathcal{X})}\right\}_{i\in I}\right\rangle_{\mathcal{H}_{\mathcal{A}}} \\
&= \sum_{i\in I} \overline{\alpha_i} \langle \phi_i, \xi\rangle_{M(E)} \\
&= \left\langle \sum_{i\in I} \phi_i \alpha_i, \xi\right\rangle_{\mathcal{X}}.
\end{aligned}$$

This shows $P^*\left(\{a_i\}_{i\in I}\right) = \sum_{i\in I} \phi_i a_i$. Moreover, P^* is an invertible element in $b\left(L_{(H_{\mathcal{A}},\mathcal{X})}\right)$. Define $\mathcal{V} := P^*P$. Hence, $\mathcal{V}$ and $\mathcal{V}^{\frac{1}{2}}$ are positive and invertible elements in $b(Hom^*_{\mathcal{A}}(\mathcal{X}))$. As see in the [11] we have

$$\left\|\mathcal{V}^{\frac{-1}{2}}\right\|_{\infty}^{-2} \langle \xi, \xi\rangle_{\mathcal{X}} \leqslant \left\langle \mathcal{V}^{\frac{1}{2}}\xi, \mathcal{V}^{\frac{1}{2}}\xi\right\rangle_{\mathcal{X}} \leqslant \left\|\mathcal{V}^{\frac{1}{2}}\right\|_{\infty}^{2} \langle \xi, \xi\rangle_{\mathcal{X}}.$$

Since $\langle P\xi, P\xi\rangle_{\mathcal{X}} = \sum_{i\in I} \langle \xi, \phi_i\rangle_{M(\mathcal{X})} \langle \phi_i, \xi\rangle_{M(\mathcal{X})}$, series is convergent in $\mathcal{A}$ and

$$\left(\left\|\mathcal{V}^{\frac{-1}{2}}\right\|_{\infty}^{-1} 1_{\mathcal{A}}\right) \langle \xi, \xi\rangle_{\mathcal{X}} \left(\left\|\mathcal{V}^{-\frac{1}{2}}\right\|_{\infty}^{-1} 1_{\mathcal{A}}\right)^* \leqslant \sum_{i\in I} \langle \xi, \phi_i\rangle_{M(\mathcal{X})} \langle \phi_i, \xi\rangle_{M(\mathcal{X})}$$
$$\leqslant \left(\left\|\mathcal{V}^{\frac{1}{2}}\right\|_{\infty} 1_{\mathcal{A}}\right) \langle \xi, \xi\rangle_{\mathcal{X}} \left(\left\|\mathcal{V}^{\frac{1}{2}}\right\|_{\infty} 1_{\mathcal{A}}\right)^*.$$

So $\{\phi_i\}_{i\in I}$ is a standard $*$-frame of multipliers in $\mathcal{X}$. □

3.3. **$*$-g-frame.**

Definition 3.5. *A sequence $\Gamma = \left\{\Gamma_i \in Hom_{\mathcal{A}}^*(\mathcal{X}, \mathcal{Y}_i)\right\}_{i\in I}$ is called a $*$-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ if for each $\xi \in \mathcal{X}$, the series $\sum_{i\in I} \langle \Gamma_i(\xi), \Gamma_i(\xi)\rangle$ is convergent in $\mathcal{A}$ and there are two strictly non-zero values C and D in $\mathcal{A}$ such that for every $\xi \in \mathcal{X}$,*

$$C\langle \xi, \xi\rangle C^* \leq \sum_{i\in I} \langle \Gamma_i(\xi), \Gamma_i(\xi)\rangle \leq D\langle \xi, \xi\rangle D^*.$$

The elements C and D are called $*$-g-frame bounds for Γ. The $*$-g-frame is called tight if $C = D$ and called Parseval if $C = D = 1$. If in the above we only need to have the upper bound, then Γ is called a $*$-g-Bessel sequence. Since the sequence $\sum_{i\in I} \langle \Gamma_i(\xi), \Gamma_i(\xi)\rangle$ is convergent in A, the $*$-g-frame can be called standard but we use this definition without the word standard if there is no risk of ambiguity. Besides, if for each $i \in I, Y_i = Y$, we call it a $*$-g-frame for X with respect to Y.

Example 3.5. *Let $\{\xi_i\}_{i\in I}$ be a $*$-frame for X with bounds C and D. For $i \in I$, consider the operator Γ_{ξ_i} defined via*

$$\Gamma_{\xi_i} : \mathcal{X} \longrightarrow \mathcal{A}; \quad \Gamma_{\xi_i}(x) = \langle \xi, \xi_i\rangle.$$

It is obvious that Γ_{ξ_i} is a bounded operator in $Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{A})$ which its adjoint is

$$\Gamma_{\xi_i}^* : \mathcal{A} \to \mathcal{X} \quad \Gamma_{\xi_i}^*(\alpha) = \alpha\xi_i.$$

Hence, $\Gamma_{\xi_i} \in Hom_{\mathcal{A}}^*(\mathcal{X}, \mathcal{A}), i \in I$. Moreover, by assumption, for each $\xi \in \mathcal{X}$

$$C\langle \xi, \xi\rangle C^* \leq \sum_{i\in I} \langle \xi, \xi_i\rangle \langle \xi_i, \xi\rangle = \sum_{i\in I} \langle \Gamma_i(\xi), \Gamma_i(\xi)\rangle \leq D\langle \xi, \xi\rangle D^*.$$

Therefore, $\Gamma = \left\{\Gamma_{\xi_i}\right\}_{i\in I}$ is a $*$-g-frame for $\mathcal{X}$ with respect to $\mathcal{A}$.

Definition 3.6. *Let $\Gamma = \{\Gamma_i : i \in I\}$ be a $*$-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$. Define the corresponding $*$-g-frame operator $S_\Gamma = T_\Gamma^* T_\Gamma : X \longrightarrow \mathcal{X}$ via $S_\Gamma(\xi) = \sum_{i\in I} \Gamma_i^* \Gamma_i \xi$. Then, S_Γ is a combination of two bounded operators and so it is a bounded operator.*

Theorem 3.4. *Let $\Gamma = \{\Gamma_i\}_{i\in I}$ be a $*$-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ with frame bounds A and B. Then, S_Γ is an invertible positive operator. Moreover, it is a self-adjoint operator such that*

$$A^*AI_{\mathcal{X}} \leq S_\Gamma \leq B^*BI_{\mathcal{X}} \tag{3.3}$$

and

$$B^{-1}(B^*)^{-1}I_{\mathcal{X}} \leq S_\Gamma^{-1} \leq A^{-1}(A^*)^{-1}I_{\mathcal{X}}, \tag{3.4}$$

where $I_{\mathcal{X}}$ *is the identity function on* $\mathcal{X}$*, and also we have*

$$\Big(p\left(A^{-1}\right)^{-2} \leq p_{\mathcal{X}}^{-}(S_\Gamma) \leq (p(B))^2.$$

Proof. According to the definition of the transform operator, for any $\xi \in \mathcal{X}$ we can write

$$\langle T_\Gamma(\xi), T_\Gamma(\xi)\rangle = \left\langle \{\Gamma_i(\xi)\}_{i\in I}, \{\Gamma_i(\xi)\}_{i\in I}\right\rangle = \sum_{i\in I}\langle \Gamma_i(\xi), \Gamma_i(\xi)\rangle.$$

By hypotheses, we get

$$A\langle \xi, \xi\rangle A^* \leq \langle T_\Gamma(\xi), T_\Gamma(\xi)\rangle \leq B\langle \xi, \xi\rangle B^*.$$

On the other hand,

$$\langle S_\Gamma(\xi), \xi\rangle = \langle T_\Gamma^* T_\Gamma(\xi), \xi\rangle = \langle T_\Gamma(\xi), T_\Gamma(\xi)\rangle = \langle \xi, T_\Gamma^* T_\Gamma(\xi)\rangle = \langle \xi, S_\Gamma(\xi)\rangle.$$

Consequently, S_Γ is a self-adjoint operator. For any $\xi \in \mathcal{X}$, we find

$$A\langle \xi, \xi\rangle A^* \leq \langle S_\Gamma(\xi), \xi\rangle \leq B\langle \xi, \xi\rangle B^*. \tag{3.5}$$

From 3.3, it follows that the $*$-g-frame operator is positive and 4.1 is obtained as well. Now, suppose that $S_\Gamma(\xi) = 0$ for any $\xi \in \mathcal{X}$. By (5), we observe that $\langle \xi, \xi\rangle = 0$, which implies S_Γ is invertible. For $\xi \in \mathcal{X}$, we have

$$A\left\langle S_\Gamma^{-1}\xi, S_\Gamma^{-1}\xi\right\rangle A^* \leq \sum_{i\in I}\left\langle \Gamma_i S_\Gamma^{-1}\xi, \Gamma_i S_\Gamma^{-1}\xi\right\rangle = \left\langle S_\Gamma^{-1}\xi, \xi\right\rangle$$

and

$$\left\langle \xi, S_\Gamma^{-1}(\xi)\right\rangle = \sum_{i\in I}\left\langle \Gamma_i S_\Gamma^{-1}(\xi), \Gamma_i S_\Gamma^{-1}(\xi)\right\rangle \leq B\left\langle S_\Gamma^{-1}(\xi), S_\Gamma^{-1}(\xi)\right\rangle B^*.$$

The last relations necessitate that for all $\xi \in \mathcal{X}$

$$B^{-1}\left\langle S_\Gamma^{-1}(\xi), \xi\right\rangle (B^*)^{-1} \leq \left\langle S_\Gamma^{-1}(\xi), S_\Gamma^{-1}(\xi)\right\rangle \leq A^{-1}\left\langle S_\Gamma^{-1}(\xi), \xi\right\rangle (A^*)^{-1}$$

and so

$$B^{-1}(B^*)^{-1}S_\Gamma^{-1} \leq \left(S_\Gamma^{-1}\right)^2 \leq A^{-1}(A^*)^{-1}S_\Gamma^{-1}.$$

Since S is a positive operator, $B^{-1}(B^*)^{-1} I_{\mathcal{X}} \leq S_{\Gamma}^{-1} \leq A^{-1}(A^*)^{-1} I_{\mathcal{X}}$. Applying the CauchyBunyakovskii inequality and Lemma 2.2 in [19], we have

$$\begin{aligned}\left(\bar{p}_{\mathcal{X}}\left(\sum_{i\in J}\Gamma_i^*\Gamma_i(\xi)\right)\right)^2 &= \left\{\sup p\left(\left\langle\sum_{i\in J}\Gamma_i^*\Gamma_i(\xi),\eta\right\rangle\right) : \eta\in\mathcal{X}, \bar{p}_{\mathcal{X}}(\eta)\leq 1\right\}^2\\ &= \left\{\sup p\left(\sum_{i\in J}\langle\Gamma_i(\xi),\Gamma_i(\eta)\rangle\right) : \eta\in\mathcal{X}, \bar{p}_{\mathcal{X}}(\eta)\leq 1\right\}^2\\ &\leq \sup_{\bar{p}_{\mathcal{X}}(\eta)\leq 1}\left(p\left(\sum_{i\in J}\langle\Gamma_i(\xi),\Gamma_i(\xi)\rangle\right)\right)\left(p\left(\sum_{i\in J}\langle\Gamma_i(\eta),\Gamma_i(\eta)\rangle\right)\right)\\ &\leq \sup_{\bar{p}_{\mathcal{X}}(\eta)\leq 1}(p(B))^2(\bar{p}_{\mathcal{X}}(\eta))^2(p(B))^2(\bar{p}_{X}(\xi))^2\\ &\leq (p(B))^4(\bar{p}_{\mathcal{X}}(\xi))^2.\end{aligned}$$

for all $p \in Se(\mathcal{A})$ and $\xi, \eta \in \mathcal{X}$. Hence,

$$\begin{aligned}(\bar{p}_{\mathcal{X}}(S_{\Gamma}(\xi)))^2 &= \left(\bar{p}_{\mathcal{X}}\left(\sum_{i\in J}\Gamma_i^*\Gamma_i(\xi)\right)\right)^2\\ &= \left\{\sup p\left(\left\langle\sum_{i\in J}\Gamma_i^*\Gamma_i(\xi),\eta\right\rangle\right) : \eta\in\mathcal{X}, \bar{p}_{\mathcal{X}}(\eta)\leq 1\right\}^2 \leq (p(B))^4(\bar{p}_{\mathcal{X}}(\xi))^2.\end{aligned}$$

Furthermore, $\bar{p}_{\mathcal{X}}(S_{\Gamma}(\xi))\left(p\left(A^{-1}\right)\right)^{-2} \leq \bar{p}_{\mathcal{X}}(S_{\Gamma}(\xi))$. Therefore

$$\left(p\left(A^{-1}\right)\right)^{-2} \leq \bar{p}_{\mathcal{X}}(S_{\Gamma}(\xi)) \leq (p(B))^2.$$

This finishes the proof. □

Theorem 3.5. *For each $i \in I$, let $\Gamma = \{\Gamma_i \in \mathrm{Hom}_A^*(\mathcal{X}, \mathcal{Y}_i)\}_{i\in I}$ and $\{\xi_{ij}\}_{j\in J_i}$ be a Parseval-frame for $\mathcal{Y}_i$. Then, the following assertions hold.*

(i) $\{\Gamma_i\}_{i\in I}$ *is a Parseval $*$-g-frame for $\mathcal{X}$ if and only if $\{\Gamma_i^*\xi_{ij}\}_{j\in J_i, i\in I}$ is a Parseval $*$-frame for X.*

(ii) *The $*$-g-frame operator of $\{\Gamma_i\}_{i\in I}$ is the $*$-frame operator of $\Gamma = \{\Gamma_i^*\xi_{ij}\}_{j\in J_i, i\in I}$*

Proof. (i) It follows from the assumptions that

$$\langle\Gamma_i(\xi),\Gamma_i(\xi)\rangle = \sum_{j\in J_i}\langle\Gamma_i(\xi),\xi_{ij}\rangle\langle\xi_{ij},\Gamma_i(\xi)\rangle.$$

Therefore

$$\sum_{i\in I}\langle\Gamma_i(\xi),\Gamma_i(\xi)\rangle = \sum_{i\in I}\sum_{j\in J_i}\langle\Gamma_i(\xi),\xi_{ij}\rangle\langle\xi_{ij},\Gamma_i(\xi)\rangle.$$

for all $\xi \in \mathcal{X}$. Since for every i, Γ_i is adjointable and so the above equality can be summarized as follow:

$$\sum_{i\in I}\langle\Gamma_i(\xi),\Gamma_i(\xi)\rangle = \sum_{i\in I}\sum_{j\in J_i}\langle\xi,\Gamma_i^*(\xi_{ij})\rangle\langle\Gamma_i^*(\xi_{ij}),\xi\rangle,$$

which shows that $\{\Gamma_i\}_{i\in I}$ is a Parseval $*$-g-frame for $\mathcal{X}$ if and only if $\left\{\Gamma_i^*\left(\xi_{ij}\right)\right\}_{j\in J_i, i\in I}$ is a Parseval $*$-frame for $\mathcal{X}$.

(ii) Let S_Γ and S_Γ be the $*$-frame operators for Γ and Γ, respectively. Then

$$S_\Gamma(\xi)=\sum_{i\in I}\sum_{j\in J_i}\left\langle \xi,\Gamma_i^*\left(\xi_{ij}\right)\right\rangle \Gamma_i^*\left(\xi_{ij}\right),\quad S_\Gamma(\xi)=\sum_{i\in I}\Gamma_i^*\Gamma_i(\xi)$$

for all $\xi\in\mathcal{X}$. On the other hand, for every $i\in I$ and $\xi\in\mathcal{X},\Gamma_i(\xi)=\sum_{j\in J_i}\left\langle\Gamma_i(\xi),\xi_{ij}\right\rangle\xi_{ij}$. Since $\Gamma_i(\xi)\in Y_i$ and the last equality is the reconstruction formula for $\Gamma_i\xi$ with respect to Parseval $*$-frame $\left\{\xi_{ij}\right\}_{j\in J_i}$, we get

$$\begin{aligned}S_\Gamma(\xi)&=\sum_{i\in I}\sum_{j\in J_i}\left\langle \xi,\Gamma_i^*\left(\xi_{ij}\right)\right\rangle \Gamma_i^*\left(\xi_{ij}\right)=\sum_{i\in I}\sum_{j\in J_i}\left\langle\Gamma_i(\xi),\xi_{ij}\right\rangle\Gamma_i^*\left(\xi_{ij}\right)\\&=\sum_{i\in I}\Gamma_i^*\left(\sum_{j\in J_i}\left\langle\Gamma_i(\xi),\xi_{ij}\right\rangle\xi_{ij}\right)=\sum_{i\in I}\Gamma_i^*\Gamma_i(\xi)=S_\Gamma(\xi).\end{aligned}$$

for all $\xi\in\mathcal{X}$. The proof of part (ii) is now complete. □

3.4. *K*-*g*-**frame.**

Definition 3.7. *Let $K\in Hom_{\mathcal{A}}^*(\mathcal{X})$. A sequence $\Gamma=\left\{\Gamma_i\in Hom_{\mathcal{A}}^*\left(\mathcal{X},\mathcal{Y}_i\right)\right\}_{i\in I}$ is called a K-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ if there exist two positive constants A and B such that for every $\xi\in\mathcal{X}$,*

$$A\langle K^*\xi,K^*\xi\rangle\leq\sum_{i\in I}\langle\Gamma_i\xi,\Gamma_i\xi\rangle\leq B\langle\xi,\xi\rangle$$

The constants A and B are called the lower and upper bounds of K-g-frames, respectively. The K-g-frame is called tight if $A=B$ and a Parseval if $A=B=1$

Definition 3.8. *Suppose that $K\in Hom_{\mathcal{A}}^*(\mathcal{X})$ and $\{\Gamma_i\}_{i\in I}$ is a K-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$. A g-frame sequence $\{\Xi_i\}_{i\in I}$ for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ is said to be a K-dual g-frame sequence of $\{\Gamma_i\}_{i\in I}$ if*

$$K\xi=\sum_{i\in I}\Gamma_i^*\Xi_i\xi,\quad\forall\xi\in\mathcal{X}$$

Lemma 3.1. *Let $\left\{U_i\in Hom_{\mathcal{A}}^*\left(\mathcal{X},\mathcal{Y}_i\right):i\in I\right\}$ be a g-orthonormal basis for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i:i\in I\}$; then the sequence $\left\{\Gamma_i\in Hom_{\mathcal{A}}^*\left(\mathcal{X},\mathcal{Y}_i\right):i\in I\right\}$ is a g-frame sequence with respect to $\{\mathcal{Y}_i:i\in I\}$ if and only if there is a unique bounded operator $Q:\mathcal{X}\to\mathcal{X}$ such that $\Gamma_i=U_iQ^*$, for all $i\in I$.*

Proof. $\Rightarrow$ Since $\left\{U_i\in Hom_{\mathcal{A}}^*\left(\mathcal{X},\mathcal{Y}_i\right):i\in I\right\}$ is a g-orthonormal basis for $\mathcal{X}$, $\{U_i\xi:i\in I\}\in\left(\sum_{i\in I}\oplus\mathcal{Y}_i\right)_{\mathcal{H}_{\mathcal{A}}}$, for any $\xi\in\mathcal{X}$. If $\left\{\Gamma_i\in Hom_{\mathcal{A}}^*\left(\mathcal{X},\mathcal{Y}_i\right):i\in I\right\}$ is a g-Bessel sequence, then the operator Q below is well-defined and bounded

$$Q:\mathcal{X}\longrightarrow\mathcal{X},\quad Q\xi=\sum_{i\in I}\Gamma_i^*U_i\xi$$

Also by using the definition of g-orthonormal basis, it is simple to observe that $U_iU_j^*\eta = \delta_{ij}\eta$. So

$$QU_j^*\eta = \sum_{i\in I}\Gamma_i^*U_iU_j^*\eta = \Gamma_j^*U_jU_j^*\eta = \Gamma_j^*\eta$$

for all $\eta \in \mathcal{Y}_j, j \in I$. Hence $QU_j^* = \Gamma_j^*$, which implies that $U_jQ^* = \Gamma_j, j \in I$. Suppose $Q_1, Q_2 \in Hom_{\mathcal{A}}^*(\mathcal{X}, \mathcal{Y}_i)$ and $U_iQ_1^* = U_iQ_2^* = \Gamma_i$ for any $i \in I$. Then for any $\xi \in \mathcal{X}, \eta_i \in \mathcal{Y}_i$, we have $\langle U_iQ_1^*\xi, \eta_i\rangle = \langle U_iQ_2^*\xi, \eta_i\rangle$, that is, $\langle Q_1^*\xi, U_i^*\eta_i\rangle = \langle Q_2^*\xi, U_i^*\eta_i\rangle$.

Since $\overline{\text{span}}\left\{U_i^*\left(\mathcal{Y}_i\right)\right\}_{i\in I} = \mathcal{X}$, $Q_1^*\xi = Q_2^*\xi$, which means that $Q_1 = Q_2$. Thus the operator Q is unique.

$\Leftarrow$: Since $\Gamma_i = U_iQ^*$, for all $i \in I$, for any $\xi \in \mathcal{X}$, we have

$$\sum_{i\in I}\langle\Gamma_i\xi, \Gamma_i\xi\rangle = \sum_{i\in I}\langle U_iQ^*\xi, U_iQ^*\xi\rangle = \langle Q^*\xi, Q^*\xi\rangle.$$

The Proposition 2.2 in [1] shows that Q^* is bounded below, therefore Q^* is invertible. Then by Theorem 3.2 in [2], we have:

$$\left\|(Q^*)^{-1}\right\|_\infty^{-2}\langle\xi, \xi\rangle \le \langle Q^*f, Q^*f\rangle \le \|Q^*\|_\infty^2\langle\xi, \xi\rangle$$

□

Lemma 3.2. *Let $\{\Gamma_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ Then $\{\Gamma_i\}_{i\in I}$ is a K-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ if and only if there exists constant $C > 0$ such that $S \ge CKK^*$, where S is the frame operator for $\{\Gamma_i\}_{i\in I}$*

Proof. $\{\Gamma_i\}_{i\in I}$ is a K-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ with bounds C, D if and only if

$$C\langle K^*\xi, K^*\rangle \le \sum_{i\in I}\langle\Gamma_i\xi, \Gamma_i\xi\rangle \le D\langle\xi, \xi\rangle, \forall\xi \in \mathcal{X},$$

that is,

$$\langle CKK^*\xi, \xi\rangle \le \langle S\xi, \xi\rangle \le \langle D\xi, \xi\rangle, \forall\xi \in \mathcal{X},$$

where S is the frame operator of K-g-frame $\{\Gamma_i\}_{i\in I}$. Therefore, the conclusion holds. □

Theorem 3.6. *Let $K \in Hom_{\mathcal{A}}^*(\mathcal{X})$ and $\left\{U_i \in Hom_{\mathcal{A}}^*(\mathcal{X}, \mathcal{Y}_i)\right\}_{i\in I}$ be a g-orthonormal basis for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$. $\{\Gamma_i\}_{i\in I}$ is a K-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ with the g-operator Q. P is the g-operator of g-frame sequence $\{T_i\}_{i\in I}$. Then $\{T_i\}_{i\in I}$ is the K-dual g-frame sequence of $\{\Gamma_i\}_{i\in I}$ if and only if $K = QP^*$.*

Proof. Suppose $\{T_i\}_{i\in I}$ is the K-dual g-frame sequence of $\{\Gamma_i\}_{i\in I}$, then $\forall\xi \in \mathcal{X}$, we have $K\xi = \sum_{i\in I}\Gamma_i^*T_i\xi$. Consider that $\{U_i\}_{i\in I}$ is the g-orthonormal basis for $\mathcal{X}$, then by Lemma 3.1, $\forall\xi \in \mathcal{X}$

$$K\xi = \sum_{i\in I}(U_iQ^*)^*(U_iP^*)\xi = Q\sum_{i\in I}U_i^*U_iP^*\xi = QP^*\xi.$$

By the arbitrariness of ξ, we obtain $K = QP^*$.

Conversely, since $\{\Gamma_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ are two g-frame sequences, therfore using Lemma 3.1, there are bounded operators Q and P, such that

$$\Gamma_i = U_i P^*, \quad T_i = U_i P^*.$$

Hence, $\forall \xi \in \mathcal{X}$

$$\sum_{i\in I} \Gamma_i^* T_i \xi = \sum_{i\in I} (U_i Q^*)^* (U_i P^*) \xi = Q \sum_{i\in I} U_i^* U_i P^* \xi = QP^*\xi = K\xi.$$

This shows that $\{T_i\}_{i\in I}$ is the K-dual g-frame sequence of $\{\Gamma_i\}_{i\in I}$. □

3.5. $*$-K-g-**frame.**

Definition 3.9. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$. We say that $\{\Gamma_i \in Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}_i)\}_{i\in I}$ is $*$-K-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ if there exist nonzero elements $A, B \in \mathcal{A}$ such that for all $\xi \in \mathcal{X}$,*

$$A\langle K^*\xi, K^*\xi\rangle A^* \leq \sum_{i\in I} \langle \Gamma_i \xi, \Gamma_i \xi\rangle \leq B\langle \xi, \xi\rangle B^* \tag{3.6}$$

The numbers A and B are called lower and upper bound of the $*$-K-g-frame, respectively. If

$$A\langle K^*\xi, K^*\xi\rangle A^* = \sum_{i\in I} \langle \Gamma_i \xi, \Gamma_i \xi\rangle, \forall \xi \in \mathcal{X}. \tag{3.7}$$

The $*$-K-g-frame is A-tight.

Example 3.6. *Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in \mathrm{N}}, v = \{v_j\}_{j\in \mathrm{N}} \in l^\infty$, we define*

$$uv = \{u_j v_j\}_{j\in \mathrm{N}}, u^* = \{\bar{u}_j\}_{j\in \mathrm{N}}, \|u\| = \sup_{j\in \mathrm{N}} |u_j|.$$

Then $\mathcal{A} = \{l^\infty, \|.\|\}$ is a C^*-algebra. Consequently $\mathcal{A} = \{l^\infty, \|.\|\}$ is pro-C^*-algebra.

Let $\mathcal{X} = C_0$ be the set of all sequences converging to zero. For any $u, v \in \mathcal{X}$ we define

$$\langle u, v\rangle = uv^* = \{u_j \bar{u}_j\}_{j\in \mathrm{N}}.$$

Then $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module.

Define $f_j = \{f_i^j\}_{i\in \mathrm{N}^*}$ by $f_i^j = \frac{1}{2} + \frac{1}{i}$ if $i = j$ and $f_i^j = 0$ if $i \neq j$ $\forall j \in \mathbf{N}^*$. Now define the adjointable operator $\Lambda_j : \mathcal{X} \to \mathcal{A}$, $\Lambda_j \xi = \langle \xi, f_j\rangle$.

then for every $\xi \in \mathcal{X}$ we have

$$\sum_{j\in \mathrm{N}} \langle \Lambda_j \xi, \Lambda_j \xi\rangle = \{\frac{1}{2} + \frac{1}{i}\}_{i\in \mathrm{N}^*} \langle \xi, \xi\rangle \{\frac{1}{2} + \frac{1}{i}\}_{i\in \mathrm{N}^*}.$$

Let $K : \mathcal{X} \to \mathcal{X}$ defined by $K\xi = \{\frac{\xi_i}{i}\}_{i\in \mathrm{N}^*}$.

Then for every $\xi \in \mathcal{X}$ we have

$$\langle K^*\xi, K^*\xi\rangle_{\mathcal{A}} \leq \sum_{j\in \mathrm{N}} \langle \Lambda_j \xi, \Lambda_j \xi\rangle = \{\frac{1}{2} + \frac{1}{i}\}_{i\in \mathrm{N}^*} \langle \xi, \xi\rangle \{\frac{1}{2} + \frac{1}{i}\}_{i\in \mathrm{N}^*}.$$

Which shows that $\{\Lambda_j\}_{j\in\mathbb{N}}$ is a $*$-K-g-frame for $\mathcal{X}$ with bounds 1 and $\{\frac{1}{2}+\frac{1}{i}\}_{i\in\mathbb{N}^*}$.

Remark 3.4.

(1) *Every $*$-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i : i \in I\}$ is an $*$-K-g-frame, for any $K \in \mathcal{L}^*_{\mathcal{A}}(\mathcal{X})$: $K \neq 0$.*

(2) *If $K \in b(\mathcal{L}_{\mathcal{A}}(\mathcal{X}))$ is a surjective operator, then every $*$-K-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i : i \in I\}$ is a $*$-g-frame.*

Example 3.7. *Let $\mathcal{X}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module. $\mathcal{L}^*_{\mathcal{A}}(\mathcal{X})$ Let $K \in \mathcal{L}^*_{\mathcal{A}}(\mathcal{X})$ an invertible element such that both are uniformly bounded and $K \neq 0$. Let $\mathcal{A}$ be a Hilbert $\mathcal{A}$-module over itself with the inner product $\langle a, b\rangle = ab^*$. Let $\{x_i\}_{i\in I}$ be an $*$-frame for $\mathcal{X}$ with bounds A and B, respectively. For each $i \in I$, we define $\Lambda_i : \mathcal{X} \to \mathcal{A}$ by $\Lambda_i\xi = \langle \xi, x_i\rangle$, $\forall \xi \in \mathcal{X}$. Λ_i is adjointable and $\Lambda_i^* a = ax_i$ for each $a \in \mathcal{A}$. And we have*

$$A\langle \xi, \xi\rangle A^* \leq \sum_{i\in I}\langle \xi, x_i\rangle\langle x_i, \xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{X}.$$

Or

$$\langle K^*\xi, K^*\xi\rangle \leq \|K\|^2_\infty\langle \xi, \xi\rangle, \forall \xi \in \mathcal{X}.$$

Then

$$\|K\|^{-1}_\infty A\langle K^*\xi, K^*\xi\rangle(\|K\|^{-1}_\infty A)^* \leq \sum_{i\in I}\langle \Lambda_i\xi, \Lambda_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{X}.$$

So $\{\Lambda_i\}_{i\in I}$ is $*$-K-g-frame for $\mathcal{X}$ with bounds $\|K\|^{-1}_\infty A$ and B, respectively.

4. K-Operator Frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$ [17]

Definition 4.1. *Let $\mathcal{X}$ be a Hilbert module over a pro-C^*-algebra $\mathcal{A}$ and let $\{T_i\}_{i\in I}$ be a family of adjointable operators for $\mathcal{X}$. $\{T_i\}_{i\in I}$ is called K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$, if there exists positive constants $A, B > 0$ such that*

$$A\langle K^*\xi, K^*\xi\rangle \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle, \forall \xi \in \mathcal{X}. \tag{4.1}$$

The numbers A and B are called lower and upper bound of the K-operator frame, respectively. If

$$A\langle K^*\xi, K^*\xi\rangle = \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle,$$

the K-operator frame is A-tight. If $A = 1$, it is called a normalized tight K-operator frame or a Parseval K-operator frame.

Example 4.1. *Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbb{N}}, v = \{v_j\}_{j\in\mathbb{N}} \in l^\infty$, we define*

$$uv = \{u_jv_j\}_{j\in\mathbb{N}}, u^* = \{\bar{u}_j\}_{j\in\mathbb{N}}, \|u\| = \sup_{j\in\mathbb{N}}|u_j|.$$

Then $\mathcal{A} = \{l^{\infty}, \|.\|\}$ is a C^*-algebra, as a result $\mathcal{A}$ is pro-C^*-algebra.

Let $\mathcal{X} = C_0$ be the set of all sequences converging to zero. For any $u, v \in \mathcal{X}$ we define

$$\langle u, v\rangle = uv^* = \{u_j \bar{u}_j\}_{j\in\mathrm{N}}.$$

Then $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module.

Now let $\{e_j\}_{j\in\mathrm{N}}$ be the standard orthonormal basis of $\mathcal{X}$. For each $j \in \mathbf{N}$ define the adjointable operator

$$T_j : \mathcal{X} \to \mathcal{X}, \quad T_j\xi = \langle \xi, e_j\rangle e_j,$$

then for every $\xi \in \mathcal{X}$ we have

$$\sum_{j\in\mathrm{N}} \langle T_j\xi, T_j\xi\rangle = \langle \xi, \xi\rangle.$$

Fix $N \in \mathbf{N}^*$ and define

$$K : \mathcal{X} \to \mathcal{X}, \quad Ke_j = \begin{cases} je_j & \text{if } j \leq N, \\ 0 & \text{if } j > N. \end{cases}$$

It is easy to check that K is adjointable and satisfies

$$K^*e_j = \begin{cases} je_j & \text{if } j \leq N, \\ 0 & \text{if } j > N. \end{cases}$$

For any $\xi \in \mathcal{X}$ we have

$$\frac{1}{N^2}\langle K^*\xi, K^*\xi\rangle \leq \sum_{j\in\mathrm{N}} \langle T_j\xi, T_j\xi\rangle = \langle \xi, \xi\rangle.$$

This shows that $\{T_j\}_{j\in\mathrm{N}}$ is a K-operator frame with bounds $\frac{1}{N^2}, 1$.

Theorem 4.1. *For an operator Bessel sequence $\{T_i\}_{i\in I} \subset Hom^*_{\mathcal{A}}(\mathcal{X})$, the following statements are equivalent:*

1. *$\{T_i\}_{i\in I}$ is K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$.*
2. *There exists $A > 0$ such that $S \geq AKK^*$, where S is the frame operator for $\{T_i\}_{i\in I}$.*
3. *$K = S^{\frac{1}{2}}Q$, for some $Q \in Hom^*_{\mathcal{A}}(\mathcal{X})$.*

5. Tensor Product

The tensor product of two C^*-algebras is a fundamental construction in the theory of C^*-algebras. In the context of pro-C^*-algebras, the tensor product can be defined in several ways, depending on the specific category of pro- C^*-algebras being considered.

One common approach is to define the tensor product of two pro- C^*-algebras as the projective limit of the tensor products of the C*-algebras in the systems defining the pro- C^*-algebras. In this way, the tensor product of pro- C^*-algebras can be viewed as a pro-C^*-algebra in a natural way. The minimal or injective tensor product of the pro-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \otimes \mathcal{B}$, is the completion of the algebraic tensor product $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}$ with respect to the topology determined by a family of C^*-seminorms. Suppose that $\mathcal{X}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{A}$ and $\mathcal{Y}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{B}$. The algebraic tensor product $\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$ is a pre-Hilbert $\mathcal{A} \otimes \mathcal{B}$-module with the action of $\mathcal{A} \otimes \mathcal{B}$ on $\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}$ defined by

$$(\xi \otimes \eta)(a \otimes b) = \xi a \otimes \eta b \text{ for all } \xi \in \mathcal{X}, \eta \in \mathcal{Y}, a \in \mathcal{A} \text{ and } b \in \mathcal{B}$$

and the inner product

$$\langle \cdot, \cdot \rangle : \left(\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}\right) \times \left(\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}\right) \to \mathcal{A} \otimes_{\text{alg}} \mathcal{B}. \text{ defined by}$$

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \otimes \langle \eta_1, \eta_2 \rangle$$

We also know that for $z = \sum_{i=1}^{n} \xi_i \otimes \eta_i$ in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ we have $\langle z, z \rangle_{\mathcal{A}\otimes\mathcal{B}} = \sum_{i,j} \langle \xi_i, \xi_j \rangle_{\mathcal{A}} \otimes \langle \eta_i, \eta_j \rangle_{\mathcal{B}} \geq 0$ and $\langle z, z \rangle_{\mathcal{A}\otimes\mathcal{B}} = 0$ iff $z = 0$.

The external tensor product of $\mathcal{X}$ and $\mathcal{Y}$ is the Hilbert module $\mathcal{X} \otimes \mathcal{Y}$ over $\mathcal{A} \otimes \mathcal{B}$ obtained by the completion of the pre-Hilbert $\mathcal{A} \otimes \mathcal{B}$-module $\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}$.

If $P \in M(\mathcal{X})$ and $Q \in M(\mathcal{Y})$ then there is a unique adjointable module morphism $P \otimes Q : \mathcal{A} \otimes \mathcal{B} \to \mathcal{X} \otimes \mathcal{Y}$ such that $(P \otimes Q)(a \otimes b) = P(a) \otimes Q(b)$ and $(P \otimes Q)^*(a \otimes b) = P^*(a) \otimes Q^*(b)$ for all $a \in A$ and for all $b \in B$ (see, for example, [11]).

Let I and J be countable index sets.

Theorem 5.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert pro-C^*-modules over unital pro-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\{T_i\}_{i\in I} \subset Hom^*_{\mathcal{A}}(\mathcal{X})$ be a K_1-operator frame for $\mathcal{X}$ and $\{R_j\}_{j\in J} \subset Hom^*_{\mathcal{B}}(\mathcal{Y})$ be a K_2-operator frame for $\mathcal{Y}$ with frame operators S_T and S_R and operator frame bounds (A, B) and (C, D) respectively. Then $\{T_i \otimes R_j\}_{i\in I, j\in J}$ is a $K_1 \otimes K_2$-operator frame for Hibert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{X} \otimes \mathcal{Y}$ with frame operator $S_T \otimes S_R$ and lower and upper operator frame bounds AC and BD, respectively.*

Proof. By the definition of K_1-operator frame $\{T_i\}_{i\in I}$ and K_2-operator frame $\{R_j\}_{j\in J}$ we have

$$A\langle K_1^*\xi, K_1^*\xi\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle T_i\xi, T_i\xi\rangle_{\mathcal{A}} \leq B\langle \xi, \xi\rangle_{\mathcal{A}}, \forall \xi \in \mathcal{X}.$$

$$C\langle K_2^*\eta, K_2^*\eta\rangle_{\mathcal{B}} \leq \sum_{j\in J} \langle R_j\eta, R_j\eta\rangle_{\mathcal{B}} \leq D\langle \eta, \eta\rangle_{\mathcal{B}}, \forall \eta \in \mathcal{K}.$$

Therefore

$$\begin{aligned}
&(A\langle K_1^*\xi, K_1^*\xi\rangle_{\mathcal{A}}) \otimes (C\langle K_2^*\eta, K_2^*\eta\rangle_{\mathcal{B}}) \\
&\leq \sum_{i\in I} \langle T_i\xi, T_i\xi\rangle_{\mathcal{A}} \otimes \sum_{j\in J} \langle R_j\eta, R_j\eta\rangle_{\mathcal{B}} \\
&\leq (B\langle \xi, \xi\rangle_{\mathcal{A}}) \otimes (D\langle \eta, \eta\rangle_{\mathcal{B}}), \forall \xi \in \mathcal{X}, \forall \eta \in \mathcal{Y}.
\end{aligned}$$

Then

$$\begin{aligned}
&AC((\langle K_1^*\xi, K_1^*\xi\rangle_{\mathcal{A}} \otimes \langle K_2^*\eta, K_2^*\eta\rangle_{\mathcal{B}}) \\
&\leq \sum_{i\in I, j\in J} \langle T_i\xi, T_i\xi\rangle_{\mathcal{A}} \otimes \langle R_j\eta, R_j\eta\rangle_{\mathcal{B}} \\
&\leq BD(\langle \xi, \xi\rangle_{\mathcal{A}} \otimes \langle \eta, \eta\rangle_{\mathcal{B}}), \forall \xi \in \mathcal{X}, \forall \eta \in \mathcal{Y}.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
&AC\langle K_1^*\xi \otimes K_2^*\eta, K_1^*\xi \otimes K_2^*\eta\rangle_{\mathcal{A}\otimes\mathcal{B}} \\
&\leq \sum_{i\in I, j\in J} \langle T_i\xi \otimes R_j\eta, T_i\xi \otimes R_j\eta\rangle_{\mathcal{A}\otimes\mathcal{B}} \\
&\leq BD\langle \xi \otimes \eta, \xi \otimes \eta\rangle_{\mathcal{A}\otimes\mathcal{B}}, \forall \xi \in \mathcal{X}, \forall \eta \in \mathcal{Y}.
\end{aligned}$$

Then for all $\xi \otimes \eta$ in $\mathcal{X} \otimes \mathcal{Y}$ we have

$$\begin{aligned}
&AC\langle (K_1 \otimes K_2)^*(\xi \otimes \eta), (K_1 \otimes K_2)^*(\xi \otimes \eta)\rangle_{\mathcal{A}\otimes\mathcal{B}} \\
&\leq \sum_{i\in I, j\in J} \langle (T_i \otimes R_j)(\xi \otimes \eta), (T_i \otimes R_j)(\xi \otimes \eta)\rangle_{\mathcal{A}\otimes\mathcal{B}} \\
&\leq BD\langle \xi \otimes \eta, \xi \otimes \eta\rangle_{\mathcal{A}\otimes\mathcal{B}}.
\end{aligned}$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ and then it's satisfied for all $z \in \mathcal{X} \otimes \mathcal{Y}$. It shows that $\{T_i \otimes R_j\}_{i\in I, j\in J}$ is a $K_1 \otimes K_2$-operator frame for Hilbert $\mathcal{A} \otimes \mathcal{B}$-module $\mathcal{X} \otimes \mathcal{Y}$ with lower and upper operator frame bounds AC and BD, respectively.
By the definition of frame operator S_T and S_R we have

$$S_T\xi = \sum_{i\in I} T_i^* T_i \xi, \forall \xi \in \mathcal{X}.$$

$$S_R\eta = \sum_{j\in J} R_j^* R_j \eta, \forall \eta \in \mathcal{K}.$$

Therefore

$$\begin{aligned}
(S_T \otimes S_R)(\xi \otimes \eta) &= S_T\xi \otimes S_R\eta \\
&= \sum_{i\in I} T_i^* T_i\xi \otimes \sum_{j\in J} R_j^* R_j \eta \\
&= \sum_{i\in I, j\in J} T_i^* T_i\xi \otimes R_j^* R_j \eta \\
&= \sum_{i\in I, j\in J} (T_i^* \otimes R_j^*)(T_i\xi \otimes R_j\eta)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i\in I, j\in J} (T_i^* \otimes R_j^*)(T_i \otimes R_j)(\xi \otimes \eta) \\
&= \sum_{i\in I, j\in J} (T_i \otimes R_j)^*)(T_i \otimes R_j)(\xi \otimes \eta).
\end{aligned}$$

Now by the uniqueness of frame operator, the last expression is equal to $S_{T\otimes R}(\xi \otimes \eta)$. Consequently we have $(S_T \otimes S_R)(\xi \otimes \eta) = S_{T\otimes R}(\xi \otimes \eta)$. The last equality is satisfied for every finite sum of elements in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ and then it's satisfied for all $z \in \mathcal{X} \otimes \mathcal{Y}$. It shows that $(S_T \otimes S_R)(z) = S_{T\otimes R}(z)$. So $S_{T\otimes R} = S_T \otimes S_R$. □

6. Dual of K-Operator Frame

In the following we define the Dual K-operator frame and we give some properties.

Definition 6.1. *[17]*

Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ and $\{T_i \in Hom^*_{\mathcal{A}}(\mathcal{X}), i \in I\}$ be a K-operator frame for the Hilbert $\mathcal{A}$-module $\mathcal{X}$. An operator Bessel sequences $\{R_i \in Hom^*_{\mathcal{A}}(\mathcal{X}), i \in I\}$ is called a K-dual operator frame for $\{T_i\}_{i\in I}$ if $K\xi = \sum_{i\in I} T_i^* R_i \xi$ for all $\xi \in \mathcal{X}$.

Example 6.1. *Let $K \in Hom_{\mathcal{A}}{}^*(\mathcal{X})$ be a surjective operator and $\{T_i \in Hom^*_{\mathcal{A}}(\mathcal{X}), i \in I\}$ be a K-operator frame for $\mathcal{X}$ with frame operator S, then S is invertible.*

For all $\xi \in \mathcal{X}$ we have :
$S\xi = \sum_{i\in I} T_i^* R_i \xi$.
So $K\xi = \sum_{i\in I} T_i^* R_i S^{-1} K\xi$.

Then the sequence $\{T_i S^{-1} K \in Hom^*_{\mathcal{A}}(\mathcal{X}), i \in I\}$ is a dual K-operator frame of $\{T_i \in Hom^*_{\mathcal{A}}(\mathcal{X}), i \in I\}$

Theorem 6.1. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ be an invertible element such that both are uniformly bounded and $Rang(K)$ is closed, and let $\{T_i\}_{i\in I}$ be K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$ with frame operator S and frame bounds A and B respectively. Then $\{T_i \pi_{S(Rang(K))} \left(S^{-1}_{|Rang(K)}\right)^* K\}$ is a K-dual of $\{T_i\}_{i\in I}$*

Proof. Let $\{T_i\}$ be a K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$. Since $S : Rang(K) \to S(Rang(K))$ is invertible, we have

$$\begin{aligned}
K\xi &= \left(S^{-1}_{|Rang(K)} S_{|Rang(K)}\right)^* K\xi \\
&= S_{|Rang(K)} \left(S^{-1}_{|Rang(K)}\right)^* K\xi \\
&= S\pi_{S(Rang(K))} \left(S^{-1}_{|Rang(K)}\right)^* K\xi \\
&= \sum_{i\in I} T_i^* T_i \pi_{S(Rang(K))} \left(S^{-1}_{|Rang(K)}\right)^* K\xi, \text{ for all } \xi \in \mathcal{X}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\sum_{i\in I} \langle T_i \pi_{S(Rang(K))} \left(S^{-1}\right)^* K\xi, T_i \pi_{S(Rang(K))} \left(S^{-1}\right)^* K\xi \rangle \\
&= \sum_{i\in I} \langle T_i^* T_i \pi_{S(Rang(K))} \left(S^{-1}\right)^* K\xi, \left(S^{-1}\right)^* K\xi \rangle \\
&= \left\langle S \left(S^{-1}\right)^* K\xi, \left(S^{-1}\right)^* K\xi \right\rangle \\
&= \left\langle K\xi, \left(S^{-1}\right)^* K\xi \right\rangle \\
&\leq A^{-1} \|K^{-1}\|^2_\infty \|K\|^2_\infty \langle \xi, \xi \rangle, \xi \in \mathcal{X}
\end{aligned}$$

Hence $\left\{T_i \pi_{Rang(K)} \left(S^{-1}\right)^* K\right\}$ is a dual of the K-operator frame $\{T_i\}$. □

7. $*$-Operator Frame for $Hom_{\mathcal{A}}^{*}(\mathcal{X})$

Definition 7.1. *A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{X}$ over a pro-C^*-algebra is said to be an $*$-operator frame for $Hom_{\mathcal{A}}^{*}(\mathcal{X})$, if there exists two strictly nonzero elements A and B in $\mathcal{A}$ such that*

$$A\langle \xi, \xi\rangle A^* \le \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \le B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{X}. \tag{7.1}$$

The elements A and B are called lower and upper bounds of the $*$-operator frame, respectively. If $A = B = \lambda$, the $*$-operator frame is λ-tight. If $A = B = 1_{\mathcal{A}}$, it is called a normalized tight $*$-operator frame or a Parseval $*$-operator frame. If only upper inequality of (7.1) hold, then $\{T_i\}_{i\in i}$ is called an $*$-operator Bessel sequence for $Hom_{\mathcal{A}}^{*}(\mathcal{X})$.

Example 7.1. *Let $\mathcal{A}$ be a Hilbert pro-C^*-module over itself with the inner product $\langle a, b\rangle = ab^*$. Let $\{\xi_i\}_{i\in I}$ be an $*$-frame for $\mathcal{A}$ with bounds A and B, respectively. For each $i \in I$, we define $T_i : \mathcal{A} \to \mathcal{A}$ by $T_i\xi = \langle \xi, \xi_i\rangle$, $\forall \xi \in \mathcal{A}$. T_i is adjointable and $T_i^* a = a\xi_i$ for each $a \in \mathcal{A}$. And we have*

$$A\langle \xi, \xi\rangle A^* \le \sum_{i\in I}\langle \xi, \xi_i\rangle\langle \xi_i, \xi\rangle \le B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{A}.$$

Then

$$A\langle \xi, \xi\rangle A^* \le \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \le B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{A}.$$

So $\{T_i\}_{i\in I}$ is an $*$-operator frame in $\mathcal{A}$ with bounds A and B, respectively.

Theorem 7.1. *Let $\{T_i\}_{i\in I} \subset Hom_{\mathcal{A}}^{*}(\mathcal{X})$ be an $*$-operator frame with lower and upper bounds A and B, respectively. The $*$-operator frame transform $R : \mathcal{X} \to l^2(\mathcal{X})$ defined by $R\xi = \{T_i\xi\}_{i\in I}$ is injective and closed range adjointable $\mathcal{A}$-module map and $\bar{p}_{\mathcal{X}}(R) \le \bar{p}_{\mathcal{X}}(B)$. The adjoint operator R^* is surjective and it is given by $R^*(\{\xi_i\}_{i\in I}) = \sum_{i\in I} T_i^*\xi_i$ for all $\{\xi_i\}_{i\in I}$ in $l^2(\mathcal{X})$.*

Proof. By the definition of norm in $l^2(\mathcal{X})$

$$\bar{p}_{\mathcal{X}}(R\xi)^2 = p(\sum_{i\in I}\langle T_i\xi, T_i\xi\rangle) \le \bar{p}_{\mathcal{X}}(B)^2 p(\langle \xi, \xi\rangle), \forall \xi \in \mathcal{X}. \tag{7.2}$$

This inequality implies that R is well defined and $\bar{p}_{\mathcal{X}}(R) \le \bar{p}_{\mathcal{X}}(B)$. Clearly, R is a linear $\mathcal{A}$-module map. We now show that the range of R is closed. Let $\{R\xi_n\}_{n\in\mathbb{N}}$ be a sequence in the range of R such that $\lim_{n\to\infty} R\xi_n = \eta$. For $n, m \in \mathbb{N}$, we have

$$p(A\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle A^*) \le p(\langle R(\xi_n - \xi_m), R(\xi_n - \xi_m)\rangle) = \bar{p}_{\mathcal{X}}(R(\xi_n - \xi_m))^2.$$

Since $\{R\xi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathcal{X}$, then

$p(A\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle A^*) \to 0$, as $n, m \to \infty$.

Note that for $n, m \in \mathbb{N}$,

$$p(\langle \xi_n - \xi_m, \xi_n - \xi_m \rangle) = p(A^{-1}A\langle \xi_n - \xi_m, \xi_n - \xi_m \rangle A^*(A^*)^{-1})$$
$$\leq p(A^{-1})^2 p(A\langle \xi_n - \xi_m, \xi_n - \xi_m \rangle A^*).$$

Therefore the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is Cauchy and hence there exists $\xi \in \mathcal{X}$ such that $\xi_n \to \xi$ as $n \to \infty$. Again by (9.2), we have

$$\bar{p}_{\mathcal{X}}(R(\xi_n - \xi_m))^2 \leq \bar{p}_{\mathcal{X}}(B)^2 p(\langle \xi_n - \xi, \xi_n - \xi \rangle).$$

Thus $p(R\xi_n - R\xi) \to 0$ as $n \to \infty$ implies that $R\xi = \eta$. It concludes that the range of R is closed. Next we show that R is injective. Suppose that $\xi \in \mathcal{X}$ and $R\xi = 0$. Note that $A\langle \xi, \xi \rangle A^* \leq \langle R\xi, R\xi \rangle$ then $\langle \xi, \xi \rangle = 0$ so $\xi = 0$ i.e. R is injective.

For $\xi \in \mathcal{X}$ and $\{\xi_i\}_{i\in I} \in l^2(\mathcal{X})$ we have

$$\langle R\xi, \{\xi_i\}_{i\in I} \rangle = \langle \{T_i\xi\}_{i\in I}, \{\xi_i\}_{i\in I} \rangle = \sum_{i\in I} \langle T_i\xi, \xi_i \rangle = \sum_{i\in I} \langle \xi, T_i^*\xi_i \rangle = \langle \xi, \sum_{i\in I} T_i^*\xi_i \rangle.$$

Then $R^*(\{\xi_i\}_{i\in I}) = \sum_{i\in I} T_i^*\xi_i$. By injectivity of R, the operator R^* has closed range and $\mathcal{X} = range(R^*)$, which completes the proof. □

Theorem 7.2. *Let $(\mathcal{X}, \mathcal{A}, \langle .,. \rangle_{\mathcal{A}})$ and $(\mathcal{X}, \mathcal{B}, \langle .,. \rangle_{\mathcal{B}})$ be two Hilbert pro-$\mathcal{C}^*$-modules and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $*$-homomorphism and θ be a map on $\mathcal{X}$ such that $\langle \theta\xi, \theta\eta \rangle_{\mathcal{B}} = \varphi(\langle \xi, \eta \rangle_{\mathcal{A}})$ for all $\xi, \eta \in \mathcal{X}$. Also, suppose that $\{T_i\}_{i\in I} \subset Hom^*_{\mathcal{A}}(\mathcal{X})$ is an $*$-operator frame for $(\mathcal{X}, \mathcal{A}, \langle .,. \rangle_{\mathcal{A}})$ with $*$-frame operator $S_{\mathcal{A}}$ and lower and upper $*$-operator frame bounds A, B respectively. If θ is surjective and $\theta T_i = T_i\theta$ for each i in I, then $\{T_i\}_{i\in I}$ is an $*$-operator frame for $(\mathcal{X}, \mathcal{B}, \langle .,. \rangle_{\mathcal{B}})$ with $*$-frame operator $S_{\mathcal{B}}$ and lower and upper $*$-operator frame bounds $\varphi(A)$, $\varphi(B)$ respectively, and $\langle S_{\mathcal{B}}\theta\xi, \theta\eta \rangle_{\mathcal{B}} = \varphi(\langle S_{\mathcal{A}}\xi, \eta \rangle_{\mathcal{A}})$.*

Proof. Let $\eta \in \mathcal{X}$ then there exists $\xi \in \mathcal{X}$ such that $\theta\xi = \eta$ (θ is surjective). By the definition of $*$-operator frames we have

$$A\langle \xi, \xi \rangle_{\mathcal{A}} A^* \leq \sum_{i\in I} \langle T_i\xi, T_i\xi \rangle_{\mathcal{A}} \leq B\langle \xi, \xi \rangle_{\mathcal{A}} B^*.$$

By lemma 3.3 we have

$$\varphi(A\langle \xi, \xi \rangle_{\mathcal{A}} A^*) \leq \varphi(\sum_{i\in I} \langle T_i\xi, T_i\xi \rangle_{\mathcal{A}}) \leq \varphi(B\langle \xi, \xi \rangle_{\mathcal{A}} B^*).$$

By the definition of $*$-homomorphism we have

$$\varphi(A)\varphi(\langle \xi, \xi \rangle_{\mathcal{A}})\varphi(A^*) \leq \sum_{i\in I} \varphi(\langle T_i\xi, T_i\xi \rangle_{\mathcal{A}}) \leq \varphi(B)\varphi(\langle \xi, \xi \rangle_{\mathcal{A}})\varphi(B^*).$$

By the relation betwen θ and φ we get

$$\varphi(A)\langle \theta\xi, \theta\xi \rangle_{\mathcal{B}} \varphi(A)^* \leq \sum_{i\in I} \langle \theta T_i\xi, \theta T_i\xi \rangle_{\mathcal{B}} \leq \varphi(B)\langle \theta\xi, \theta\xi \rangle_{\mathcal{B}} \varphi(B)^*.$$

By the relation betwen θ and T_i we have

$$\varphi(A)\langle\theta\xi,\theta\xi\rangle_{\mathcal{B}}\varphi(A)^* \leq \sum_{i\in I}\langle T_i\theta\xi, T_i\theta\xi\rangle_{\mathcal{B}} \leq \varphi(B)\langle\theta\xi,\theta\xi\rangle_{\mathcal{B}}\varphi(B)^*.$$

Then

$$\varphi(A)\langle\eta,\eta\rangle_{\mathcal{B}}(\varphi(A))^* \leq \sum_{i\in I}\langle T_i\eta, T_i\eta\rangle_{\mathcal{B}} \leq \varphi(B)\langle\eta,\eta\rangle_{\mathcal{B}}(\varphi(B))^*, \forall\eta\in\mathcal{X}.$$

On the other hand we have

$$\begin{aligned}\varphi(\langle S_{\mathcal{A}}\xi,\eta\rangle_{\mathcal{A}}) &= \varphi(\langle\sum_{i\in I} T_i^*T_i\xi,\eta\rangle_{\mathcal{A}})\\ &= \sum_{i\in I}\varphi(\langle T_i\xi, T_i\eta\rangle_{\mathcal{A}})\\ &= \sum_{i\in I}\langle\theta T_i\xi,\theta T_i\eta\rangle_{\mathcal{B}}\\ &= \sum_{i\in I}\langle T_i\theta\xi, T_i\theta\eta\rangle_{\mathcal{B}}\\ &= \langle\sum_{i\in I} T_i^*T_i\theta\xi,\theta\eta\rangle_{\mathcal{B}}\\ &= \langle S_{\mathcal{B}}\theta\xi,\theta\eta\rangle_{\mathcal{B}}.\end{aligned}$$

Which completes the proof. □

8. Tensor Product

The minimal or injective tensor product of the pro-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A}\otimes\mathcal{B}$, is the completion of the algebraic tensor product $\mathcal{A}\otimes_{\text{alg}}\mathcal{B}$ with respect to the topology determined by a family of C^*-seminorms. Suppose that $\mathcal{X}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{A}$ and $\mathcal{Y}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{B}$. The algebraic tensor product $\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$ is a pre-Hilbert $\mathcal{A}\otimes\mathcal{B}$-module with the action of $\mathcal{A}\otimes\mathcal{B}$ on $\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}$ defined by

$$(\xi\otimes\eta)(a\otimes b) = \xi a\otimes\eta b \text{ for all } \xi\in\mathcal{X}, \eta\in\mathcal{Y}, a\in\mathcal{A} \text{ and } b\in\mathcal{B}$$

and the inner product

$$\langle\cdot,\cdot\rangle : \left(\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}\right)\times\left(\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}\right)\to\mathcal{A}\otimes_{\text{alg}}\mathcal{B}. \text{ defined by}$$

$$\langle\xi_1\otimes\eta_1,\xi_2\otimes\eta_2\rangle = \langle\xi_1,\xi_2\rangle\otimes\langle\eta_1,\eta_2\rangle$$

We also know that for $z=\sum_{i=1}^n\xi_i\otimes\eta_i$ in $\mathcal{X}\otimes_{alg}\mathcal{Y}$ we have $\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}} = \sum_{i,j}\langle\xi_i,\xi_j\rangle_{\mathcal{A}}\otimes\langle\eta_i,\eta_j\rangle_{\mathcal{B}}\geq 0$ and $\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}} = 0$ iff $z=0$.

The external tensor product of $\mathcal{X}$ and $\mathcal{Y}$ is the Hilbert module $\mathcal{X}\otimes\mathcal{Y}$ over $\mathcal{A}\otimes\mathcal{B}$ obtained by the completion of the pre-Hilbert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}$.

If $P\in M(\mathcal{X})$ and $Q\in M(\mathcal{Y})$ then there is a unique adjointable module morphism $P\otimes Q:\mathcal{A}\otimes\mathcal{B}\to\mathcal{X}\otimes\mathcal{Y}$ such that $(P\otimes Q)(a\otimes b) = P(a)\otimes Q(b)$ and $(P\otimes Q)^*(a\otimes b) = P^*(a)\otimes Q^*(b)$ for all $a\in A$ and for all $b\in B$ (see, for example, cite The minimal or injective tensor product of the pro-C^*-algebras

$\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A}\otimes\mathcal{B}$, is the completion of the algebraic tensor product $\mathcal{A}\otimes_{\text{alg}}\mathcal{B}$ with respect to the topology determined by a family of C^*-seminorms. Suppose that $\mathcal{X}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{A}$ and $\mathcal{Y}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{B}$. The algebraic tensor product $\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$ is a pre-Hilbert $\mathcal{A}\otimes\mathcal{B}$-module with the action of $\mathcal{A}\otimes\mathcal{B}$ on $\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}$ defined by

$$(\xi\otimes\eta)(a\otimes b)=\xi a\otimes\eta b \text{ for all } \xi\in\mathcal{X},\eta\in\mathcal{Y}, a\in\mathcal{A} \text{ and } b\in\mathcal{B}$$

and the inner product

$$\langle\cdot,\cdot\rangle:\left(\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}\right)\times\left(\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}\right)\to\mathcal{A}\otimes_{\text{alg}}\mathcal{B}. \text{ defined by}$$

$$\langle\xi_1\otimes\eta_1,\xi_2\otimes\eta_2\rangle=\langle\xi_1,\xi_2\rangle\otimes\langle\eta_1,\eta_2\rangle$$

We also know that for $z=\sum_{i=1}^{n}\xi_i\otimes\eta_i$ in $\mathcal{X}\otimes_{alg}\mathcal{Y}$ we have $\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}}=\sum_{i,j}\langle\xi_i,\xi_j\rangle_{\mathcal{A}}\otimes\langle\eta_i,\eta_j\rangle_{\mathcal{B}}\geq 0$ and $\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}}=0$ iff $z=0$.
The external tensor product of $\mathcal{X}$ and $\mathcal{Y}$ is the Hilbert module $\mathcal{X}\otimes\mathcal{Y}$ over $\mathcal{A}\otimes\mathcal{B}$ obtained by the completion of the pre-Hilbert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}$.

If $P\in M(\mathcal{X})$ and $Q\in M(\mathcal{Y})$ then there is a unique adjointable module morphism $P\otimes Q:\mathcal{A}\otimes\mathcal{B}\to\mathcal{X}\otimes\mathcal{Y}$ such that $(P\otimes Q)(a\otimes b)=P(a)\otimes Q(b)$ and $(P\otimes Q)^*(a\otimes b)=P^*(a)\otimes Q^*(b)$ for all $a\in A$ and for all $b\in B$ (see, for example, [11]) Let I and J be countable index sets.

Theorem 8.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert pro-C^*-modules over pro-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\{T_i\}_{i\in I}\subset Hom^*_{\mathcal{A}}(\mathcal{X})$ and $\{L_j\}_{j\in J}\subset Hom^*_{\mathcal{B}}(\mathcal{Y})$ be two $*$-operator frames for $\mathcal{X}$ and $\mathcal{Y}$ with $*$-frame operators S_T and S_L and $*$-operator frame bounds (A,B) and (C,D) respectively. Then $\{T_i\otimes L_j\}_{i\in I,j\in J}$ is an $*$-operator frame for Hibert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{X}\otimes\mathcal{Y}$ with $*$-frame operator $S_T\otimes S_L$ and lower and upper $*$-operator frame bounds $A\otimes C$ and $B\otimes D$, respectively.*

Proof. By the definition of $*$-operator frames $\{T_i\}_{i\in I}$ and $\{L_j\}_{j\in J}$ we have

$$A\langle\xi,\xi\rangle_{\mathcal{A}}A^*\leq\sum_{i\in I}\langle T_i\xi,T_i\xi\rangle_{\mathcal{A}}\leq B\langle\xi,\xi\rangle_{\mathcal{A}}B^*,\forall\xi\in\mathcal{X},$$

and

$$C\langle\eta,\eta\rangle_{\mathcal{B}}C^*\leq\sum_{j\in J}\langle L_j\eta,L_j\eta\rangle_{\mathcal{B}}\leq D\langle\eta,\eta\rangle_{\mathcal{B}}D^*,\forall\eta\in\mathcal{Y}.$$

Therefore

$$\begin{aligned}&(A\langle\xi,\xi\rangle_{\mathcal{A}}A^*)\otimes(C\langle\eta,\eta\rangle_{\mathcal{B}}C^*)\\&\leq\sum_{i\in I}\langle T_i\xi,T_i\xi\rangle_{\mathcal{A}}\otimes\sum_{j\in J}\langle L_j\eta,L_j\eta\rangle_{\mathcal{B}}\\&\leq(B\langle\xi,\xi\rangle_{\mathcal{A}}B^*)\otimes(D\langle\eta,\eta\rangle_{\mathcal{B}}D^*),\forall\xi\in\mathcal{X},\forall\eta\in\mathcal{Y}.\end{aligned}$$

Then

$$(A \otimes C)(\langle \xi, \xi \rangle_{\mathcal{A}} \otimes \langle \eta, \eta \rangle_{\mathcal{B}})(A^* \otimes C^*)$$
$$\leq \sum_{i \in I, j \in J} \langle T_i \xi, T_i \xi \rangle_{\mathcal{A}} \otimes \langle L_j \eta, L_j \eta \rangle_{\mathcal{B}}$$
$$\leq (B \otimes D)(\langle \xi, \xi \rangle_{\mathcal{A}} \otimes \langle \eta, \eta \rangle_{\mathcal{B}})(B^* \otimes D^*), \forall \xi \in \mathcal{X}, \forall \eta \in \mathcal{Y}.$$

Consequently we have

$$(A \otimes C)\langle \xi \otimes \eta, \xi \otimes \eta \rangle_{\mathcal{A} \otimes \mathcal{B}}(A \otimes C)^*$$
$$\leq \sum_{i \in I, j \in J} \langle T_i \xi \otimes L_j \eta, T_i \xi \otimes L_j \eta \rangle_{\mathcal{A} \otimes \mathcal{B}}$$
$$\leq (B \otimes D)\langle \xi \otimes \eta, \xi \otimes \eta \rangle_{\mathcal{A} \otimes \mathcal{B}}(B \otimes D)^*, \forall \xi \in \mathcal{X}, \forall \eta \in \mathcal{Y}.$$

Then for all $\xi \otimes \eta \in \mathcal{X} \otimes \mathcal{Y}$ we have

$$(A \otimes C)\langle \xi \otimes \eta, \xi \otimes \eta \rangle_{\mathcal{A} \otimes \mathcal{B}}(A \otimes C)^*$$
$$\leq \sum_{i \in I, j \in J} \langle (T_i \otimes L_j)(\xi \otimes \eta), (T_i \otimes L_j)(\xi \otimes \eta) \rangle_{\mathcal{A} \otimes \mathcal{B}}$$
$$\leq (B \otimes D)\langle \xi \otimes \eta, \xi \otimes \eta \rangle_{\mathcal{A} \otimes \mathcal{B}}(B \otimes D)^*.$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ and then it's satisfied for all $z \in \mathcal{X} \otimes \mathcal{Y}$. It shows that $\{T_i \otimes L_j\}_{i \in I, j \in J}$ is $*$-operator frame for Hibert $\mathcal{A} \otimes \mathcal{B}$-module $\mathcal{X} \otimes \mathcal{Y}$ with lower and upper $*$-operator frame bounds $A \otimes C$ and $B \otimes D$, respectively.

By the definition of $*$-frame operator S_T and S_L we have:

$$S_T \xi = \sum_{i \in I} T_i^* T_i \xi, \forall \xi \in \mathcal{X},$$

and

$$S_L \eta = \sum_{j \in J} L_j^* L_j \eta, \forall \eta \in \mathcal{Y}.$$

Therefore

$$\begin{aligned}
(S_T \otimes S_L)(\xi \otimes \eta) &= S_T \xi \otimes S_L \eta \\
&= \sum_{i \in I} T_i^* T_i \xi \otimes \sum_{j \in J} L_j^* L_j \eta \\
&= \sum_{i \in I, j \in J} T_i^* T_i \xi \otimes L_j^* L_j \eta \\
&= \sum_{i \in I, j \in J} (T_i^* \otimes L_j^*)(T_i \xi \otimes L_j \eta) \\
&= \sum_{i \in I, j \in J} (T_i^* \otimes L_j^*)(T_i \otimes L_j)(\xi \otimes \eta) \\
&= \sum_{i \in I, j \in J} (T_i \otimes L_j)^*)(L_i \otimes L_j)(\xi \otimes \eta).
\end{aligned}$$

Now by the uniqueness of $*$-frame operator, the last expression is equal to $S_{T \otimes L}(\xi \otimes \eta)$. Consequently we have $(S_T \otimes S_L)(\xi \otimes \eta) = S_{T \otimes L}(\xi \otimes \eta)$. The last equality is satisfied for every finite sum of elements in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ and then it's satisfied for all $z \in \mathcal{X} \otimes \mathcal{Y}$. It shows that $(S_T \otimes S_L)(z) = S_{T\otimes L}(z)$. So $S_{T\otimes L} = S_T \otimes S_L$. □

9. $*$-K-Operator Frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$

Definition 9.1. *[18] Let $\{T_i\}_{i\in I}$ be a family of adjointable operators on a Hilbert $\mathcal{A}$-module $\mathcal{X}$ over a unital pro-C^*-algebra, and let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$. $\{T_i\}_{i\in I}$ is called a $*$-K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{H})$, if there exists two nonzero elements A and B in $\mathcal{A}$ such that*

$$A\langle K^*\xi, K^*\xi\rangle A^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{X}. \tag{9.1}$$

The elements A and B are called lower and upper bounds of the $*$-K-operator frame, respectively. If

$$A\langle K^*\xi, K^*\xi\rangle^* = \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle,$$

the $*$-K-operator frame is an A-tight. If $A = 1$, it is called a normalized tight $*$-K-operator frame or a Parseval $*$-K-operator frame.

Example 9.1. *Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in \mathrm{N}}, v = \{v_j\}_{j\in \mathrm{N}} \in l^\infty$, we define*

$$uv = \{u_j v_j\}_{j\in \mathrm{N}}, u^* = \{\bar{u}_j\}_{j\in \mathrm{N}}, \|u\| = \sup_{j\in \mathrm{N}} |u_j|.$$

Then $\mathcal{A} = \{l^\infty, \|.\|\}$ is a $\mathbb{C}^*$-algebra. Then $\mathcal{A}$ is pro-$\mathbb{C}^*$-algebra.

Let $\mathcal{X} = C_0$ be the set of all null sequences. For any $u, v \in \mathcal{X}$ we define

$$\langle u, v\rangle = uv^* = \{u_j \bar{u}_j\}_{j\in \mathrm{N}}.$$

Therefore $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module.

Define $f_j = \{f_i^j\}_{i\in \mathrm{N}^*}$ by $f_i^j = \frac{1}{2} + \frac{1}{i}$ if $i = j$ and $f_i^j = 0$ if $i \neq j$ $\forall j \in \mathbf{N}^*$.

Now define the adjointable operator $T_j : \mathcal{X} \to \mathcal{X}$, $T_j\{(\xi_i)_i\} = (\xi_i f_i^j)_i$.

Then for every $x \in \mathcal{X}$ we have

$$\sum_{j\in \mathrm{N}}\langle T_j\xi, T_j\xi\rangle = \{\frac{1}{2} + \frac{1}{i}\}_{i\in \mathrm{N}^*}\langle \xi, \xi\rangle\{\frac{1}{2} + \frac{1}{i}\}_{i\in \mathrm{N}^*}.$$

So $\{T_j\}_j$ is a $\{\frac{1}{2} + \frac{1}{i}\}_{i\in \mathrm{N}^*}$-tight $*$-operator frame.

Let $K : \mathcal{H} \to \mathcal{H}$ defined by $K\xi = \{\frac{\xi_i}{i}\}_{i\in \mathrm{N}^*}$.

Then for every $\xi \in \mathcal{X}$ we have

$$\langle K^*\xi, K^*\xi\rangle \leq \sum_{j\in \mathrm{N}}\langle T_j\xi, T_j\xi\rangle = \{\frac{1}{2} + \frac{1}{i}\}_{i\in \mathrm{N}^*}\langle \xi, \xi\rangle\{\frac{1}{2} + \frac{1}{i}\}_{i\in \mathrm{N}^*}.$$

This shows that $\{T_j\}_{j\in \mathrm{N}}$ is an $*$-K-operator frame with bounds $1, \{\frac{1}{2} + \frac{1}{i}\}_{i\in \mathrm{N}^*}$.

Remark 9.1.

(1) *Every $*$-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$ is an $*$-K-operator frame, for any $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$: $K \neq 0$.*
(2) *If $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ is a surjective operator, then every $*$-K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$ is an $*$-operator frame.*

Example 9.2. *Let $\mathcal{X}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module. $Hom^*_{\mathcal{A}}(\mathcal{X})$. Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ an invertible element such that both are uniformly bounded and $K \neq 0$. Let $\{T_i\}_{i\in I}$ be an $*$-operator frame for $\mathcal{X}$ with bounds A and B, respectively. We have*

$$A\langle \xi, \xi\rangle A^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{X}.$$

Or

$$\langle K^*\xi, K^*\xi\rangle \leq \|K\|^2_\infty\langle \xi, \xi\rangle, \forall \xi \in \mathcal{X}.$$

Then

$$\|K\|^{-1}_\infty A\langle K^*\xi, K^*\xi\rangle(\|K\|^{-1}_\infty A)^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{X}.$$

So $\{T_i\}_{i\in I}$ is $*$-K-operator frame for $\mathcal{X}$ with bounds $\|K\|^{-1}_\infty A$ and B, respectively.

Theorem 9.1. *Let K be a surjective operators in $Hom^*_{\mathcal{A}}(\mathcal{X})$. If $\{T_i\}_{i\in I}$ is an $*$-K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$, then the frame operator S is positive, invertible and adjointable. In addition we have the reconstruction formula, $\xi = \sum_{i\in I} T_i^*T_iS^{-1}\xi$, $\forall \xi \in \mathcal{X}$.*

Proof. We start by showing that, S is a self-adjoint operator. By definition we have $\forall \xi, \eta \in \mathcal{H}$

$$\begin{aligned}
\langle S\xi, \eta\rangle &= \left\langle \sum_{i\in I} T_i^*T_i\xi, \eta\right\rangle \\
&= \sum_{i\in I}\langle T_i^*T_i\xi, \eta\rangle \\
&= \sum_{i\in I}\langle \xi, T_i^*T_i\eta\rangle \\
&= \left\langle \xi, \sum_{i\in I} T_i^*T_i\eta\right\rangle \\
&= \langle \xi, S\eta\rangle.
\end{aligned}$$

Then S is a selfadjoint.

The operator S is clearly positive.

By (2) in Remark 9.1 $\{T_i\}_{i\in I}$ is an $*$-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$.

The definition of an $*$-operator gives

$$A_1\langle \xi, \xi\rangle A_1^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*.$$

Thus by the definition of norm in $l^2(\mathcal{X})$

$$\bar{p}_{\mathcal{X}}(R\xi)^2 = \bar{p}_{\mathcal{X}}(\sum_{i\in I}\langle T_i\xi, T_i\xi\rangle) \leq \bar{p}_{\mathcal{X}}(B)^2 p(\langle\xi,\xi\rangle), \forall\xi\in\mathcal{X}. \tag{9.2}$$

Therefore R is well defined and $\bar{p}_{\mathcal{X}}(R) \leq \bar{p}_{\mathcal{X}}(B)$. It's clear that R is a linear $\mathcal{A}$-module map. We will then show that the range of R is closed. Let $\{R\xi_n\}_{n\in\mathbb{N}}$ be a sequence in the range of R such that $\lim_{n\to\infty} R\xi_n = \eta$. For $n, m \in \mathbb{N}$, we have

$$p(A\langle\xi_n - \xi_m, \xi_n - \xi_m\rangle A^*) \leq p(\langle R(\xi_n - \xi_m), R(\xi_n - \xi_m)\rangle) = \bar{p}_{\mathcal{X}}(R(\xi_n - \xi_m))^2.$$

Seeing that $\{R\xi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathcal{X}$, then

$p(A\langle\xi_n - \xi_m, \xi_n - \xi_m\rangle A^*) \to 0$, as $n, m \to \infty$.

Note that for $n, m \in \mathbb{N}$,

$$\begin{aligned} p(\langle\xi_n - \xi_m, \xi_n - \xi_m\rangle) &= p(A^{-1}A\langle\xi_n - \xi_m, \xi_n - \xi_m\rangle A^*(A^*)^{-1}) \\ &\leq p(A^{-1})^2 p(A\langle\xi_n - \xi_m, \xi_n - \xi_m\rangle A^*). \end{aligned}$$

Thus the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is Cauchy and hence there exists $\xi \in \mathcal{X}$ such that $\xi_n \to \xi$ as $n \to \infty$. Again by (9.2), we have

$$\bar{p}_{\mathcal{X}}(R(\xi_n - \xi_m))^2 \leq \bar{p}_{\mathcal{X}}(B)^2 p(\langle\xi_n - \xi, \xi_n - \xi\rangle).$$

Thus $p(R\xi_n - R\xi) \to 0$ as $n \to \infty$ implies that $R\xi = \eta$. It is therefore concluded that the range of R is closed. We now show that R is injective. Let $\xi \in \mathcal{X}$ and $R\xi = 0$. Note that $A\langle\xi,\xi\rangle A^* \leq \langle R\xi, R\xi\rangle$ then $\langle\xi,\xi\rangle = 0$ so $\xi = 0$ i.e. R is injective.

For $\xi \in \mathcal{X}$ and $\{\xi_i\}_{i\in I} \in l^2(\mathcal{X})$ we have

$$\langle R\xi, \{\xi_i\}_{i\in I}\rangle = \langle\{T_i\xi\}_{i\in I}, \{\xi_i\}_{i\in I}\rangle = \sum_{i\in I}\langle T_i\xi, \xi_i\rangle = \sum_{i\in I}\langle\xi, T_i^*\xi_i\rangle = \langle\xi, \sum_{i\in I} T_i^*\xi_i\rangle.$$

Then $R^*(\{\xi_i\}_{i\in I}) = \sum_{i\in I} T_i^*\xi_i$. Since R is injective, then the operator R^* has closed range and $\mathcal{X} = range(R^*)$, therefore $S = R^*R$ is invertible

□

Let $K \in Hom_{\mathcal{A}}^*(\mathcal{X})$, in the following theorem we constructed an $*$-K-operator frame by using an $*$-operator frame.

Theorem 9.2. *Let $\{T_i\}_{i\in I}$ be an $*$-K-operator frame in $\mathcal{X}$ with bounds A, B and $K \in Hom_{\mathcal{A}}^*(\mathcal{X})$ be an invertible element such that both are uniformly bounded. Then $\{T_iK\}_{i\in I}$ is an $*$-K^*-operator frame in $\mathcal{X}$ with bounds A, $\|K\|_\infty B$. The frame operator of $\{T_iK\}_{i\in I}$ is $S' = K^*SK$, where S is the frame operator of $\{T_i\}_{i\in I}$.*

Proof. From

$$A\langle\xi,\xi\rangle A^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle\xi,\xi\rangle B^*, \forall\xi\in\mathcal{X}.$$

We get for all $\xi \in \mathcal{X}$,

$$A\langle K\xi, K\xi\rangle A^* \leq \sum_{i\in I}\langle T_iK\xi, T_iK\xi\rangle \leq B\langle K\xi, K\xi\rangle B^* \leq \|K\|_\infty B\langle \xi, \xi\rangle(\|K\|_\infty B)^*.$$

Then $\{T_iK\}_{i\in I}$ is an $*$-K^*-operator frame in $\mathcal{X}$ with bounds A, $\|K\|_\infty B$.

By definition of S,we have $SK\xi = \sum_{i\in I} T_i^*T_iK\xi$. Then

$$K^*SK = K^*\sum_{i\in I} T_i^*T_iK\xi = \sum_{i\in I} K^*T_i^*T_iK\xi.$$

Hence $S^{'} = K^*SK$. □

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