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Weighted Polynomial Approximation Error, Szegö Curve and Growth Parameters of Analytic Functions

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Abstract. The Szegö curve is denoted by $S_{R_o} = \{z \in \mathbf{C} : |ze^{1-z}| = R_o,\ |z| \leq 1\}$ and let H_R be the class of functions analytic in G_R but not in $G_{R'}$ if $R < R'$, $G_{R_o} = \text{int } S_{R_o},\ 0 < R_o < R < 1$. In this paper we have studied growth parameters in terms of weighted polynomial approximation errors on S_{R_o} for the functions $f \in H_R$ having rapidly increasing maximum modulus so that the order of $f(z)$ is infinite.

1. Introduction

The problem evolved from Lorentz's approximation by incomplete polynomials on the real line was discussed by G.G. Lorentz [12] and has been developed into the general theory of approximation with varying weights. It is noted that the weighted approximation of functions in the complex plane has not studied so extensively in comparison to study on real line in the current approximation theory literature with the exception of the papers by Borwein and Chen [3], Pritsker and Varga ([15], [16]) and Kumar and Basu [10]. Therefore, in this paper, we tried to bridge this gap and obtained some new results concerning the growth parameters of analytic functions in terms of weighted polynomials $\{e^{-nz}P_n(z)\}_{n=0}^{\infty}$ approximation errors.

The normalized partial sum $s_n(nz)$ introduced by Szegö [22] satisfies the following equation:

$$e^{-nz}s_n(nz) = 1 - \frac{\sqrt{n}}{\tau_n\sqrt{2\pi}}\int_0^z \left(\zeta e^{1-\zeta}\right)^n d\zeta, \quad n \geq 1, \quad z \in \mathbf{C}, \tag{1.1}$$

where $s_n(z) = \sum_{k=0}^{n}(z^k/k!)$ and from Sterling's asymptotic series formula, see Henrici [4]

$$\tau_n = \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \simeq 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots,\ n \to \infty$$

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so that $\tau_n \to 1$ as $n \to \infty$.

The curve $\{z \in \mathbf{C} : |\phi(z)| = 1\}, \phi(z) = ze^{1-z}$ introduced by Szegö divides the complex plane $\mathbf{C}$ into three domains: One of them is the bounded domain G contained in the unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$ whose boundary consists of that part of the Szegö curve, *i.e.*,

$$S = \{z \in \mathbf{C} : |ze^{1-z}| = 1,\ |z| \leq 1\}$$

which is contained in the closed unit disc. The Szegö curve S is a piecewise analytic Jordan curve with one corner point at $z = 1$. It is proved in Szegö [22] that G, in the z-plane is mapped conformally onto the unit disc D, in the w-plane, by the function $w = \phi(z)$, and that the unbounded domains, also determined from the Szegö curve, are given by $\Omega_o = \{z : |\phi(z)| < 1,\ |z| > 1\}$ and $\Omega_\infty = \{z : |\phi(z)| > 1\}$.

For each R_o with $0 < R_o \leq 1$, the set

$$S_{R_o} = \{z \in \mathbf{C} : |\phi(z)| = R_o,\ |z| \leq 1,\ 0 < R_o \leq 1\}$$

is an associated level curve of the mapping ϕ. Clearly, $S_{R_o} \subset G$ for any R_o with $0 < R_o < 1$, and $S_1 = S$. Also, we have $G_{R_o} = int\ S_{R_o}$, where $int\ S_{R_o}$ means the interior points of the curve S_{R_o} and $G_1 = G$.

Let

$$E_n^w\left(f, \overline{G}_{R_o}\right) = \inf_{P_n \in \pi_n} \left\| e^{-nz} P_n(z) - f(z) \right\|_{S_{R_o}}, \tag{1.2}$$

be the error of the best weighted approximation on S_{R_o}, or equivalently, $\overline{G}_{R_o}$ for a function f analytic in G_R, where $0 < R_o < R < 1$ and π_n is the set of all polynomials of degree $\leq n$.

For classifying analytic functions by their growth, the concept of order was introduced. If the order is a (finite) positive number, then the concept of type permits a subclassification. For the cases of order $\rho = 0$ and $\rho = \infty$ no subclassification is possible. For this particular subclassification the type of f can be defined by using the concept of index-pair (p, q) introduced by Juneja et al. [6]. The concept of index q, the q-order and q-type were introduced by Bajpai et al. [1] in order to obtain a measure of growth of the maximum modulus, when it is rapidly increasing.

Kasana and Kumar [9] have studied the growth parameters in terms of Chebyshev and interpolation errors for entire functions of index-pair (p, q). Rice [18] and Winiarski [25] have obtained these results for $(p, q) = (2, 1)$. Also, Bernstein [2], Juneja [5], Reddy [17], Shah [21] and Varga [23] have studied the rate of decay of these errors for entire functions. All these results do not give any information about the rates of decay of these errors when f is not entire. However, Rizvi and Nautiyal [19] studied the rates of decay of approximation and interpolation errors when f is not entire. But the results contained in [19] do not give any specific information about the growth of $f(z)$ if maximum modulus of $f(z)$ is increasing so rapidly that the order of $f(z)$ is infinite. In the present paper, we have studied growth parameters in terms of weighted polynomial approximation errors defined by (1.2) for the functions having rapidly increasing maximum modulus.

Let H_R be the class of functions analytic in G_R but not in $G_{R'}$ if $R < R'$.

Thus we define the growth parameters for a function $f \in H_R$, $0 < R < 1$ as follows:

A function $f \in H_R$, $0 < R < 1$, will be said to be of q-order $\rho_0(q)$
$(\rho_0(q) < \infty, \rho_0(q-1) = \infty, q = 2, 3, \ldots)$ if

$$\rho_0(q) = \limsup_{r \to R^-} \frac{\log^{[q]} M(r)}{\log(R/(R-r))}. \tag{1.3}$$

In case $0 < \rho_0(q) < \infty$, the q-type $T_0(q) (0 \le T_0(q) < \infty)$ of f is defined as

$$T_0(q) = \limsup_{r \to R^-} \frac{\log^{[q-1]} M(r)}{(R/(R-r))^{\rho_0(q)}}, \tag{1.4}$$

where

$$M(r) \equiv M(r, f) = \max_{z \in G_r} \left| f(z) \right|$$

and

$$\log^{[0]} M(r) = M(r), \log^{[q]} M(r) = \log \log^{[q-1]} M(r).$$

Pritsker and Varga [16] have given a necessary and sufficient condition for the validity of the locally uniform approximation of any function $f(z)$ which is analytic in an open bounded set G^* in the complex plane by weighted polynomials of the form $\{w_*^n(z) P_n(z)\}_{n=0}^{\infty}$, where $w_*(z)$ is analytic and different from zero in G^*. Also, they have generalized the Theorems 3.8 and 4.3 of [15].

It is significant to mention that our main results are different from those of Pritsker and Varga ([15], [16]).

For the weighted normalized partial sums $e^{-nz} s_n(z)$, the following inequality is valid

$$\left| e^{-nz} s_n(z) - 1 \right| \le \frac{4}{\sqrt{2n\pi} |z-1|}, \quad z \in \overline{G} \setminus \{1\}, \quad n \ge 1. \tag{1.5}$$

The detailed proof of (1.5) is available in [15].

In view of (1.5), a consequence of (1.1) is that $e^{-nz} s_n(z)$ converges to $f(z) \equiv 1$, locally uniformly in G, (*i.e.*, uniformly on every compact subset of G). This raises the question of possibility of uniform approximation of any function analytic in G by weighted polynomials $\{e^{-nz} P_n(z)\}$, where P_n is a complex polynomial of degree $\le n$, for each $n \ge 0$.

The harmonic measure at the point $z = 0$ with respect to G is defined as the pre-image of the normalized are-length measure on $\Gamma = \{w \in \mathbf{C} : |w| = 1\}$ under the mapping $w = \phi(z)$, where $\phi(z) = ze^{1-z}$, *i.e.*,

$$w(0, B, G) = m(\phi(B \cap S)) \tag{1.6}$$

for any Borel set $B \subset \mathbf{C}$. Here $dm = d\theta / 2\pi$. From (1.6), note that $w(0, \cdot, G)$ is a unit Borel measure which is supported on S, *i.e.*, $w(0, S, G) = 1$ and supp $w(0, \cdot, G) = S$. For any polynomial $P_n(z)$, the normalized counting measure of its zeros is defined by

$$\nu_n(P_n) = \frac{1}{n} \sum_{P_n(z_i)=0} \delta_{z_i}, \tag{1.7}$$

where δ_z is the unit point mass at z where all zeros are considered with multiplicity.

2. Auxiliary Results

This section contains various results which have been utilized to prove the main theorems.

Lemma 2.1. *Let* (R_o, R) *a pair of numbers with* $0 < R_o < R \leq 1$. *If a function* f *is analytic in* G_R, *then there exists a sequence of polynomials* $\{P_n(z)\}$ *such that*

$$\begin{aligned} E_n^w(f, \overline{G}_{R_o}) &\leq \left\| f(z) - e^{-nz} P_n(z) \right\|_{\overline{G}_{R_o}} \leq K'' \frac{1}{R - \varepsilon - R_0} M(R - \varepsilon, f) \\ & (R_o / R - \varepsilon)^{n+1}, \end{aligned} \tag{2.1}$$

where $M(R - \varepsilon, f) = \max_{z \in G_{R-\varepsilon}} |f(z)|$ *and* K'' *is a large number.*

Proof. The following weighted equilibrium problem provides important tools in the derivation of the proof. For the weighted energy integral

$$I_E(\mu) = \int\int ln \frac{1}{|z - t| w(z) w(t)} \, d\mu(z) \, d\mu(t), \quad \mu \in M(E), \tag{2.2}$$

find

$$\tau_E^* = \inf_{\mu \in M(E)} I_E(\mu). \tag{2.3}$$

and identify the extremal measure $\mu_E \in M(E)$ for which the infimum in (2.3) is attained. Here $M(E)$ denotes the class of all positive Borel measure μ on $\mathbf{C}$ which are supported on E and have total mass unity, *i.e.*, $\mu(\mathbf{C}) = 1$.

The logarithmic potential of a Borel measure μ, with compact support which is defined as

$$V^\mu(z) = \int \ln \frac{1}{|z - t|} d\mu(t).$$

It follows from Theorem I.3.3 of [20] that the solution of the weighted energy problem in (2.3) for the weight function $w(z) = e^{Re\, z}$, $z \in \mathbf{C}$ of (2.2) on $\overline{G}_{R_o}$, $0 < R_o \leq 1$, is given by

$$\mu_{\overline{G}_{R_o}} = w(0, \cdot, G_{R_o}),$$

and

$$V^{\mu_{\overline{G}_{R_o}}}(z) + Q(z) = \begin{cases} 1 - lnR_o, & z \in \overline{G}_{R_o} \\ 1 - ln|\phi(z)|, & z \in \mathbf{C} \setminus G_{R_o}, \end{cases} \tag{2.4}$$

where $Q(z) = Re\, z$ and $\phi(z) = ze^{1-z}$.

Now, suppose that $f(z)$ is analytic in G_R. For each $n \geq 0$, let $z_1^{(n+1)}, z_2^{(n+1)}, \ldots, z_{n+1}^{(n+1)}$ be $n + 1$ points in G_R. In view of the Hermite interpolation formula, the polynomial $P_n(z)$ which interpolates $e^{nz} f(z)$ at these points is given by, see [24],

$$e^{nz} f(z) - P_n(z) = \frac{w_{n+1}(z)}{2\pi i} \int_{S_{R-\varepsilon}} \frac{f(t) e^{nt}}{(t - z) w_{n+1}(t)} \, dt, \tag{2.5}$$

where $w_{n+1}(z) = \prod_{k=1}^{n+1}(z - z_k^{(n+1)})$ and $z \in G_{R-\varepsilon}$; $\varepsilon > 0$ is small enough so that the set $\{z_1^{(n+1)}, z_2^{(n+1)}, \ldots, z_{n+1}^{(n+1)}\}$ is contained in $G_{R-\varepsilon}$. Division by e^{nz} in (2.5) yields

$$f(z) - e^{-nz}P_n(z) = \frac{e^{-nz}w_{n+1}(z)}{2\pi i}\int_{S_{R-\varepsilon}} \frac{f(t)\,dt}{(t-z)e^{-nt}w_{n+1}(t)},\ \ z \in G_{R-\varepsilon}.$$

Let $\nu_n(w_n)$ be the normalized counting measure of zeros of $w_n(z)$ defined as (see (1.7)),

$$\nu_n(w_n) = \frac{1}{n}\sum_{k=1}^{n}\delta_{z_k}(n), \quad n \geq 1.$$

Obviously,

$$|w_n(z)| = \exp\{-nV^{\nu_n(w_n)}(z)\},\ \ n \geq 1. \tag{2.6}$$

For each R_o with $0 < R_o < R$, choosing an interpolation in (2.5) which satisfies

$$\left\{z_k^{(n+1)}\right\}_{k=1}^{n+1} \subset S_{R_o} \tag{2.7}$$

and

$$\nu_n(w_n) \to w(0,\cdot,G_{R_o}) \ \text{ as } \ n \to \infty. \tag{2.8}$$

Note that at (2.8) the convergence is weak-convergence. As an example of an interpolation where (2.7) and (2.8) are valid, one can take the pre-images of equally spaced points on $|w| = R_o$ under the conformal map $w = \phi(z) = ze^{1-z}$, *i.e.*, for $\eta = \phi^{-1}$, we define

$$z_k^{(n)} = \eta\left(R_o e^{i2\pi k/n}\right),\ 1 \leq k \leq n,\ n = 1, 2, \ldots.$$

In view of (2.6)-(2.8), we have

$$\lim_{n\to\infty}\left|w_n(z)\right|^{1/n} = \lim_{n\to\infty}\exp\{-V^{\nu_n(w_n)}(z)\} = \exp\{-V^{w(0,\cdot,G_{R_o})}(z)\}, \tag{2.9}$$

which holds locally uniformly in $\mathbf{C} \setminus \overline{G}_{R_o}$. Taking any $\varepsilon > 0$ small enough so that $R_o + \varepsilon < R - \varepsilon$ in (2.5) we obtain

$$\left\|f(z) - e^{-nz}P_n(z)\right\|_{\overline{G}_{R_o}} \leq \frac{\|e^{-nz}w_{n+1}(z)\|_{S_{R_o}}\|f\|_{S_{R-\varepsilon}}}{2\pi\ \mathrm{dist}\,(S_{R-\varepsilon}, S_{R_o})\min_{t\in S_{R-\varepsilon}}\left|e^{-nt}w_{n+1}(t)\right|}.$$

We see that the immediate outcome of (2.4) is

$$V^{w(0,\cdot,G_{R_o})}(z) = -\ln|z|, \tag{2.10}$$

where $z \in \mathbf{C} \setminus G_{R_O}$. From (2.4), it is obvious that $V^{w(0,\cdot,G_{R_o})}(z)$ is continuous on $S_{R_o} = \mathrm{supp}\ w(0,\cdot,G_{R_o})$ and, therefore, is continuous in $\mathbf{C}$ by Theorem II.3.5 of [20], see also Theorem 1.7 in [11]. Assume that $z \in S_{R_o}$, so that from (2.4), $-\ln|z| + \mathrm{Re}\,z = 1 - \ln R_o$. Then with (2.10), it gives

$$V^{w(0,\cdot,G_{R_o})}(z) + Q(z) = -\ln|z| + \mathrm{Re}\,z = 1 - \ln R_o, \quad Q(z) = \mathrm{Re}\,z.$$

It can be easily seen from the above definition that $V^{w(0,\cdot,G_{R_o})} + Q(z)$ is harmonic in G_{R_o} and is identically constant on the boundary S_{R_o}. From (2.4) we have

$$V^{w(0,\cdot,G_{R_o})} + Q(z) = 1 - \ln R_o, \quad z \in \overline{G}_{R_o}. \tag{2.11}$$

Using (2.9) and (2.11), we obtain

$$\| f(z) - e^{-nz}P_n(z) \|_{\overline{G}_{R_o}} \leq K' \frac{1}{R-\varepsilon-R_0} M(R-\varepsilon, f) \frac{e^{R_e z}(e^{ln(R_0)-1})^{n+1}}{e^{R_e t}(e^{ln(R_0-\varepsilon)-1})^{n+1}},$$

which follows from :

$$e^{-nz}w_{n+1}(z) = e^z(e^{-z}w_{n+1}(z)^{1/(n+1)})^{n+1},$$

so

$$\left| e^z(e^{-z}w_{n+1}(z)^{1/(n+1)})^{n+1} \right|$$

is behaves like

$$|e^z|(e^{-Q-V})^{n+1} = e^{R_e z}(e^{ln(R_0)-1})^{n+1},$$

by (2.4) and (2.9).

So we get

$$\|f(z) - e^{-nz}P_n(z)\|_{\overline{G}_{R_0}} \leq K'' \frac{1}{R-\varepsilon-R_0} M(R-\varepsilon, f) \left(\frac{R_0}{R-\varepsilon}\right)^{n+1}.$$

□

Remark 2.1. *In proving Lemma 2.1, we have used the technique developed by Pritsker and Varga in [15].*

Lemma 2.2. *A function f has a singularity on S_R if, and only if,*

$$\limsup_{n\to\infty} \left[E_n^w(f, \overline{G}_{R_o})\right]^{1/n} = \frac{R_o}{R}.$$

Proof. If $f(z)$ is analytic in G_R, then by Lemma 2.1,

$$\limsup_{n\to\infty} \left(E_n^w(f, \overline{G}_{R_o})\right)^{1/n} \leq \frac{R_o}{R-\varepsilon},$$

for all $R-\varepsilon$ sufficiently near to R and so

$$\limsup_{n\to\infty} \left(E_n^w(f, \overline{G}_{R_o})\right)^{1/n} \leq \frac{R_o}{R}, \tag{2.12}$$

However, the strict inequality in (2.12) is equivalent to the analyticity of $f(z)$ in G_ρ for some ρ with $R < \rho < 1$, which is a contradiction. Thus $f(z)$ has a singularity on S_R if and only if equality holds in (2.12).

□

Lemma 2.3. *For any polynomial $P_n(z)$ of degree $\leq n$, we have*

$$\left|e^{-nz}P_n(z)\right| \leq \|e^{-nz}P_n(z)\|_{S_{R_o}} \left(\frac{|\phi(z)|}{R_o}\right)^n,$$

where $z \in \mathbf{C} \setminus G_{R_O}$, $n \geq 0$ and $0 < R_o \leq 1$.

Proof. Since $ln(|\phi(z)|/R_o)$ is the green function, the statement of the lemma follows on the similar lines as that of the Bernstein-Walsh lemma (see [22]). □

Lemma 2.4. *Let $f \in H_R$ and let R_o be a fixed number $(0 < R_o < R)$. Then the function $g(z) = \sum_{n=0}^{\infty} E_n^w(f, \overline{G}_{R_o})z^n$ is analytic in a disk centered at origin whose radius is R/R_0 and for every $r, R_o \leq r < R$,we have*

$$M(r, f) \leq a_o + 2g(r/R_o), \tag{2.13}$$

where a_o is not a constant it depends on $r = |z|$.

Proof. Since $f \in H_R$ so $f(z)$ is analytic in G_R. Consider the function

$$g(z) = \sum_{n=0}^{\infty} E_n^w\left(f, \overline{G}_{R_o}\right) z^n.$$

As $\lim_{n\to\infty} \left[E_n^w\left(f, \overline{G}_{R_o}\right)\right]^{1/n} = R_o/R$, by Lemma 2.2. It follows that g(z) is analytic in a disk centered at the origin whose radius is R/R_o.

By uniform convergence on $\overline{G}_{R_o}$, the function $f(z)$ can be represented in telescopic series:

$$f(z) = e^{-nz}P_n(z) + \sum_{k=n}^{\infty} \left(e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)\right),\ z \in \overline{G}_{R_o}.$$

where $P_{n's}$ are best approximation polynomials for which

$$\|f - e^{-kz}P_k(z)\|_{\overline{G}_{R_0}} = E_k^w.$$

Thus,

$$|f(z)| \leq |e^{-nz}P_n(z)| + \sum_{k=n}^{\infty} \left|(e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z))\right| \tag{2.14}$$

and

$$\begin{aligned}
\left|e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)\right| &\leq \left\|e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)\right\|_{\overline{G}_{R_o}} \\
&\leq \left\|f - e^{-(k+1)z}P_{k+1}(z)\right\|_{\overline{G}_{R_o}} \\
&+ \left\|f - e^{-kz}P_k(z)\right\|_{\overline{G}_{R_o}} \\
&= E_{k+1}^w\left(f, \overline{G}_{R_o}\right) + E_k^w\left(f, \overline{G}_{R_O}\right) \\
&\leq 2E_k^w\left(f, \overline{G}_{R_o}\right).
\end{aligned}$$

In view of Lemma 2.3, we get

$$\left|e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)\right| \leq 2E_k^w\left(f, \overline{G}_{R_o}\right)\left(\frac{|\phi(z)|}{R_o}\right)^k, k \geq n, z \in \mathbf{C} \setminus G_{R_0}.$$

Hence the inequality (2.14) yields

$$|f(z)| \leq a_o + 2\sum_{k=n}^{\infty} E_k^w\left(f, \overline{G}_{R_o}\right)\left(\frac{|\phi(z)|}{R_o}\right)^k.$$

If $z \in S_r$, i.e.,$|\phi(z)| = r$,then

$$|f(z)| \leq a_o + 2\sum_{k=n}^{\infty} E_k^w\left(f, \overline{G}_{R_o}\right)\left(\frac{r}{R_o}\right)^k. \tag{2.15}$$

The last series in (2.15) converges inside G_R since we can majorate it by g. Now (2.15) implies (2.13). Hence the proof is completed.

□

In order to prove the main results we need the concepts of q order and q type of a function of a single complex variable which is analytic in the disc $|z| < R$.

The q order $\rho_0(q)$ and type $T_0(q)$ of $f(z)$ are defined in a analogous manner to (1.3) and (1.4). The coefficient characterization of $\rho_0(q)$ and $T_0(q)$ for $f(z)$ are as follows;

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < R$, $0 < R < 1$. and have q -order $\rho_0(q)$ $(\rho_0(q) > 0, q > 2)$. Then

$$\rho_0(q) + A(q) = \limsup_{n\to\infty} \frac{\log^{[q-1]}|a_n| R^n}{\log n - \log^+ \log^+ |a_n| R^n}, 0 \leq \rho_0(q) \leq \infty, \tag{2.16}$$

where $A(q) = 1$ if $q = 2$, $A(q) = 0$ if $q \geq 3$ and for $x > 0$, we put $\log^+ x = \max(\log x, 0)$.

The function f is of q-order $\rho_0(q)(0 < \rho_0(q) < \infty)$ and q-type $T_0(q)$ if, and only if,

$$V(q) = T_0(q)B(q), \tag{2.17}$$

where $B(q) = \frac{(\rho_0+1)^{\rho_0+1)}}{\rho_0^{\rho_0}}$ for $q = 2$ and $B(q) = 1$ if $q = 3, 4, \ldots$ and

$$V(q) = \limsup_{n\to\infty} (\log^{[q-2]} n)(\log^+ |a_n| R^n)^{\rho_0(q)+A(q)}.$$

The above coefficients characterizations of $f(z)$ are due to c.f. ([8], Theorems 1 and 5).

3. Main Results

Theorem 3.1. *Let $f \in H_R, 0 < R_0 < R < 1$, be of order $\rho_0(q)$. Then*

$$\rho_0(q) + A(q) = \limsup_{n\to\infty} \frac{\log^{[q-1]} n}{\log n - \log^+ \log^+ E_n^w(f, \overline{G}_{R_o})(R/R_o)^n}. \tag{3.1}$$

Proof. Let

$$\liminf_{n\to\infty} \frac{\log n - \log^+ \log^+ E_{n,1}(f) R^n}{\log^{q-1} n} = \alpha.$$

Obviously $0 \leq \alpha \leq \infty$. First suppose that $0 < \alpha < \infty$. Then, by the definition of α, there exists a sequence $\{n_k\}$ of positive integers tending to infinity such that

$$\log E_n^w(f, \overline{G}_{R_o})(R/R_o)_k^n > n_k(\log^{[q-2]} n_k)^{(-\alpha+\varepsilon)} \quad for \quad k = 1, 2, 3, \ldots \tag{3.2}$$

Using (2.1) and with (3.2), we obtain

$$\log M(R_0, f) \geq n_k(\log^{[q-2]} n_k)^{(-\alpha+\epsilon)} + n_k \log(R_0/R) - \log A \tag{3.3}$$

for the sequence $\{n_k\}$ and all $R_0(< R)$ sufficiently close to R. Let $\{R_{0k}\}$ be a sequence defined by

$$n_k = \exp^{[q-2]}\{e\log(R/R_{0k})\}, k = 1,2,3,\dots, then \quad R_{0k} \to R \quad as \quad k \to \infty.$$

Thus, using (3.2) and (3.3), for all sufficiently large values of k, we get

$$\begin{aligned} \log M(R_{0k}, f) &\geq (1-e)n_k(\log^{[q-2]} n_k)^{-(\alpha+\epsilon)}[1+0(1)] \\ &= e(e-1)\left\{\exp^{[q-2]}(e\log(R/R_{0k}))^{-1/(\alpha+\epsilon)}\right\}. \\ &\quad \log(R/R_{0k})^{-1}[1+0(1)]. \end{aligned} \tag{3.4}$$

Since $\log(R/(R-R_{0k}))\ -\log\log(R/R_{0k})$ *as* $k \to \infty$, after a simple calculation the above inequality gives

$$\rho_0(q) + A(q) \geq 1/\alpha. \tag{3.5}$$

This inequality is trivially true if $\alpha = 0$ or $\alpha = \infty$.

For reverse inequality in (3.1), we just apply (2.16) to $g(z)$, defined in Lemma 2.4. This completes the proof.

□

Theorem 3.2. *Let* $f \in H_R, 0 < R_0 < R < 1$, *and have* $q-order$ $\rho_0(q)(0 < \rho_0(q) < \infty)$, $q-type$ $T_0(q)$, *then*

$$G(q) = T_0(q)B_0(q), \tag{3.6}$$

where $B_0(2) = \frac{(\rho_0(2)+1)^{\rho_0(2)+1}}{(\rho_0(2)/2)^{\rho_0(2)}}, A(2) = 1$ *and* $B_0(q) = 1, A(q) = 0$ *if* $q = 3,4,\dots$

$$G(q) = \limsup_{n\to\infty}(\log^{[q-2]} n)\left(\frac{\log^+ E_n^w(f,\overline{G}_{R_o})(R/R_o)^n}{n}\right)^{\rho_0(q)+A(q)}. \tag{3.7}$$

Proof. Let $G(q) < \infty$. For given $\varepsilon > 0$, by (3.7) we have

$$\left(\log^{[q-2]} n\right)\left(\frac{\log^+ E_n^w(f,\overline{G}_{R_o})(R/R_o)^n}{n}\right)^{\rho_0(q)+A(q)} < G(q)+\varepsilon,$$

for all $n > n_0 \equiv n_0(\varepsilon)$, it gives

$$\begin{aligned} \log^{[q-1]} n &+ (\rho_0(q)+A(q))\left[\log^+\log^+ E_n^w(f,\overline{G}_{R_o})(R/R_o)^n - \log n\right] \\ &< \log(G(q)+\varepsilon), \end{aligned}$$

or

$$\begin{aligned} \rho_0(q)+A(q) &> \frac{\log^{[q-1]} n}{\log n - \log^+\log^+ E_n^w(f,\overline{G}_{R_o})(R/R_o)^n} \\ &- \frac{\log(G(q)+\varepsilon)}{\log n - \log^+\log^+ E_n^w(f,\overline{G}_{R_o})(R/R_o)^n}. \end{aligned}$$

Let $0 < T_0(q) < \infty$. For given $\varepsilon > 0$, by definition, we have

$$\log M(R_0, f) < \exp^{[q-2]}\left\{(T_0(q)+\varepsilon)(R/(R-R_0))^{\rho_0(q)}\right\} \tag{3.8}$$

for all R_0 such that $0 < r_0 = r_0(\varepsilon) < R_0 < R$.

Thus using Lemma 2.1, (3.8) gives

$$\begin{aligned} \log^+ E_n^w(f, \overline{G}_{R_o})(R/R_o)^n &\leq \exp^{[q-2]}\left\{(T_0(q)+\varepsilon)(R/(R-R_0))^{\rho_0(q)}\right\} \\ &+ n\log(R/R_0) + \log^+ A. \end{aligned} \tag{3.9}$$

The maximum value of right hand side of (3.9) is uniquely determined by the value of R_0 given by

$$\prod_{i=0}^{q-2} \exp^{[i]}\left\{(T_0(q)+\varepsilon)(R/(R-R_0))^{\rho_0(q)}\right\} = \frac{n(R-R_0)}{R\rho_0(q)}. \tag{3.10}$$

For q = 2, using (3.10) in (3.9), we get

$$\begin{aligned} \log^+ E_n^w(f, \overline{G}_{R_o})(R/R_o)^n &\leq \frac{(T_0(q)+\varepsilon)^{1/(\rho_0(q)+1)}(2n)^{\rho_0(q)/(1+\rho_0(q))}}{(\rho_0(q))^{\rho_0(q)/(1+\rho_0(q)+1)}} \\ &(1+\rho_0(q)+o(1)), \end{aligned}$$

for all sufficiently large value of n. On proceeding to limits, the above inequality gives (3.6) for $q = 2$.

Next, for $q = 3, 4, \ldots$ (3.10) gives

$$\frac{R}{R-R_0} \simeq \left(\frac{\log^{[q-2]} n}{T_0(q)+\varepsilon}\right)^{1/\rho_0(q)} \quad as \quad n \to \infty.$$

Thus for $n > n_0$, (3.9) gives

$$\log^+ E_n^w(f, \overline{G}_{R_o})(R/R_o)^n < n + n\log(R/R_0) + log^+ A,$$

or

$$\log^{[q-2]} n^{(1+o(1))}\left(\frac{\log^+ E_n^w(f, \overline{G}_{R_o})(R/R_o)^n}{n}\right)^{\rho_0(q)} < (T_0(q)+\varepsilon)(1+o(1)).$$

Proceeding to limits as $n \to \infty$, the above inequality gives

$$T_0(q) \geq G(q) \quad for \quad q \geq 3.$$

The reverse inequality is obtained by applying (2.17) to $g(z)$ of Lemma 2.4. If $G(q)$ is infinite then $T_0(q) = \infty$ and f is of growth $(\rho_0(q), \infty)$. Hence the proof is completed.

□

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