International Journal of Analysis and Applications

On Some Graphs Based on the Ideals of JU-Algebras

Ali H. Hakami, Moin A. Ansari*, Azeem Haider

Department of Mathematics, College of Science, Jazan University, Jazan 45142, Saudi Arabia

**Corresponding author:* maansari@jazanu.edu.sa

Abstract. We will construct few types of simple graphs (with no multiple edges or loops) based on the ideal annihilator, right ideal annihilator, left ideal annihilator for JU-algebras. We will also study some graph invariants, such as connectivity, regularity, and planarity for these graphs.

1. Introduction

The motivation of logical algebras arises from the work on BCI/BCK algebras by Imai and Iseki [4] that is actually generalizations of the set theoretic difference and proportional calculi.
In algebraic combinatorics we employ the concepts and workings of modern algebra in many directions of combinatorics. We can associate graphs with algebraic systems and it becomes a research subject and it can be interesting for others attention. The research work in this direction aims to expose the connections on either side of algebraic structures and graph theoretic concepts that actually advancing the applications and uses of one to another. Algebraic graph theory with a zero divisor in the commutative ring R with identity was introduced by Beck [1] in 1998. It was mentioned there that $\Gamma(R)$ is a graph subject to the condition that if two vertices are elements of R so that they will be adjacent to each other if and only if $pq = 0$. Recently, graphs with the zero divisor concept of partially ordered sets are introduced by Halas and Jukl in [3]. In this work, we have considered graphs with zero divisor concepts in JU-algebras based on some of their ideals.
Whereas BCK-algebra was introduced by Imai and Iseki [4] in 1966 and parallely BCI-algebra was introduced by Iseki [5] as a superclass of BCK-algebras. Idea and knowledge based on associated graph of BCK-algebra was introdued by Jun and Lee [6] where some graph is defined and verified. Chordality of the graph was studied by Tahmasbpour in [9] that was defined by Zahiri and Borzooei. Initially it was introduced and constructed for four different graphs of BCK-algebras

Received: Oct. 18, 2023.

2020 *Mathematics Subject Classification.* 13A70.

Key words and phrases. JU-algebras; ideals; graphs.

https://doi.org/10.28924/2291-8639-22-2024-1
ISSN: 2291-8639

based on equivalence classes that was determined by I. Furthermore, Tahmasbpour [9] introduced some graphs of BCK-algebras on fuzzy ideals μ_1. Furthermore, Tahmasbpour [10] defined 12 different graphs of lattice implication algebras on filter and LI-ideal.

Another class of logical algebras, namely, KU-algebra is introduced by Prabpayak and Leerawat [11]. Some of basic properties and homomorphic structural properties of KU-algebras is given in [12]. Later on KU-algebra was widely studied by several authors and the contribution continued to the study through different areas, e.g. in the direction of fuzzy algebras, neutrosophic and intuitionistic algebras with softness and roughness.

Cubic KU-ideals of KU-algebras was introduced by Naveed et al. [13]. Mostafa et al. [14] defined and studied fuzzy ideals of KU-algebras. Furthermore, interval-valued fuzzy KU-ideals in KU-algebras was defined by Mostafa et al. [15]. Roughness in KU-algebras [18] was taken under consideration by Moin and Ali. A pseudo-metric on KU-algebras was constructed and studied some of its properties by Ali et al. [19].

In a consequence of works based on different logical algebras, Moin [16] studied rough set theory on JU-algebras, whereas Moin et al. [17] introduced JU-algebras and p-closure ideals. Whereas Usman et al. [20] introduced pseudo valuations and their metric on JU-algebras.

In this paper. few types of graphs based on the annihilator of ideals, right ideals and left ideals for a JU-algebras is constructed and some graph invariants, such as; connectivity, regularity, and planarity for these graphs.

There are 6 sections in this paper. Section 2, is based on some necessary definitions on the concepts of JU-algebras and graph theory e.g. planar graphs, outer planar graphs, connected graphs, Eulerian graphs and chromatic numbers, among others. Also this section contains simply introductory part of JU-algebras in which we start with the discussion of the concepts of JU-algebras and then investigate their elementary and fundamental properties. Some basic concepts, e.g. ideals and ideal annihilators are given then after. Section 3, is graphs based on the ideal-annihilator of a JU-algebra. Section 4 is study of graphs of JU-algebras based on left and right ideal-annihilator that we denoted by $\phi_1(P)$. Section 5 is related to graphs on the ideals of JU-algebras based on the binary operations $\wedge$ that we have associated with the graph $Q(P)$ that is constructed from binary operations $\wedge$ and $\vee$. Lastly in Section 6, conclusion is given.

2. Preliminaries of JU-Algebras and Graph Theory

Basic definitions, notations and properties related to JU-algebras are considered in this section.

Definition 2.1. *JU-algebra say $(P, \diamond, 1)$ is an algebra of type $(2,0)$ that contains a single binary operation $\diamond$ and satisfy the following (for any $p, q, r \in P$),*

(JU_1) $(q \diamond r) \diamond [(r \diamond p) \diamond (q \diamond p)] = 1$,

(JU_2) $1 \diamond p = p$,

(JU_3) $p \diamond q = q \diamond p = 1$ *implies* $p = q$.

1 is called fixed element of P. We shall write P for $(P, \diamond, 1)$ just to show a JU-algebra. An ordered relation $'' \leq ''$ in P is defined as $q \leq p \Leftrightarrow p \diamond q = 1$. We have that a JU-algebra is generalization of a KU-algebras.

Lemma 2.1. *If P denotes a JU-algebra, then $(P, \leq)$ is POS i.e.,*
(J_4) $p \leq p$,
(J_5) $p \leq q, q \leq p$, *imply* $p = q$,
(J_6) $p \leq r, r \leq q$, *imply* $p \leq q$.

Proof. If $q = r = 1$ in (JU_1) we get $p \diamond p = 1$, i.e. $p \leq p$ which proves (J_4). (J_5) directly follows from (JU_3). For (J_6) take $p \leq r$ and $r \leq q$ implies that $r \diamond p = 1$ and $q \diamond r = 1$. By (JU_1) we have $q \diamond p = 1$ implies that $p \leq q$. □

Lemma 2.2. *If P is a JU-algebra, then following inequalities holds for any $p, q, r \in P$:*
(J_7) $p \leq q$ *implies* $q \diamond r \leq p \diamond r$,
(J_8) $p \leq q$ *implies* $r \diamond p \leq r \diamond q$,
(J_9) $(r \diamond p) \diamond (q \diamond p) \leq q \diamond r$,
(J_{10}) $(q \diamond p) \diamond p \leq q$,

Proof. (J_7), (J_8) and (J_9) follows from (JU_1) by adequate substitution of elements. (JU_1) and (JU_2) implies (J_{10}). □

Lemma 2.3. *In a JU-algebra P for any $p, q, r \in P$, we have the following*
(J_{11}) $p \diamond p = 1$,
(J_{12}) $r \diamond (q \diamond p) = q \diamond (r \diamond p)$,
(J_{13}) *If* $(p \diamond q) \diamond q = 1$, *then P is a KU-algebra,*
(J_{14}) $(q \diamond p) \diamond 1 = (q \diamond 1) \diamond (p \diamond 1)$.

Proof. Letting $q = r = 1$ in JU_1, we have; $p \diamond p = 1$ that proves (J_{11}). For (J_{12}) we have $(r \diamond p) \diamond p \leq r$ by substituting $q = 1$ in (JU_1) and now using (J_7) we get

$$r \diamond (q \diamond p) \leq ((r \diamond p) \diamond p) \diamond (q \diamond p). \tag{2.1}$$

Replace r by $r \diamond p$ in (JU_1) we get, $[q \diamond (r \diamond p)] \diamond [((r \diamond p) \diamond p) \diamond (q \diamond p)] = 1$ that shows

$$((r \diamond p) \diamond p) \diamond (q \diamond p) \leq q \diamond (r \diamond p). \tag{2.2}$$

From (2.1), (2.2) and Lemma 2.1 (J_6) we get,

$$r \diamond (q \diamond p) \leq q \diamond (r \diamond p). \tag{2.3}$$

Next to that we replace $q \to r$ and $r \to q - 1$ in (2.3) we get

$$q \diamond (r \diamond p) \leq r \diamond (q \diamond p). \tag{2.4}$$

Now (2.3), (2.4) and (J_5) yields, $r \diamond (q \diamond p) = q \diamond (r \diamond p)$.
Now to prove (J_{13}) we need to show that $p \diamond 1 = 1, \quad \forall \quad p \in X$. Replacing $q \to 1, p \to 1, r \to p$ in

(JU_1), we obtained, $(1 \diamond p) \diamond [(p \diamond 1) \diamond (1 \diamond 1)] = 1 \Rightarrow p \diamond [(p \diamond 1) \diamond 1] = 1 \Rightarrow p \diamond 1 = 1$ (by using $q = 1$ in the given condition of (J_{13})).
Using (J_{12}) for any $p, q \in P$ in the followings we see that,
$(q \diamond 1) \diamond (p \diamond 1) = (q \diamond 1) \diamond [p \diamond [(q \diamond p) \diamond (q \diamond p)] = (q \diamond 1) \diamond [(q \diamond p) \diamond (p \diamond (q \diamond p))]$
$= (q \diamond p) \diamond [(q \diamond 1) \diamond (q \diamond (p \diamond p))] = (q \diamond p) \diamond [(q \diamond 1) \diamond (q \diamond 1)] = (q \diamond p) \diamond 1.$
Hence (J_{14}) holds. □

Example 2.1. *[14] Consider $P = \{1, 2, 3, 4, 5\}$ then we construct the following table*

$\diamond$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

Clearly P is a JU-algebra.

An example for a JU algebra that may not be a KU-algebra is given here:

Example 2.2. *[17] Let $P = \{1, 2, 3, 4\}$, we construct the following table*

$\diamond$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

It is clear that P is a JU-algebra but not a KU-algebra. Construction of next table shows that at the same time P is a KU-algebra and a JU-algebra both by using a different operation say $\diamond'$.

Example 2.3. *[17] With $P = \{1, 2, 3, 4\}$ and $\diamond$ as binary operation we have the following table*

$\diamond'$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

Definition 2.2. *If $q \diamond p \in J \forall p, q \in J$ so that J is non-void subset of P then it is called a JU subalgebra of P. The set $P_X := \{p \in X | (p \diamond 1) \diamond 1 = p\}$ is known as p-semisimple. P is said to be a p-semisimple JU-algebra if $(p \diamond 1) \diamond 1 = p \forall p \in X$. If j is an elemen of P then we say it is the minimal element of P if $p \leq j$ returns $p = j$ for all $p \in X$. For such a $j \in P$, it is defined by $K(j) := \{p \in X | p \geq j\}$. The set $B_P = \{p \in X | p \diamond 1 = 1\}$ is said to be the JU-part of P.*

Definition 2.3. *A non empty subset J of P is called JU-ideal if*

1. $1 \in J$,
2. $\forall p, q \in X,\ p, p \diamond q$ *imply* $q \in J$.

Definition 2.4. *Let J is a subset of a JU-Algebra P, then is said to be a p-ideal of P if $1 \in J$; and $q, (r \diamond q) \diamond (r \diamond p) \in J$ imply $p \in J$ for any $p, q, r \in X$.*

Definition 2.5. *An ideal J of a JU-Algebra P is called strong if $p \in J$ and $q \notin J$ imply $q \diamond x \notin J$ for any $p, q \in X$.*

Example 2.4. *[17] With $P = \{1, 2, 3, 4, 5, 6\}$ we construct the following table:*

$\diamond$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	1
6	1	1	2	1	1	1

It is clear that $(P, \diamond, 1)$ is a JU-algebra, $A = \{1, 2\}$ and $B = \{1, 2, 3, 4, 5\}$ are ideals of JU-algebra P.

For each ideal of a JU-algebra we can determine a congruence $\sim$ on X so that $p \sim q \iff p \diamond q$ and $q \diamond p \in J$ for $p, q \in P$. we'll use the symbol P/J in place of quotient algebra $P/\sim$, that is actually a JU-algebra.

Likewise as classical concept, in JU-algebras say P, an ideal may not be subalgebra of P. A closed ideal of P is both a subalgebra and an ideal of P. Consider J to be a subset of P, then the smallest ideal of P containing J is called the generated ideal of P by J. Generated ideal is indicated by $\langle J \rangle$.

A mapping $f : X \to p'$ is defined as homomorphism of a JU-algebras $(P, \diamond, 1)$ into a JU-algebra $(P', \diamond', 1')$ if $f(p \diamond q) = f(p) \diamond' f(q)$ for all $p, q \in X$. Clearly, $f(1) = 1'$. Every ideal A of P determines a congruence $\sim$ on P in the sense that $p \sim q$ if and only if $p \diamond q$ and $q \diamond p \in J$ for some $p, q \in X$. P/A stands for quotient algebra of P in stead of $P/\sim$, which is a JU-algebra.

3. Graphs Based on the Ideal-Annihilator of a JU-Algebra

In this section we have mentioned few graphs that are based on JU-ideals and some properties of those graphs are shown.

Definition 3.1. *For a non-void subset A of a JU-algebra P and an ideal I of P. The set of all zero-divisors of A by I is defined as:*

$$Ann_I(A) = \{u \in X |\ a \centerdot u \in I \text{ or } u \centerdot a \in I, \forall a \in A\}.$$

Proposition 3.1. *For any two nonempty subsets A and B of a JU-algebra P and an ideal I of P, the following hold:*

(1) $\{1\} \subseteq Ann_I(A)$.
(2) $I \subseteq Ann_I(A)$.
(3) *If* $A \subseteq B$, *then* $Ann_I B \subseteq Ann_I(A)$.
(4) *If* $1 \in A$, *then* $Ann_I(A) = Ann_1(A - \{1\})$.
(5) $Ann_I(I) = X$.
(6) *If* $I = \{1\}$, *then we have;* $Ann_I(A) = \{p|$ *p is comparable to every element in* $A\}$.

Proof. (1) By (ku2) and Definition 2.2 (1), $a \centerdot 1 = 1 \in I$ for all $a \in A$ and hence $\{1\} \subseteq Ann_I(A)$.

(2) Let $u \in I$, then by Definition 2.1 we have $a \centerdot u \in I, forall a \in A$. Also, $1 \centerdot u = 0, forall u \in P$, So $I \cup \{1\} \subseteq Ann_I(A)$.
(3) Suppose that $u \in Ann_I B$, then $b \centerdot u \in I$ or $u \centerdot b \in I, forall b \in B$, but $A \subseteq B$, therefore $b \centerdot u \in I$ or $u \centerdot b \in I, forall b \in A$. i.e $u \in Ann_I(A)$, hence $Ann_I B \subseteq Ann_I(A)$.
(4) According to Definition 2.1 we have $Ann_I(A) = \cap_{a \in A} Ann_I a$. Also, $Ann_I\{1\} = X$. Then $Ann_I(A) = Ann_I(A - \{1\})$.
(5) Let $u \in P$, we know by Definition 2.1, $u \centerdot a \in I, forall a \in I$, then $u \in Ann_I(I)$, hence $Ann_I(I) = X$.
(6) Follows from the definitions.

□

Definition 3.2. *Let I is an ideal of P and* $\phi_I(P)$ *is a simple graph, where P is vertex set. We have that two different vertices p and q of P are adjacent if and only if* $Ann_I\{x, y\} = I \cup \{1\}$.

Example 3.1. *We construct the following table with consideration of* $P = \{1, a, b, c\}$ *and the operation* $\centerdot$:

$\centerdot$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1
1	1	1	1	1

It is clear that $(P, \centerdot, 1)$ *is a bounded JU-algebra and that* $E(\phi_{\{1\}}(P)) = \{ab, bc, ac\}$.

Theorem 3.1. *We consider I to be an ideal of P, then* $N_G(\{1\}) = \phi$, *where* $G = \phi_I(P)$.

Proof. We know $Ann_I\{1\} = P$ and for all $p \in X, p \neq 1$, we get, $I \cup \{p, 1\} \subseteq Ann_I\{p\}$. Then $I \cup \{x, 1\} \subseteq Ann_I\{p\}$ and $I \cup \{x, 1\} \subseteq Ann_I\{x, 1\}$, for all $p \in X, x \neq 1$. So, by using Definition 2.1 of graph $\phi_I(P)$, for all $p \in X, x \neq 1, p$ is connected to element 1 if and only if $p \in I$, since $p \in I$, therefore by using Proposition 3.1 $Ann_I\{p\} = X$. So the element 1 are not connected to p, for all $p \in X$. □

Theorem 3.2. *Let* $P = \{1\} \cup Atom(P), I = \{1\}$ *be an ideal of* P.

Proof. We know $Ann_{\{1\}}\{1\} = P$ by Proposition 3.1 since $P = Atom(P) \cup \{1\}$, we have, for all $P \in Atom(P)$, $Ann_{\{1\}}\{p\} = \{1, x\}$. It is also clear that $Ann_{\{1\}}\{x, y\} = Ann_{\{1\}}\{p\} \cap Ann_{\{1\}}\{q\}$. Then by Definition 3.2 of graph $\phi_{\{1\}}(P)$, p and q are adjacent if and only if $p, y \in Atom(P)$. □

Theorem 3.3. *Let $P = \{1\} \cup Atom(P)$. Then we have:*
$\omega(\phi_{\{1\}}(P)) = |Atom(P)|$.

Proof. Followed by Theorem 3.2. □

Theorem 3.4. *Consider $I = \{1\}$ is an ideal of P, then $N_G(p) = \{y; y \text{ is uncomparable with } x\}$, and $G = \phi_I(P), x \neq 1$.*

Proof. For every $p \in X, x \neq 1$, we get $Ann_{\{1\}}\{x\} = \{y; y \text{ is uncomparable with } x\}$. Next to that, we know $Ann_{\{1\}}\{x, y\} = Ann_{\{1\}}\{p\} \cap Ann_{\{1\}}\{q\}$. Then by Definition 3.2 of graph $\phi_{\{1\}}(P)$, p and q are adjacent $\Leftrightarrow$ p and q are uncomparable. □

Theorem 3.5. *I being ideal of P. We get $\alpha(\phi_I(P)) \geq |I|$.*

Proof. Letting $p, y \in I$. Using Proposition 3.1(5) we get, $Ann_I\{x\} = P$ and $Ann_I\{y\} = X$. Therefore, by Definition 3.2 of graph $\phi_I(P)$, $q \centerdot x \notin E(\phi_I(p))$. Therefore, we have $\alpha(\phi_I(P)) \geq |I|$. □

Theorem 3.6. *Let $|X| > 2$ and I to be a prime ideal, then $\phi_I(P)$ is a null graph.*

Proof. On the contrary suppose that $\phi_I(P)$ is a nonempty graph. Then there $\exists p, y \in P$, such that $py \in E(\phi_I(P))$. Using 3.2 of graph $\phi_I(P)$, we get, $Ann_I\{x, y\} = I \cup \{1\}$. Also, since $|X - I| > 1$, we can select $r \in X, z \notin 1, z \neq 1$. Since I is prime, so $p \centerdot z \in I$ or $r \centerdot x \in I$, and $q \centerdot z \in I$ or $r \centerdot y \in I$, hence $r \in Ann_I\{x, y\}$ that contradict. □

4. Graphs of JU-Algebras Based on Left and Right Ideal-Annihilator

Definition 4.1. *I being an ideal of P, the sets $Ann_I^R\{x\} = \{y \in X; x \centerdot y \in I\}$, and $Ann_I^L\{p\} = \{y \in X; y \centerdot x \in I\}$ are called annihilators of right ideals and left ideals of p, respectively.*

Definition 4.2. *I being ideal of P then, we get: $\Sigma_I(P)$ and $\Delta_I(P)$ are two simple graphs, whose vertex set is P whose two different vertices are p and q that are adjacent in $\Sigma_I(P)$ if and only if $Ann_I^R\{p\} \subseteq Ann_I^R\{q\}$ or $Ann_I^R\{q\} \subseteq Ann_I^R\{p\}$. Also, there is an edge between p and q in the graph $\Delta_I(P)$ if and only if $Ann_I^L\{p\} \subseteq Ann_I^L\{q\}$ or $Ann_I^L\{q\} \subseteq Ann_I^L\{p\}$.*

Example 4.1. *We construct the table with $P = \{1, a, b, c, d\}$ and the operation $\centerdot$ as below:*

$\centerdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	c
b	1	1	1	c	c
c	1	1	a	1	a
d	1	1	1	1	1

It is clear that $(P, \centerdot, 1)$ *is a bounded JU-algebra of* P.
We can see the graphs $\Sigma_{\{1\}}(P)$ *and* $\Delta_{\{1\}}(P)$ *will represent the same graph that is shown in the following figure.*

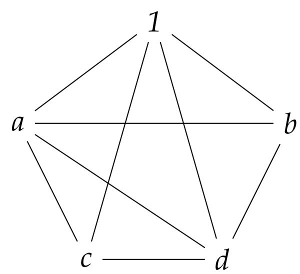

Figure 1: Graph of $\Sigma_{\{1\}}(P)$ *and* $\Delta_{\{1\}}(P)$

Proposition 4.1. *I being an ideal of* P. *We have:*

(1) $\omega(\Sigma_I(P)) \geq max\{|A|; A \text{ is a chain in } X\}$
(2) $\omega(\Delta_I(P)) \geq max\{|A|; A \text{ is a chain in } X\}$

Proof. (1) According to Definition 2.1 if $p \leq q$ then, $r \centerdot x \leq z \centerdot y$. Next, let $p \leq q, z \in Ann_I^R\{q\}$. Using Definition 4.1, $r \centerdot y \in I$. Thus, with Definition 2.2 of ideal, $r \centerdot x \in I$. So, $r \in Ann_I^R\{p\}$, we get, $Ann_I^R\{q\} \subseteq Ann_I^R\{p\}, x \centerdot y \in E(\Sigma_I(P))$.
(2) Similar as part (1).

□

Theorem 4.1. *I being ideal of* P. *We have:*

(1) $\Sigma_I(P)$ *is connected,* $diam(\Sigma_I(P)) \leq 2$, $gr(\Sigma_I(P)) = 3$.
(2) $\Delta_I(P)$ *is connected,* $diam(\Delta_I(P)) \leq 2$, $gr(\Delta_I(P)) = 3$.

Proof. (1) For every $p \in X, x \leq 1$, then by Proposition 4.1, the element 1 is connected to every element in P. Therefore, $\Sigma_I(P)$ is connected and hence $diam(\Sigma_I(P)) \leq 2$. Moreover, $gr(\Sigma_I(P)) = 3$.
(2) Similar as part (1).

□

Theorem 4.2. *I being ideal of* P. *We have:*

(1) $\Sigma_I(P)$ *is regular if and only if it is complete.*
(2) $\Delta_I(P)$ *is regular if and only if it is complete.*

Proof. (1) Consider $\Sigma_I(P)$ to be a regular graph. But $deg(1) = |X| - 1$, therefore, for all $p \in X, deg(p) = |X| - 1$. Hence, $\Sigma_I(P)$ is a complete graph. As an indirect part, we can say a complete graph is always regular.
(2) Similar as part (1).

□

Proposition 4.2. *Let P be a chain, I be an ideal of P. Then the graphs $\Sigma_I(P)$ and $\Delta_I(P)$ are planar if and only if $|X| \leq 4$.*

Proof. Using Proposition 4.1, the graphs $\Sigma_I(P)$ and $\Delta_I(P)$ are complete graphs K_2 and K_3 for $|x| = 3$ and $|X| = 4$, respectively and hence they are planar for $|X| \leq 4$. Now if $|X| \geq 5$, then $\Sigma_I(P)$ and $\Delta_I(P)$ have a subgraph isomorphic to K_5, consequently, by using Kuratowski's Theorem the graphs $\Sigma_I(P)$ and $\Sigma_I(P)$ are nonplanar. Indirectly, it is known that there are five vertices in K_5, thus if any graph $\Sigma_I(P)$ or $\Delta_I(P)$, is not planar, and hence there are at least five vertices in the graphs $\Sigma_I(P)$ and $\Sigma_I(P), (P)$, which is not true and hence $|X| \leq 4$. □

Proposition 4.3. *Let P be a chain, I be an ideal of P. Then the graphs $\Sigma_I(P)$ and $\Delta_I(P)$ are outer planar if and only if $|X| \leq 3$.*

Proof. According to Proposition 4.1, the graphs $\Sigma_I(P)$ and $\Delta_I(P)$ are complete graphs, now if $|X| \geq 4$, then both graphs $\Sigma_I(P)$ and $\Delta_I(P)$ have a subgraph that is isomorphic to K_4 and hence by [8], the graphs $\Sigma_I(P)$ and $\Delta_I(P)$ are not outer planar. It is known that K_4 has four vertices hence if any of the graphs $\Sigma_I(P)$ or $\Delta_I(P)$ are non outer planar, hence there are at least four vertices, in the graph $\Sigma_I(P)$ and $\Delta_I(P)$, that is actually contrary to the fact that $|X| \leq 3$. □

Proposition 4.4. *Let P be a chain, I be an ideal of P. Then the graphs $\Sigma_I(P)$ and $\Delta_I(P)$ are toroidal graphs if and only if $|X| \leq 7$.*

Proof. By using the Proposition 4.1, the graphs $\Sigma_I(P)$ and $\Delta_I(P)$ are complete graphs. If $|X| \geq 8$, then the both graphs $\Sigma_I(P)$ and $\Delta_I(P)$ have a subgraph that is isomorphic to K_8. Now by [8], both graphs $\Sigma_I(P)$ and $\Delta_I(P)$ are not toroidal. Conversely, since K_8 has eight vertices, therefore, both the graphs $\Sigma_I(P)$ and $\Delta_I(P)$ are not toroidal and hence both the graphs $\Sigma_I(P)$ and $\Delta_I(P)$ has at least eight vertices, that is contrary to the fact that $|X| \leq 7$. □

5. Graphs on the Ideals of JU-Algebras Based on the Binary Operations $\wedge$

For this section we assume that, the set P will represent a bounded and commutative JU-algebra.

Definition 5.1. *Letting I to be an ideal of P, we can construct a simple graph $\Upsilon_I(P)$ with vertex set P and two distinct vertices p and q are adjacent if and only if $p \wedge y \in I$.*

Example 5.1. *We construct a table with $P = \{1, a, b, c, d, e\}$ and the operation **.** as follows:*

.	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
e	1	1	1	1	1	1

Thus, $(P, \cdot, 1)$ *is both a bounded and commutative KU-algebra. We can simply check that* $I = \{1, a\}$ *is an ideal of* P.

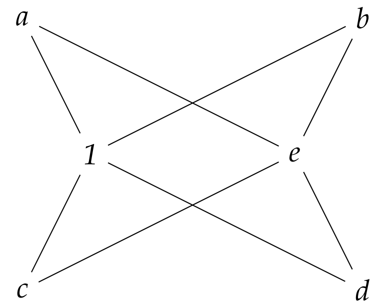

Figure 2: The graph of $\Upsilon_I(P)$.

Lemma 5.1. *Let* I *be an ideal of* P. *Then* $deg(p) = |X| - 1$ *for all* $p \in I$ *in the graph* $\Upsilon_I(P)$.

Proof. Let $p \in I$ and q be an arbitrary element of P. Then $(p \cdot q) \cdot q \in I$. Since $(p \cdot q) \cdot q \leq p$, as I is an ideal of P. So, $q \cdot x \in E(\Upsilon_I(P))$. □

Theorem 5.1. *Let* I *being an ideal of* P, *the graph* $\Upsilon_I(P)$ *would be regular if and only if it is complete.*

Proof. Consider $\Upsilon_I(P)$ as a regular graph. By using Lemma 5.1, we shall get $deg(1) = |X| - 1$. It is known here that $\Upsilon_I(P)$ is regular, hence, for some $p \in X, deg(p) = |X| - 1$. That shows $\Upsilon_I(P)$ is a complete graph. Indirectly, if a graph is complete it is always regular. □

The following Proposition 5.1 and Theorem 5.2 are follows from Lemma 5.1.

Proposition 5.1. *I being an ideal of* P. *We get that* $\omega(Q_I(P)) \geq |I|$.

Theorem 5.2. *I being an ideal of* P. *We get that* $\Upsilon_I(P)$ *is connected and* $diam(\Upsilon_I(P)) \leq 2$.

Theorem 5.3. *Let* I *be an ideal of* P. *Then* $gr(\Upsilon_I(P)) = 3$.

Proof. Let $a \neq 1 \in I$ and p be an arbitrary element in P. Then easily we have that $1 - a - x - 1$ is a cycle of length 3 in $\Upsilon_I(P)$. □

Proposition 5.2. *With an ideal* I *of* P *we have the below statements hold:*

(1) *In case of* $\Upsilon_I(P)$ *to be a planar graph,* $|I| \leq 4$.
(2) *In case of* $\Upsilon_I(P)$ *to be an outer planar,* $|I| \leq 3$.
(3) *In case of* $\Upsilon_I(P)$ *to be a toroidal,* $|I| \leq 7$.

Proof. (1) From Lemma 5.1, the graph $\Upsilon_I(P)$ is complete graph on I. Next, if we have $|I| \geq 5$, then $\Upsilon_I(P)$ we get that there is a subgraph of it that is isomorphic to K_5 that further by Kuratowski's theorem, is non planar.

(2) From Lemma 5.1, the graph $\Upsilon_I(P)$ is complete graph on I. Next, if we have $|I| \geq 4$, then $\Upsilon_I(P)$ we get that it has a subgraph isomorphic to K_4 that is by [8], returns that the graph $\Upsilon_I(P)$ is not outer planar.

(3) Again by uaing Lemma 5.1, we see that the graph $\Upsilon_I(P)$ is a complete graph. Further if $|I| \geq 7$, then for $\Upsilon_I(P)$ there is always a subgraph that is isomorphic to K_8. Thus using [8], the graph $\Upsilon_I(P)$ is not toroidal.

□

Theorem 5.4. *For an ideal I of P if $\Upsilon_I(P)$ is an Euler graph, then $|X|$ is odd.*

Proof. From Lemma 5.1, for all $p \in I$, $deg(p) = |X| - 1$. If $\Upsilon_I(P)$ is an Euler graph, and so degree of every vertex in I is even. Implies, $|X|$ is an odd number. □

Theorem 5.5. *I being an ideal of P and $I = \cap_{1\leq i\leq n} P_i$. For every $1 \leq j \leq n$, the ideals $I \neq \cap_{1\leq i\leq n, i\neq j} P_i$, where P_i are prime ideals of P, then $\omega(Q_I(P)) = n = \chi(Q_I(P))$.*

Proof. For every j having $1 \leq j \leq n$, we suppose p_j in $(\cap_{1\leq i\leq n, i\neq j} P_i) - P_j$. We have $A = \{p, ..., x_n\}$ is a clique in $\Upsilon_I(P)$. Thus $\omega(\Upsilon_I(P)) \geq n$. Next to show that, $\chi(\Upsilon_I(P)) \leq n$. Define a coloring f by choosing $f(p) = min\{i; x \notin P_i\}$. Suppose $f(p) = k, p$ and q are two adjacent vertices. Naturally, $p \notin P_k$ and $p \wedge y \in I$. Since we had let P_k prime, hence $q \in P_k$, and thus $f(q) \neq k$. Again, since $\omega(\Upsilon_1(P)) \leq \chi(\Upsilon_I(P))$, implies that the result hold. □

Theorem 5.6. *Consider I is an ideal of P. If $I = \cap_{j\in J} P_j$, where P_j are prime ideals of P and J is an infinite set for each $i \in J$, also $I \neq \cap_{j\neq i} P_j$, then $\omega(\Upsilon_I(P)) = \infty = \chi(\Upsilon_I(P))$.*

Proof. For every $i \in J$, we have $p_i \in (\cap_{j\neq i} P_j - P_i)$. It is easy to see that the set of p_i yields an infinite clique in $\Upsilon_I(P)$. Since $\omega(\Upsilon_I(P)) \leq \chi(\Upsilon_I(P))$, it proves the required result. □

6. Conclusion

In this article we have studied and discussed annihilators based on right-ideals, left-ideals and ideals for JU-algebras. Furthermore, construction of some main classes of graphs in a bounded JU-algebra $(P, \bullet, 0)$ related to ideals that are denoted by $\Phi_1(P)$, $\Delta_1(P)$ *and* $\Sigma_1(P)$ are taken under consideration. Then some graphical properties such as planarity, regularity, and connectivity on the structure of these graphs are studied. We have constructed the graph $Q_I(P)$ and have studied their properties with these aspects. The dual of all the above concepts can be study in the future work.

Acknowledgment: The authors extend their appreciation to the Deanship of Scientific Research, Jazan University for supporting this research work through the Research Units Support Program, Support Number: RUP2-02.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] I. Beck, Coloring of Commutative Rings, J. Algebra. 116 (1988), 208–226. https://doi.org/10.1016/0021-8693(88)90202-5.

[2] R. Diestel, Graph Theory, Springer, New York, 1997.

[3] R. Halas, M. Jukl, On Beck's Coloring of Posets, Discr. Math. 309 (2009), 4584–4589. https://doi.org/10.1016/j.disc.2009.02.024.

[4] Y. Imai, K. Iseki, On Axiom Systems of Propositional Calculi, XIV, Proc. Japan Acad. Ser. A Math. Sci. 42 (1966), 19–22. https://doi.org/10.3792/pja/1195522169.

[5] K. Iseki, An Algebra Related With a Propositional Calculus, Proc. Japan Acad. Ser. A Math. Sci. 42 (1966), 26–29. https://doi.org/10.3792/pja/1195522171.

[6] Y.B. Jun, K.J. Lee, Graphs Based on BCK/BCI-Algebras, Int. J. Math. Math. Sci. 2011 (2011), 616981. https://doi.org/10.1155/2011/616981.

[7] A. Mohammadian, A. Erfanian, M. Farrokhi, Planar, Toroidal and Projective Generalized Peterson Graphs, In: Proceeding of the 7th Algebraic Combinatorics Conference of Iran, October 16-17, 2014, Ferdowsi University of Mashhad, Mashhad, Iran, 2014.

[8] F. Shahsavar, M. Afkhami, K. Khashyarmanesh, On End-Regular, Planar and Outer Planar of Zero-Divisor Graphs of Posets, In: Proceeding of the 6th Algebraic Conference, Iran, 30-31 October 2013, University of Mazandaran, Babolsar, Iran, 2013.

[9] A.T. Meikola, Graphs of BCI/BCK-Algebras, Turk. J. Math. 42 (2018), 1272-1293.

[10] A.T. Meikola, Twelve Kinds of Graphs of Lattice Implication Algebras Based on Filter and LI-Ideal, Cankaya Univ. J. Sci. Eng. 17 (2020), 11–40.

[11] C. Prabpayak, U. Leerawat, On Ideals and Congruences in KU-Algebras, Sci. Magna. 5 (2009), 54–57.

[12] C. Prabpayak, U. Leerawat, On Isomorphisms of KU-Algebras, Sci. Magna. 5 (2009), 25–31.

[13] N. Yaqoob, S.M. Mostafa, M.A. Ansari, On Cubic KU-Ideals of KU-Algebras, ISRN Algebra. 2013 (2013), 935905. https://doi.org/10.1155/2013/935905.

[14] S.M. Mostafa, M.A. Abd-Elnaby, M.M.M. Yousef, Fuzzy Ideals of KU-Algebras, Int. Math. Forum. 6 (2011), 3139–3149.

[15] S.M. Mostafa, M.A. Abd-Elnaby, O.R. Elgendy, Interval Valued Fuzzy KU-Ideals in KU-Algebras, Int. Math. Forum. 6 (2011), 3151–3159.

[16] M.A. Ansari, Rough Set Theory Applied to JU-Algebras, Int. J. Math. Comput. Sci. 16 (2021), 1371-1384.

[17] M.A. Ansari, A. Haider, A.N. Koam, On JU-Algebras and p-Closure Ideals, Int. J. Math. Comput. Sci. 15 (2020), 135–154.

[18] M.A. Ansari, A.N.A. Koam, Rough Approximations in KU-Algebras, Italian J. Pure Appl. Math. 40 (2018), 679–691.

[19] A.N.A. Koam, A. Haider, M.A. Ansari, Pseudo-Metric on KU-Algebras, Korean J. Math. 27 (2019), 131–140. https://doi.org/10.11568/KJM.2019.27.1.131.

[20] U. Ali, M.A. Ansari, M.U. Rehman, Pseudo-Valuations and Pseudo-Metric on JU-Algebras, Open J. Math. Sci. 3 (2019), 440–446. https://doi.org/10.30538/oms2019.0086.