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Approximation of Periodic Functions by Wavelet Fourier Series

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Abstract. This paper aims to examine the expansion of periodic functions using wavelet bases. M. Skopina [8] obtained a Wavelet analog of the classical Jackson's theorem for trigonometric approximation. Our result generalizes the result of M. Skopina [8] and V. Karanjgaokar *et al*. [15].

1. Introduction

Wavelets were initially introduced by A. Grossman and J. Morlet [6] as functions that possess the ability to be translated and dilated, thereby enabling their utilization for expansions in $L^2(\mathbb{R})$. The introduction of wavelet analysis was initially aimed in improving seismic signal processing by replacement of short-time Fourier analysis with new better algorithms that can effectively identify and analyze abrupt changes in signals. The application of wavelet approximation has emerged as a novel tool in the fields of Mathematics, Physics, and Engineering. From the standpoint of approximation theory and harmonic analysis, the wavelet theory holds substantial significance on various aspects. This approach provides straightforward and refined unconditional wavelet

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bases for various function spaces, such as Lebesgue, Sobolev, Besov, and others. A recent advancement in approximation theory involves the approximation of any given function through wavelet polynomials.

The Wavelet Approximation technique is a contemporary tool that is utilized for the identification and examination of sudden alterations in seismic signal processing. The Haar Wavelet has been determined to be an effective method for Wavelet Approximation studied by Devore [11], Debnath [6], Meyer [18], Morlet [4] and Lal and Kumar [12]. The present study aims to investigate the expansions of periodic functions with respect to Wavelet bases. To achieve this, we introduce a general monotonically decreasing function Pn(x) and generalize the results of M. Skopina [8] and V. Karanjgaokar *et al*.

In this paper the section 2 contains some definitions and preliminaries. Section 3 consists of Theorems and Lemmas which are used in the proof the main theorems. Section 4 contains the two main theorems of our paper and the proof of these theorems are given in section 5. Section 6 includes some corollaries of our main theorems, and lastly, the references used to support the content of this paper have been included.

2. Definitions and Preliminaries

2.1. **Periodic Multiresolution Analysis (PMRA) (V. Karanjgaokar *et al.* [15]).** The concept of PMRA had been defined and used in Deng Feng and Si Long [2], Prestin and Selig [5] and M. Skopina [9]. Let $\phi \in L^2(\mathbb{R})$ and $\psi \in L^2(\mathbb{R})$ be scaling function of MRA and wavelet function respectively given by

$$\hat{\phi}(x) = m_0\left(\frac{x}{2}\right)\hat{\phi}\left(\frac{x}{2}\right) \tag{2.1}$$

and

$$\hat{\psi}(x) = m_0\left(\frac{x+1}{2}\right)\hat{\phi}\left(\frac{x}{2}\right)e^{i\pi x} \tag{2.2}$$

where $m_0 \in L^2(\mathbb{T})$ is a low pass filter. The normalized integer shifts and scales of ψ given by

$$\psi_{j,n}(x) = 2^{\frac{j}{2}}\psi(2^j x + n),\ j,n \in \mathbb{Z}$$

constitute an orthonormal basis in $L^2(\mathbb{R})$. If both the functions ϕ and ψ have sufficient decay, then the functions

$$\Phi_{j,n}(x) = 2^{\frac{j}{2}}\sum_{l\in\mathbb{Z}}\phi(2^j x + 2^j l + n) \tag{2.3}$$

and

$$\Psi_{j,n}(x) = 2^{\frac{j}{2}}\sum_{l\in\mathbb{Z}}\psi(2^j x + 2^j l + n) \tag{2.4}$$

are in $L^2(\mathbb{T})$ and the systems $\{\Phi_{j,n}\}_{n=0}^{2^j-1}$ and $\{\Psi_{j,n}\}_{n=0}^{2^j-1}$ are orthonormal for each $j = 0,1,2,\dots$ The spaces

$$V_j = span\{\Phi_{j,n},\ n = 0,1,2,\dots,2^j-1\}$$

and

$$W_j = span\{\Psi_{j,n},\ n = 0, 1, 2, \ldots, 2^j - 1\}$$

satisfy the properties:

$$V_0 = \{const\}, V_j \subset V_{j+1}, V_{j+1} = V_j \oplus W_j, \text{for all } j = 0, 1, 2, \ldots,$$

and

$$\cup_{j=0}^{\infty} V_j = L^2(\mathbb{T}).$$

The collection $\{V_j\}_{j=0}^{\infty}$ is called a periodic multiresolution analysis generated by Φ.

2.2. **Decay of wavelet function and scaling function (V. Karanjgaokar *et al.* [15]).** The scaling function ϕ and wavelet function ψ in (2.3) and (2.4) have sufficient decay, if they satisfy

$$max\{|\phi(x)|, |\psi(x)|\} \leq C/(1 + |x|^{1+\epsilon})\ (\epsilon > 0), \tag{2.5}$$

with the mother function $\psi \in L^2$ ($\mathbb{R}$) and $\phi \in L^2(\mathbb{R})$ are given by equations (2.1) and (2.2).

2.3. **Wavelet Fourier Series (M. Skopina [8]).** If $f \in L^2(\mathbb{T})$.Then

$$< f, \Phi_{0,0} > \Phi_{0,0} + \sum_{j=0}^{\infty} \sum_{n=0}^{2^j-1} < f, \Psi_{j,n} > \Psi_{j,n} \tag{2.6}$$

is called Wavelet Fourier series . The double sum in (2.6) can be transformed in single sum by redenoting periodic wavelets as

$$w_0 = \Phi_{0,0}\ , w_{2^j+L} = \Psi_{j,L} \quad 0 \leq L \leq 2^j - 1,$$

and the series (2.6) can be rewritten as

$$\sum_{k=0}^{\infty} < f, w_k > w_k. \tag{2.7}$$

Let $S_N(f)$ denote the N^{th} partial sum of (2.7), with $N = 2^j + L$, $0 \leq L < 2^j - 1$ and $S_{2^j-1}(f)$ is an orthogonal projection of f onto V_j with $\{\Phi_{j,n}\}_{n=0}^{2^j-1}$ as orthonormal basis in V_j, then

$$S_{2^j-1}(f) = \sum_{n=0}^{2^j-1} < f, \Phi_{j,n} > \Phi_{j,n}, \tag{2.8}$$

$$S_N(f) = \sum_{n=0}^{2^j-1} < f, \Phi_{j,n} > \Phi_{j,n} + \sum_{n=0}^{L} < f, \Psi_{j,n} > \Psi_{j,n}. \tag{2.9}$$

Set $f = w_0 = 1$ in (2.7) and since $< f, w_k >= \delta_{0,k}$, we have $S_N(f) = 1$ for all N, $j = 0, 1, 2, \cdots$, Hence

$$\int_0^1 \sum_{k=0}^{N} w_k(x)\overline{w_k(t)}dt \equiv 1, \qquad \int_0^1 \sum_{k=0}^{2^j-1} \Phi_{j,k}(x)\overline{\Phi_{j,k}(t)}dt \equiv 1. \tag{2.10}$$

2.4. **Modulus of Smoothness (V. Karanjgaokar *et al.* [15]).** For details about modulus of continuity $w(f,h)$, integral modulus of continuity $w_r(f,h)$ and integral modulus of smoothness $w_r^2(f,h)$ one can refer L.N. Mishra ([7], [17]) and P. Chandra [10].

The Smoothness of a function is measured by the order of the derivative of the function which are continuous. The r^{th} modulus of smoothness is given by

$$w_r(f,h)_p = \sup_{|t|\le h} \|\Delta_t^r f\|_p. \tag{2.11}$$

The error of best approximation of order N, is given by

$$E_N(f)_p = inf\|f - T\|_p, \tag{2.12}$$

where infimum is taken over all "wavelet polynomials"

$$T = \sum_{k=0}^{N} a_k w_k.$$

3. Theorems and Lemmas

In 2000, M. Skopina [8] , proved the following theorem, which is a wavelet analog of Classical Jackson theorem for trigonometric approximation:

Theorem 3.1. (M. Skopina [8]):*Let ϕ satisfies (2.5), $\psi \in C^m(R)$ with $\psi^{(l)}$ bounded for $l \le m$,*

$$|\psi(x)| \le C/(1 + |x|^n) \qquad n > m + 1, \tag{3.1}$$

$p \in [1, \infty)$. Then

$$E_N(f) \le \|f - S_N(f)\|_p \le C(p,n,m) w_r(f, 1/N)_p \qquad N = 1, 2, 3... \tag{3.2}$$

for all $f \in L_p(T)$ ($f \in C(T)$ for $p = \infty$) and for all positive integers r,
$r \le m + 1$,$r < n - 1$.

Theorem 3.2. (M. Skopina [8]):*Let $\phi \in C^{(m)}(\mathbb{R})$ satisfy (2.5) and $|\phi^{(m)}| \le \frac{C}{(1+|x|^{1+\epsilon})}$, $\epsilon > 0$, $p \in [1, \infty]$. Then*

$$\|f^m\|_p \le C(p,m)\ 2^{mj}\ \|f\|_p, \tag{3.3}$$

for all $f \in V_j$, $j = 0, 1.....$
This theorem is a wavelet analog of Bernstein's inequality for trigonometric polynomials.

Theorem 3.3. (V. Karanjgaokar *et al*. [15]) *: Let ϕ, $\psi \in L^2(\mathbb{R})$ and $n > 1$ such that.*

$$|\phi(x)|, |\psi(x)| \leq C/(1+|x|)^n, \tag{3.4}$$

and if $f(x) = 0\ \forall x \in [x_0 - \delta, x_0 + \delta]$, where $0 < \delta < 1/2$, $x_0 \in R$ and C is a constant , then

$$S_N(f, x_0) = O(N^{1-n}) \text{ as } N \to \infty. \tag{3.5}$$

Theorem 3.4. (V. Karanjgaokar *et al*. [15]) *: Let $f \in L_p(\mathbb{T}), 1 \leq p \leq \infty$. ($f \in C(\mathbb{T})$ for $p = \infty$), $\psi \in C^m(\mathbb{R})$ with $\psi^{(l)}$ bounded for $l \leq m$ and satisfy*

$$|\psi(x)| \leq C/(1+|x|)^n \qquad n > m+1. \tag{3.6}$$

Also ϕ satisfy

$$max|\phi(x)| \leq C/(1+|x|)^n \qquad n > 1. \tag{3.7}$$

Then

$$inf\|f - T\|_p \leq \|f - S_N f\|_p \leq C(p, n, m) w_r(f, 1/N)_p \quad N = 1, 2, \cdots. \tag{3.8}$$

for all wavelet polynomials $T = \sum_{k=0}^{N} a_k w_k$ and $w_r(f, 1/N)_p$ denotes r^{th} modulus of smoothness, r is a positive integer with $r \leq m+1$, $r < n-1$.

Lemma 3.1. (V. Karanjgaokar and N. Shrivastav [14]) *Let g and h be functions defined on $\mathbb{R}$, with*

$$\max(|g(x)|, |h(x)|) = O(P_n(x))$$

where $P_n(x)$ is a function of x for each fixed positive integer n and is a positive monotonic decreasing function of $|x|$, with the series $\sum_{k=0}^{\infty} P_n(x)$ converges for fixed $n > 1$. Then

$$\int_0^1 f(t) \sum_{k=0}^{L} \sum_{l' \in \mathbb{Z}} g(2^j x + 2^j l' + k) \sum_{l \in \mathbb{Z}} \overline{h(2^j t + 2^j l + k)} dt = \int_{-\infty}^{\infty} f(t) \sum_{v \in Z(j,L)} g(2^j x + v) \overline{h(2^j t + v)} dt,$$

where $Z(j, L) = \{v \in \mathbb{Z} : v = 2^j l + k, l \in \mathbb{Z}, k = 0, 1, \ldots, L\}$. The proof of this lemma is trivial and one can see the lemma for $P_n(x) = \frac{C}{1+|x|^n}$, $n > 1$ in M. Skopina [8] .

Lemma 3.2. (Kelly *et al*. [13])*: Let μ be a bounded decreasing and integrable function in $[0, \infty)$. Then for all $x, y \in \mathbb{R}$,*

$$\sum_{k \in \mathbb{Z}} |\mu(x+k)||\mu(y+k)| \leq C\mu\left(\frac{|x-y|}{4}\right)$$

where C is the constant depending only on μ.
The proof of this lemma is simple, it's proof can be seen in M. Skopina [9] *and* Kelly *et al*. [13]. *The proof of this lemma for $\mu(x) = \frac{1}{(1+|x|)^{1+\epsilon}}$, can be seen in* V. Karanjgaokar [16].

Lemma 3.3. (M. Skopina [8]) *If g, h satisfy the hypothesis of Lemma 3.1 , then*

$$2^j \int_0^1 \Big| \sum_{k=0}^{L} \sum_{l' \in \mathbb{Z}} g(2^j x + 2^j l' + k) \sum_{l \in \mathbb{Z}} \overline{h(2^j t + 2^j l + k)} \Big| dt \leq C, \tag{3.9}$$

where C is a constant depending only on the functions g, h and μ.

To prove (3.9), we should apply Lemma 3.1 for $f \equiv 1$ and Lemma 3.2.

4. Main Theorem

Theorem 4.1. *Let $f \in L_p(\mathbb{T}), 1 \le p < \infty$. ($f \in C(\mathbb{T})$ for $p = \infty$),$\psi \in C^m(\mathbb{R})$ with $\psi^{(l)}$ bounded for $l \le m$ and satisfy*

$$|\psi(x)| = O(P_n(x)) \qquad n > m+1, \tag{4.1}$$

and ϕ satisfy

$$max|\phi(x)| = O(P_n(x)) \qquad n > 1.$$

Where, for each fixed positive integer n, $P_n(x)$ is a function of x, which is positive and monotonic decreasing with x. For $n > 1$ and a positive integer r with $r < n-1$ & $r \le n+1$,

$$\int_{-\infty}^{\infty} |x|^r P_n(x)dx. \tag{4.2}$$

is convergent and integrable on $\mathbb{R}$.
Then

$$inf\|f-T\|_p \le \|f - S_N f\|_p \le C(p,n,m)w_r(f,1/N)_p \quad N = 1,2,\cdots. \tag{4.3}$$

for all wavelet polynomials $T = \sum_{k=0}^{N} a_k w_k$ and $w_r(f,1/N)_p$ denotes r^{th} modulus of smoothness.

Note: This result generalizes the result of M. Skopina [8] for $P_n(x) = \frac{C}{1+|x|^n}$, $n > 1$ and V. Karanjgaokar *et al*. [15] for $P_n(x) = \frac{C}{(1+|x|)^n}$, $n \ge 1$.

Theorem 4.2. *Let $f \in V_j \subseteq L^2(\mathbb{T})$, $\phi \in C^m(R)$ and for $m \ge 0$*

$$|\phi^{(m)}(x)| = O(P_n(x))$$

Where, for each fixed positive integer n. $P_n(x)$ is a positive and monotonic decreasing function of x with $n \ge 1$ and a positive integer r with $r < n-1$ & $r \le m+1$, satisfy (4.2). Then

$$\|f^m\|_p \le 2^{mj}\, C(p,r,n)\, \|f\|_p \tag{4.4}$$

for all $f \in V_j$, $j = 0,1,.....$and $p \in [1,\infty]$.

5. Proof of the Theorem

5.1. **Proof of the Theorem 4.1.** We shall prove the following inequality for the trigonometric polynomial f

$$\|f - S_N(f)\|_p \le C(p,r,n)\|f^r\|_p/N^r, \tag{5.1}$$

and then we will get (3.8) using Theorem 3.3.
By the definition of S_{2^j-1} , (2.6), (2.7) and (2.8), for $2^j \le N \le 2^{j+1}$,we have

$$\lim_{k\to\infty} \|f - S_{2^k-1}\|_p = 0,$$

Therefore, we can write

$$f - S_N(f) = (S_{2^j-1}(f) - S_N(f)) + \sum_{i=j}^{\infty}(S_{2^{i+1}-1}(f) - S_{2^i-1}(f)). \tag{5.2}$$

Thus in order to prove (5.1), it is sufficient to prove that

$$\|S_{2^j+L}(f) - S_{2^j-1}(f)\|_p \le C(p,r,n)\ 2^{-jr}\ \|f^r\|_p, \tag{5.3}$$

for all $j = 0, 1, \cdots, L = 0, \cdots, 2^j - 1$. We use Lemma 3.1 , Taylor's formula

$$f(t) = \sum_{k=0}^{r-1}\frac{f^{(k)}(x)}{k!}(t-x)^k + \frac{1}{(r-1)!}\int_x^t f^r(\tau)(t-\tau)^{r-1}d\tau,$$

and the fact that all the moments of $\psi \in L^2(\mathbb{R})$ upto order m are zero i.e.

$$\int_{-\infty}^{\infty} x^l\psi(x)dx = 0, l = 0, 1, \cdots, m. \tag{5.4}$$

see Daubechies ([3], ch.5). Thus we get

$$\begin{aligned}
S_{2^j+L}(f,x) - S_{2^j-1}(f,x) &= 2^j\int_0^1 f(t)\sum_{k=0}^{L}\sum_{l'\in Z}\psi(2^jx + 2^jl' + k)\sum_{l\in Z}\overline{\psi(2^jt + 2^jl + k)}\ dt\\
&= 2^j\int_{-\infty}^{\infty} f(t)\sum_{v\in Z(j,L)}\psi(2^jx + v)\overline{\psi(2^jt + v)}\ dt\\
&= \left(2^j/(r-1)!\right)\int_{-\infty}^{\infty}\int_x^t f^{(r)}(\tau)(t-\tau)^{r-1}d\tau\sum_{v\in Z(j,L)}\psi(2^jx + v)\ \overline{\psi(2^jt + v)}dt.
\end{aligned}$$

Thus using Lemma 3.2 and the condition (4.1) on ψ , we will get

$$\Big|\sum_{v\in Z(j,L)}\psi(2^jx + v)\overline{\psi(2^jt + v)}\Big| \le\ CP_n(2^j(t-x)),$$

and hence

$$\begin{aligned}
|S_{2^j+L}(f,x) - S_{2^j-1}(f,x)| &\le 2^jC\int_{-\infty}^{\infty}\Big|\int_x^t f^{(r)}(\tau)(t-\tau)^{r-1}d\tau\Big|P_n(2^j(t-x))\ dt\\
&= 2^jC\int_{-\infty}^{\infty}\int_x^{x+t}|f^{(r)}(\tau)||t + x - \tau|^{r-1}d\tau\ P_n(2^jt)dt
\end{aligned} \tag{5.5}$$

Using Jensen's inequality for $p < \infty$, we obtain

$$\begin{aligned}
&\|\ S_{2^j+L}(f) - S_{2^j-1}(f)\ \|_p \\
&\le 2^j\int_{-\infty}^{\infty} dt\ P_n(2^jt)\Big(\int_0^1\Big|\int_x^{x+t}|f^{(r)}(\tau)|\ |x + t - \tau|^{r-1}d\tau\Big|^p dx\Big)^{1/p}\\
&\le C(p,r,n)\ 2^j\int_{-\infty}^{\infty} dtP_n(2^jt)\Big(\int_0^1 |t|^{r(p-1)}\ \int_x^{x+t}|f^{(r)}(\tau)|^p|x + t - \tau|^{r-1}d\tau\ dx\Big)^{1/p}.
\end{aligned} \tag{5.6}$$

Now three cases arise (i) $|t| \ge 1$, (ii) $0 \le t < 1$, (iii) $-1 < t < 0$.

Case(i) $|t| \ge 1$. As $f^{(r)}$ is periodic we have

$$\begin{aligned}\int_x^{x+t}|f^{(r)}(\tau)|^p|x+t-\tau|^{r-1}d\tau &\leq |t|^{r-1}\int_x^{x+t}|f^{(r)}(\tau)|^p d\tau\\ &\leq 2|t|^r\|f^{(r)}\|_p^p.\end{aligned}$$

Thus in (5.6)

$$\int_0^1 dx\int_x^{x+t}|f^{(r)}(\tau)|^p|x+t-\tau|^{r-1}d\tau \leq C(r)|t|^r\|f^{(r)}\|_p^p. \tag{5.7}$$

Case (ii) $0 \leq t < 1$. using change of order of integration

$$\begin{aligned}\int_0^1 dx\int_x^{x+t}|f^{(r)}(\tau)|^p|x+t-\tau|^{r-1}d\tau &\leq \int_0^{t+1}|f^{(r)}(\tau)|^p d\tau\int_{\tau-t}^{\tau}(t+x-\tau)^{r-1}dx\\ &\leq (2/r)\|f^{(r)}\|_p^p\, t^r.\end{aligned}$$

Therefore (5.7) holds in both the cases.

Case (iii) If $-1 < t < 0$, then

$$\begin{aligned}\Big|\int_x^{x+t}|f^{(r)}(\tau)|^p|x+t-\tau|^{r-1}d\tau\Big| &\leq \int_t^1|f^{(r)}(\tau)|^p d\tau\int_\tau^{\tau-t}(\tau-x-t)^{r-1}dx\\ &\leq (2/r)\|f^{(r)}\|_p^p\ \ |t|^r.\end{aligned}$$

Therefore (5.7) holds for every value of t. Thus substituting from (5.7) to (5.6), we get

$$\|S_{2^j+L}(f)-S_{2^j-1}(f)\|_p \leq C(p,r,n)\|f^{(r)}\|_p\, 2^j\int_{-\infty}^{\infty}\Big(|t|^rP_n(2^jt)\Big)dt,$$

for $p<\infty$ and for $p=\infty$, from (5.3) it follows that

$$\|S_{2^j+L}(f)-S_{2^j-1}(f)\|_\infty \leq C(p,r,n)\|f^{(r)}\|_\infty\, 2^j\int_{-\infty}^{\infty}\Big(|t|^rP_n(2^jt)\Big)dt.$$

$$\|S_{2^j+L}(f)-S_{2^j-1}(f)\|_\infty \leq C(p,r,n)\|f^{(r)}\|_\infty\, 2^{-jr}\int_{-\infty}^{\infty}\Big(|x|^rP_n(x)\Big)dx.$$

These relations gives (5.3) and finally (5.1).

This gives (4.3) for trigonometric polynomial. Our aim is to show (4.3) for an arbitrary function $f \in L^{(p)}(\mathbb{T})$ $(for\ p=\infty, f\in C(\mathbb{T}))$. For this we approximate f in the norm by a trigonometric polynomial and as (4.3) holds for each trigonometric polynomial and we use

$$\|S_N(f)\|_p \leq C(p)\|f\|_p$$

and

$$w_r(f,1/N) \leq C(r)\|f\|_p.$$

Finally we prove (4.3) for each $f \in L_p(T)$ $(C(T)\ \ for\ \ p=\infty)$. This completes the proof of the Theorem.

5.2. **Proof of the Theorem 4.2.** Let $f \in V_j \subseteq L^2(\mathbb{T})$, since $f = S_{2^j-1}(f)$ by (2.8) , we have

$$f(x) = 2^j \int_0^1 f(t) \sum_{k=0}^{2^j-1} \sum_{l' \in Z} \phi(2^j x + 2^j l' + k) \sum_{l \in Z} \overline{\phi(2^j t + 2^j l + k)}\ dt. \tag{5.8}$$

Hence

$$f^m(x) = 2^{j(m+1)} \int_0^1 f(t) K_j(x,t)\ dt. \tag{5.9}$$

Where

$$K_j(x,t) = \sum_{k=0}^{2^j-1} \sum_{l' \in Z} \phi^m(2^j x + 2^j l' + k) \sum_{l \in Z} \overline{\phi(2^j t + 2^j l + k)}. \tag{5.10}$$

For $p = \infty$ (4.4) follow from (5.10) immediately, due to Lemma (3.3) with $g = \phi$, $h = \phi^m$. For $p < \infty$. Using by Jensen's Inequality, (5.9) implies

$$\begin{aligned} f^m(x) &= 2^{j(m+1)} \int_0^1 f(t) \sum_{k=0}^{2^j-1} \sum_{l' \in Z} \phi^m(2^j x + 2^j l' + k) \sum_{l \in Z} \overline{\phi(2^j t + 2^j l + k)}\ dt. \\ &= 2^{j(m+1)} \int_{-\infty}^{\infty} f(t)\ |h(2^j x + v)\overline{g(2^j t + v)}|\ dt \\ f^m(x) &\leq 2^{j(m+1)} \int_{-\infty}^{\infty} f(t) C P_n(2^j(t-x)) dt \end{aligned}$$

$$\|f^m(x)\|_p \leq 2^{j(m+1)} C(p,r,n) \|f^r\|_p\ 2^j \int_{-\infty}^{\infty} (|t|^r P_n(2^j t)) dt \tag{5.11}$$

$$\|f^m(x)\|_p \leq 2^{j(m+1)} C(p,r,n) \|f^r\|_p\ 2^{-jr} \int_{-\infty}^{\infty} (|x|^r P_n(x)) dx. \tag{5.12}$$

Finally using (4.2), we obtain

$$\|f^m(x)\|_p \leq C(p,r,n) 2^{mj} \|f\|_p \tag{5.13}$$

for $j = 0, 1, 2...$

6. Corollaries of Theorem 4.1 and Theorem 4.2

Corollary 6.1. *In our theorem if we take $P_n(x) = e^{-nx}$,*
Then

$$\int_{-\infty}^{\infty} \Big(|x|^r e^{-nx} \Big) dx.$$

is convergent and integrable on $\mathbb{R}$.

Proof: If $P_n(x) = e^{-nx}$

$$\int_{-\infty}^{\infty} \Big(|x|^r P_n(x) \Big) dx = \int_{-\infty}^{\infty} \Big(|x|^r e^{-nx} \Big) dx$$

is convergent for $r - n + 1 < 0$ and integrable on $\mathbb{R}$.

Corollary 6.2. *In our theorem if we take* $P_n(x) = x^{-n}(\log x)^{-n}$*,*
Then

$$\int_{-\infty}^{\infty}\Big(|x|^r x^{-n}(\log x)^{-n}\Big)dx.$$

is convergent and integrable on $\mathbb{R}$*.*

Proof: If $P_n(x) = x^{-n}(\log x)^{-n}$

$$\begin{aligned}\int_{-\infty}^{\infty}\Big(|x|^r P_n(x)\Big)dx &= \int_{-\infty}^{\infty}\Big(|x|^r x^{-n}(\log x)^{-n}\Big)dx\\ &= \int_{-\infty}^{\infty}\Big(|x|^{r-n}\frac{x}{x(\log x)^n}\Big)dx\\ &= \int_{-\infty}^{\infty}\Big(|x|^{r-n+1}\frac{1}{x(\log x)^n}\Big)dx.\end{aligned}$$

is convergent for $r-n+1<0$ and integrable on $\mathbb{R}$.

Corollary 6.3. *In our theorem if we take* $P_n(x) = \frac{1}{x^n}\log(1+\frac{n}{x})$*,*
Then

$$\int_{-\infty}^{\infty}\Big(|x|^r\,\frac{1}{x^n}\log\Big(1+\frac{n}{x}\Big)\Big)dx.$$

is convergent and integrable on $\mathbb{R}$*.*

Proof : If $P_n(x) = \frac{1}{x^n}\log(1+\frac{n}{x})$

$$\begin{aligned}\int_{-\infty}^{\infty}\Big(|x|^r P_n(x)\Big)dx &= \int_{-\infty}^{\infty}\Big(|x|^r\frac{1}{x^n}\log(1+\frac{n}{x})\Big)dx\\ &= \int_{-\infty}^{\infty}\Big(|x|^{r-n}\log(1+\frac{n}{x})\Big)dx.\end{aligned}$$

$$\int_{-\infty}^{\infty}\Big(|x|^r P_n(x)\Big)dx \leq \int_{-\infty}^{\infty}\Big(|x|^{r-n}(1+\frac{n}{x})\Big)dx. \qquad \log x < x$$

is convergent for $r-n+1<0$ and integrable on $\mathbb{R}$.

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