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On the Stability of Quadratic-Quartic (Q_2Q_4) Functional Equation over Non-Archimedean Normed Space

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Abstract. In the present work the stability of Hyers-Ulam mixed type of quadratic-quartic Cauchy functional equation

$$\mathfrak{g}(2x+y)+\mathfrak{g}(2x-y)=4\mathfrak{g}(x+y)+4\mathfrak{g}(x-y)+2\mathfrak{g}(2x)-8\mathfrak{g}(x)-6\mathfrak{g}(y)$$

has been proved over Non-Archimedean normed space.

1. Introduction

The stability of functional equations arose in 1940 from a question by Ulam [18] on the stability of group homomorphisms.

Given two groups H_1 and H_2 with the metric $d(.,.)$ on H_2 and for $\varepsilon > 0$, does there exist $\delta > 0$ such that if a mapping $G : H_1 \to H_2$ satisfies the inequality $d(G(a,b), G(a)G(b)) < \delta$ for all $a, b \in H_1$, then there exist a homomorphism $G' : H_1 \to H_2$ with $d(G(a), G'(a)) < \varepsilon$ for every $a \in H_1$?

Hyers [12] gave the very first positive response to Ulam's question for Banach spaces such that

$$\|\mathfrak{g}(x+y)-\mathfrak{g}(x)-\mathfrak{g}(y)\| \leq \delta \tag{1.1}$$

for all $x, y \in E$ and for some $\delta > 0$. Then there is a unique additive mapping $\top : E \to E'$ such that

$$\|\mathfrak{g}(x)-\top(x)\| \leq \delta \tag{1.2}$$

for all $x \in E$. Furthermore, if $\mathfrak{g}(tx)$ is continuous at $t \in R$ for any fixed $x \in E$, then $\top$ is linear (For instance [3]).

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For the quadratic functional equation

$$\mathfrak{g}(x+y)+\mathfrak{g}(x-y)=2\mathfrak{g}(x)+2\mathfrak{g}(y), \tag{1.3}$$

we note that the quadratic function $\mathfrak{g}(x)=x^2$ is a solution of (1.3). So one usually calls the above functional equation is quadratic and every solution of (1.3) is said to be a quadratic mapping.

Aoki [1] generalized the Hyers theorem for additive mappings. Hyers theorem was generalized by Rassias [16] by allowing the Cauchy difference to be unbounded. Gajada [7] responded to the question for the case $p>1$, posed by Rassias. Moslehian and Rassias [14] proved generalized HUS of the Cauchy functional equation and the $\mathcal{Q}_2F$ equation in NAN spaces.

In this present article, to the best of our knowledge there is no discussion so far concerning the HUS of the following $\mathcal{Q}_2$-$\mathcal{Q}_4F$ equation.

$$\mathfrak{g}(2x+y)+\mathfrak{g}(2x-y)=4\mathfrak{g}(x+y)+4\mathfrak{g}(x-y)+2\mathfrak{g}(2x)-8\mathfrak{g}(x)-6\mathfrak{g}(y) \tag{1.4}$$

in NAN space. It is easy to show that the function $f(x)=ax^2+bx^4$ is a solution of the functional equation (1.4), which is called a mixed type $\mathcal{Q}_2$-$\mathcal{Q}_4F$ equation. For more detailed definitions of mixed type functional equations, we can refer to [4–6, 8–10, 13, 15].

Throughout this article, assume that $\mathcal{G}$ is an additive group, $\mathcal{X}$ is a complete NAN space and V_1, V_2 are vector spaces. Denote the Q_2-Q_4F equation as given below.

$$\mathcal{Q}\mathfrak{g}(x,y)=\mathfrak{g}(2x+y)+\mathfrak{g}(2x-y)-4\mathfrak{g}(x+y)-4\mathfrak{g}(x-y)-2\mathfrak{g}(2x)+8\mathfrak{g}(x)+6\mathfrak{g}(y) \text{ for every } x,y\in\mathcal{G}. \tag{1.5}$$

Consider this functional inequality

$$\|\mathcal{Q}\mathfrak{g}(x,y)\|\leq\varphi(x,y) \tag{1.6}$$

for an upper bound a function φ from $\mathcal{G}^2\to[0,\infty)$.

2. Preliminaries

In 1897, Hensel [11] has introduced a normed space which does not have the Archimedean property. It turnout that Non-Archimedean spaces have many nice applications [17, 19]. The basic definition and properties of Non-Archimedean space are as follows.

Definition 2.1. *[2] A Non-Archimedean field is a field $\mathcal{K}$ equipped with a function (valuation) $|.|$ from $\mathcal{K}$ into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r||s|$, and $|r+s|\leq\max\{|r|,|s|\}$ for all $r,s\in\mathcal{K}$. Clearly $|1|=|-1|=1$ and $|\eta|\leq 1$ for all $\eta\in\mathcal{N}$.*

Definition 2.2. *Let $\mathcal{X}$ be a linear space over a Non-Archimedean field $\mathcal{K}$ with a non-trivial valuation $|.|$. A function norm from $\mathcal{X}$ to $\mathcal{R}$ is a Non-Archimedean norm if it satisfies the following conditions:*

(NA1) $\|r\|\geq 0$ *and* $=0$ *iff* $r=0$,

(NA2) $\|\alpha r\|=|\alpha|\|r\|, \alpha\in\mathcal{K}, r\in\mathcal{X}$,

(NA3) $\|r+s\|\leq\max\{\|r\|,\|s\|\}, r,s\in\mathcal{X}$.

Then $(\mathcal{X},\|.\|)$ is called a Non-Archimedean normed space.

Because of this,

$\|x_p - x_q\| \le \max\{\|x_{r+1} - x_r\| : q \le r \le p-1\}$ $(p > q)$

A sequence $\{x_p\}$ is Cauchy iff $\{x_{p+1} - x_p\}$ converges to zero in a NAN space. By a complete NAN space every Cauchy sequence is convergent.

The most important examples of Non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for $x, y > 0$, there exists $\eta \in \mathcal{N}$ such that $x < \eta y$.

3. Main Results

In this section, using the direct method, we investigate the HUS for the quadratic-quartic ($\mathcal{Q}_2\mathcal{Q}_4$) functional equation (1.4) in Non-Archimedean normed space. We start this section with the following lemma.

Lemma 3.1. *[9] If a function $\mathfrak{g}$ from V_1 to V_2 satisfies (1.4), then $\mathfrak{g}$ is quadratic-quartic.*

Theorem 3.1. *Let a function φ from $\mathcal{G}^2 \to [0,\infty)$ be such that*

$$\lim_{\eta\to\infty} \frac{1}{|2|^{2\eta}}\varphi(2^{\eta+1}x, 2^{\eta+1}y) = 0 = \lim_{n\to\infty} \frac{1}{|2|^{2\eta}}\bar{\varphi}(2^{\eta}x) \text{ for every } x, y \in \mathcal{G}. \tag{3.1}$$

Let $\mathfrak{g}$ be an even function from $\mathcal{G}$ to $\mathcal{X}$ that satisfies (1.6) and $\mathfrak{g}(0) = 0$.
Then, uniqueness of quadratic function $\mathcal{Q}_2$ from $\mathcal{G} \to \mathcal{X}$ exists and

$$\|\mathfrak{g}(2x) - 16\mathfrak{g}(x) - \mathcal{Q}_2(x)\| \le \frac{1}{|2|^2}\psi_{q_2}(x) \text{ for each } x \in \mathcal{G}. \tag{3.2}$$

where

$$\psi_{q_2}(x) = \lim_{\eta\to\infty} \max\left\{\frac{1}{|2|^{2\jmath}}\bar{\varphi}(2^{\jmath}x) : 0 \le \jmath < \eta\right\} \tag{3.3}$$

and

$$\bar{\varphi}(x) = \max\left\{|4|\varphi(x,x), \varphi(x,2x)\right\} \text{ exists for each } x \in \mathcal{G}. \tag{3.4}$$

Proof. Putting y by x in (1.6), we get

$$\|\mathfrak{g}(3x) - 6\mathfrak{g}(2x) + 15\mathfrak{g}(x)\| \le \varphi(x,x)$$

$$\|4\mathfrak{g}(3x) - 24\mathfrak{g}(2x) + 60\mathfrak{g}(x)\| \le |4|\ \varphi(x,x) \tag{3.5}$$

In equation (1.6) substituting $y = 2x$, we obtain

$$\|\mathfrak{g}(4x) - 4\mathfrak{g}(3x) + 4\mathfrak{g}(2x) + 4\mathfrak{g}(x)\| \le \varphi(x,2x) \tag{3.6}$$

Using (3.5) it follows that,

$$\|\mathfrak{g}(4x) - 20\mathfrak{g}(2x) + 64\mathfrak{g}(x)\| \le \bar{\varphi}(x) \text{ for each } x \in \mathcal{G}. \tag{3.7}$$

Let h_1 be a mapping from $\mathcal{G}$ into $\mathcal{X}$ defined by $h_1(x) = \mathfrak{g}(2x) - 16\mathfrak{g}(x)$ for each $x \in \mathcal{G}$. we conclude this (3.7) as

$$\|h_1(2x) - 4h_1(x)\| \leq \bar{\varphi}(x) \ for\ each\ x \in \mathcal{G}. \tag{3.8}$$

Replacing x by $2^{\eta-1}x$ in (3.8), we arrive

$$\left\|\frac{h_1(2^\eta x)}{2^{2\eta}} - \frac{h_1(2^{\eta-1}x)}{2^{2(\eta-1)}}\right\| \leq \frac{1}{|2|^{2\eta}}\ \bar{\varphi}(2^{\eta-1}x) \ for\ each\ x \in \mathcal{G}. \tag{3.9}$$

It follows from (3.1) and (3.9) the sequence $\left\{\frac{h_1(2^\eta x)}{2^{2\eta}}\right\}$ is Cauchy. Since $\mathcal{X}$ is complete, we conclude $\left\{\frac{h_1(2^\eta x)}{2^{2\eta}}\right\}$ is convergent.

$$Let\ Q_2(x) = \lim_{\eta\to\infty} \frac{h_1(2^\eta x)}{2^{2\eta}} \ for\ each\ x \in \mathcal{G}. \tag{3.10}$$

From (3.8) and (3.9) it follows by induction that

$$\begin{aligned}
\left\|\frac{h_1(2^\eta x)}{2^{2\eta}} - h_1(x)\right\| &\leq \max\left\{\frac{1}{|2|^{2\eta}}\bar{\varphi}(2^{\eta-1}x)\ldots\frac{1}{|2|^2}\bar{\varphi}(2^0x)\right\} \\
\left\|\frac{h_1(2^\eta x)}{2^{2\eta}} - h_1(x)\right\| &\leq \frac{1}{|2|^2}\max\left\{\frac{1}{|2|^{2\jmath}}\bar{\varphi}(2^\jmath x) \ : 0 \leq \jmath < \eta\right\}
\end{aligned} \tag{3.11}$$

for each $\eta \in \mathcal{N}$ and for all $x \in \mathcal{G}$. As $\eta \to \infty$ in (3.11) and using (3.3), we get (3.2). Now to prove Q_2 is quadratic. It follows from (3.1), (3.9) and (3.10) we obtain

$$\begin{aligned}
\|Q_2(2x) - 4Q_2(x)\| &= \lim_{\eta\to\infty}\left\|\frac{h_1(2^\eta 2x)}{2^{2\eta}} - 2^2\frac{h_1(2^\eta x)}{2^{2\eta}}\right\| \\
&\leq |2|^2 \lim_{\eta\to\infty}\frac{1}{|2|^{2(\eta+1)}}\ \bar{\varphi}(2^\eta x) \\
&= \lim_{n\to\infty}\frac{1}{|2|^{2\eta}}\ \bar{\varphi}(2^\eta x) \\
&= 0 \ for\ each\ x \in \mathcal{G}.
\end{aligned}$$

Hence,

$$Q_2(2x) = 4Q_2(x) \ for\ each\ x \in \mathcal{G}. \tag{3.12}$$

On the other hand (1.6), (3.1) and (3.10) implies that

$$\begin{aligned}
\|Q_{Q_2}(x,y)\| &= \left\|\lim_{\eta\to\infty}\frac{Qh_1(2^\eta x, 2^\eta y)}{2^{2\eta}}\right\| \\
&= \lim_{\eta\to\infty}\frac{1}{|2|^{2\eta}}\left\|Qh_1(2^\eta x, 2^\eta y)\right\|
\end{aligned}$$

$$= \lim_{\eta\to\infty} \frac{1}{|2|^{2\eta}} \left\| Q\mathfrak{g}(2^{\eta+1}x, 2^{\eta+1}y) - 16Q\mathfrak{g}(2^{\eta}x, 2^{\eta}y) \right\|$$

$$\leq \lim_{\eta\to\infty} \frac{1}{|2|^{2\eta}} \max\left\{ \varphi(2^{\eta+1}x, 2^{\eta+1}y), |16|\varphi(2^{\eta}x, 2^{\eta}y) \right\}$$

$$= 0 \ for\ each\ x \in \mathcal{G}.$$

Therefore, Q_2 satisfies (1.4). By Lemma (3.1), $|Q_2(2x) - 16Q_2(x)|$ is quadratic. Therefore, Q_2 is quadratic.

Uniqueness: Let there exist another quadratic function Q_2'

$$\|Q_2(x) - Q_2'(x)\| = \lim_{\iota\to\infty} \left\| \frac{Q_2(2^{\iota}x)}{2^{2\iota}} - \frac{Q_2'(2^{\iota}x)}{2^{2\iota}} \right\|$$

$$\leq \lim_{\iota\to\infty} \frac{1}{|2|^{2\iota}} \max\left\{ \left\| Q_2(2^{\iota}x) - h_1(2^{\iota}x)\|, \|h_1(2^{\iota}x) - Q_2'(2^{\iota}x) \right\| \right\}$$

$$\|Q_2(x) - Q_2'(x)\| \leq \frac{1}{|2|^2} \lim_{\iota\to\infty} \lim_{\eta\to\infty} \max\left\{ \frac{1}{|2|^{2\jmath}} \bar{\varphi}(2^{\jmath}x) : \iota \leq \jmath < \eta + \iota \right\}$$

$$= 0 \ for\ each\ x \in \mathcal{G}.$$

Therefore $Q_2(x) = Q_2'(x)$. □

Theorem 3.2. *Let a function φ from $\mathcal{G}^2 \to [0, \infty)$ be such that*

$$\lim_{\eta\to\infty} |2|^{2\eta} \varphi\left(\frac{x}{2^{\eta+1}}, \frac{y}{2^{\eta+1}}\right) = 0 = \lim_{\eta\to\infty} |2|^{2\eta} \bar{\varphi}\left(\frac{x}{2^{\eta}}\right) for\ every\ x, y \in \mathcal{G}.$$

Let $\mathfrak{g}$ be an even function from $\mathcal{G}$ to $\mathcal{X}$ that satisfies (1.6) and $\mathfrak{g}(0) = 0$.
Then, uniqueness of quadratic function Q_2 from $\mathcal{G} \to \mathcal{X}$ exists and

$$\|\mathfrak{g}(2x) - 16\mathfrak{g}(x) - Q_2(x)\| \leq |2|^2 \psi_{q_2}(x) \ for\ each\ x \in \mathcal{G}.$$

where

$$\psi_{q_2}(x) = \lim_{\eta\to\infty} \max\left\{ |2|^{2\jmath} \bar{\varphi}\left(\frac{x}{2^{\jmath}}\right) : 0 \leq \jmath < \eta \right\}$$

and

$$\bar{\varphi}(x) = \max\left\{ |4|\varphi(x, x), \varphi(x, 2x) \right\} exists\ for\ each\ x \in \mathcal{G}.$$

Proof. The proof is similar to Theorem (3.1). □

Corollary 3.1. *Let r, s and δ are positive real numbers. Define a function $\mathfrak{g}$ from $\mathcal{G}$ to $\mathcal{X}$ and if a quadratic mapping satisfying the inequality*

$$\|Q\mathfrak{g}(x,y)\| \leq \delta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^{r}\|y\|^{s}) \text{ for all } x, y \in \mathcal{G}.$$

$$\|\mathfrak{g}(2x) - 16\mathfrak{g}(x) - Q_2(x)\| \leq \frac{1}{|2|^2}\psi_{q_2}(x) \text{ for each } x \in \mathcal{G}.$$

where

$$\psi_{q_2}(x) = \delta(1 + |2|^{r+s} + |2|^{s})\|x\|^{r+s}$$

Then,

(i) For $r + s > 2$, there is a unique quadratic mapping $Q_2(x) : \mathcal{G} \to \mathcal{X}$ such that

$$\|\mathfrak{g}(2x) - 16\mathfrak{g}(x) - Q_2(x)\| \leq \frac{\delta}{|2|^2}(1 + |2|^{r+s} + |2|^{s})\|x\|^{r+s}$$

(ii) For $r + s < 2$, there is a unique quadratic mapping $Q_2(x) : \mathcal{G} \to \mathcal{X}$ such that

$$\|\mathfrak{g}(2x) - 16\mathfrak{g}(x) - Q_2(x)\| \leq |2|^2\delta(1 + |2|^{r+s} + |2|^{s})\|x\|^{r+s} \text{ for each } x \in \mathcal{G}.$$

For the case $r + s = 2$, we have the following counter example.

Example 3.1. *Let $p > 2$, be a prime number and $\mathfrak{g} : Q_p \to Q_p$ be defined by $\mathfrak{g}(x) = x^2 + 1$. Since $|2^{\eta}| = 1$ for all $\eta \in \mathcal{N}$. Then for all $\delta > 0$,*

$$\|Q\mathfrak{g}(x,y)\| = 6 \leq \delta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^{r}\|y\|^{s}) \text{ for all } x, y \in \mathcal{G}.$$

and

$$\left\| \frac{h_1(2^{\eta}x)}{2^{2\eta}} - \frac{h_1(2^{\eta-1}x)}{2^{2(\eta-1)}} \right\| = |3|^2.|5| \neq 0.$$

Hence $\left\{2^{-2\eta}h_1(2^{\eta}x)\right\}$ is not a Cauchy sequence. Where $h_1(x) = \mathfrak{g}(2x) - 16\mathfrak{g}(x)$.

Theorem 3.3. *Let a function φ from $\mathcal{G}^2 \to [0, \infty)$ be such that*

$$\lim_{\eta\to\infty} \frac{1}{|2|^{4\eta}}\varphi(2^{\eta+1}x, 2^{\eta+1}y) = 0 = \lim_{\eta\to\infty} \frac{1}{|2|^{4\eta}}\bar{\varphi}(2^{\eta}x) \text{ for each } x \in \mathcal{G} \tag{3.13}$$

Let $\mathfrak{g}$ be an even function from $\mathcal{G}$ to $\mathcal{X}$ that satisfies (1.6) and $\mathfrak{g}(0) = 0$.
Then, uniqueness of quartic function Q_4 from $\mathcal{G} \to \mathcal{X}$ exists and

$$\|\mathfrak{g}(2x) - 4\mathfrak{g}(x) - Q_4(x)\| \leq \frac{1}{|2|^4}\psi_{q_4}(x) \text{ for each } x \in \mathcal{G}. \tag{3.14}$$

where

$$\psi_{q_4}(x) = \lim_{\eta\to\infty} \max\left\{\frac{1}{|2|^{4\jmath}}\bar{\varphi}(2^{\jmath}x) : 0 \leq \jmath < \eta\right\} \tag{3.15}$$

and $\bar{\varphi}(x)$ follows as in (3.4) for all $x \in \mathcal{G}$.

Proof. The proof of this theorem is similar to Theorem (3.1), we get

$$\|\mathfrak{g}(4x) - 20\mathfrak{g}(2x) + 64\mathfrak{g}(x)\| \le \bar{\varphi}(x) \ for\ each\ x \in \mathcal{G}. \tag{3.16}$$

Let h_2 mapping from $\mathcal{G}$ into $\mathcal{X}$ defined by $h_2(x) = \mathfrak{g}(2x) - 4\mathfrak{g}(x)$ for each $x \in \mathcal{G}$.
we conclude this (3.16), we obtain

$$\|h_2(2x) - 16h_2(x)\| \le \bar{\varphi}(x) \ for\ each\ x \in \mathcal{G}. \tag{3.17}$$

Replacing x by $2^{\eta-1}x$ in (3.17), we get

$$\left\| \frac{h_2(2^\eta x)}{2^{4\eta}} - \frac{h_2(2^{\eta-1}x)}{2^{4(\eta-1)}} \right\| \le \frac{1}{|2|^{4\eta}}\, \bar{\varphi}(2^{\eta-1}x) \ for\ each\ x \in \mathcal{G}. \tag{3.18}$$

It follows from (3.13) and (3.18) the sequence $\left\{\frac{h_2(2^\eta x)}{2^{4\eta}}\right\}$ is Cauchy. Since $\mathcal{X}$ is complete, we conclude $\left\{\frac{h_2(2^\eta x)}{2^{4\eta}}\right\}$ is convergent.

$$Let\ \mathcal{Q}_4(x) = \lim_{\eta\to\infty} \frac{h_2(2^\eta x)}{2^{4\eta}} \ for\ each\ x \in \mathcal{G}. \tag{3.19}$$

From (3.17) and (3.18) it follows by induction that

$$\left\| \frac{h_2(2^\eta x)}{2^{4\eta}} - h_2(x) \right\| \le \max\left\{ \frac{1}{|2|^{4(\jmath+1)}} \bar{\varphi}(2^\jmath x) \ : 0 \le \jmath < \eta \right\}$$

$$\left\| \frac{h_2(2^\eta x)}{2^{4\eta}} - h_2(x) \right\| \le \frac{1}{|2|^4} \max\left\{ \frac{1}{|2|^{4\jmath}} \bar{\varphi}(2^\jmath x) \ : 0 \le \jmath < \eta \right\} \tag{3.20}$$

for each $\eta \in \mathcal{N}$ and for all $x \in \mathcal{G}$. As $\eta \to \infty$ in (3.20) and using (3.15), we get (3.14). Now to prove $\mathcal{Q}_4$ is quartic. It follows from (3.13), (3.18) and (3.19) we obtain

$$\begin{aligned}
\|\mathcal{Q}_4(2x) - 16\mathcal{Q}_4(x)\| &= \lim_{\eta\to\infty} \left\| \frac{h_2(2^\eta 2x)}{2^{4\eta}} - 2^4 \frac{h_2(2^\eta x)}{2^{4\eta}} \right\| \\
&\le |2|^4 \lim_{\eta\to\infty} \frac{1}{|2|^{4(\eta+1)}}\, \bar{\varphi}(2^\eta x) \\
&= \lim_{\eta\to\infty} \frac{1}{|2|^{4\eta}}\, \bar{\varphi}(2^\eta x) \\
&= 0 \ for\ each\ x \in \mathcal{G}.
\end{aligned}$$

Hence,

$$\mathcal{Q}_4(2x) = 16\mathcal{Q}_4(x) \ for\ each\ x \in \mathcal{G}. \tag{3.21}$$

On the other hand (1.6), (3.13) and (3.19) implies that

$$\begin{aligned}
\|Q_{Q_4}(x,y)\| &= \left\| \lim_{\eta\to\infty} \frac{Qh_2(2^\eta x, 2^\eta y)}{2^{4\eta}} \right\| \\
&= \lim_{\eta\to\infty} \frac{1}{|2|^{4\eta}} \left\| Qh_2(2^\eta x, 2^\eta y) \right\| \\
&= \lim_{\eta\to\infty} \frac{1}{|2|^{4\eta}} \left\| Q\mathfrak{g}(2^{\eta+1}x, 2^{\eta+1}y) - 4Q\mathfrak{g}(2^\eta x, 2^\eta y) \right\| \\
&\le \lim_{\eta\to\infty} \frac{1}{|2|^{4\eta}} \max\left\{ \varphi(2^{\eta+1}x, 2^{\eta+1}y), |4|\varphi(2^\eta x, 2^\eta y) \right\} \\
&= 0. \textit{ for each } x \in \mathcal{G}.
\end{aligned}$$

Therefore, Q_4 satisfies (1.4). By Lemma (3.1), $|Q_4(2x) - 4Q_4(x)|$ is quartic.

Therefore, Q_4 is quartic.

Uniqueness: Let there exist another quartic function Q_4'

$$\begin{aligned}
\|Q_4(x) - Q_4'(x)\| &= \lim_{\iota\to\infty} \left\| \frac{Q_4(2^\iota x)}{2^{4\iota}} - \frac{Q_4'(2^\iota x)}{2^{4\iota}} \right\| \\
&\le \lim_{\iota\to\infty} \frac{1}{|2|^{4\iota}} \max\left\{ \left\| Q_4(2^\iota x) - h_2(2^\iota x) \|, \|h_2(2^\iota x) - Q_4'(2^\iota x) \right\| \right\} \\
\|Q_4(x) - Q_4'(x)\| &\le \frac{1}{|2|^4} \lim_{\iota\to\infty} \lim_{\eta\to\infty} \max\left\{ \frac{1}{|2|^{4\jmath}} \bar{\varphi}(2^\eta x) : \iota \le \jmath < \eta + \iota \right\} \\
&= 0 \textit{ for each } x \in \mathcal{G}.
\end{aligned}$$

Therefore, $Q_4(x) = Q_4'(x)$. □

Theorem 3.4. *Let a function φ from $\mathcal{G}^2 \to [0,\infty)$ be such that*

$$\lim_{\eta\to\infty} |2|^{4\eta} \varphi\left(\frac{x}{2^{\eta+1}}, \frac{y}{2^{\eta+1}}\right) = 0 = \lim_{\eta\to\infty} |2|^{4\eta} \bar{\varphi}\left(\frac{x}{2^\eta}\right) \textit{ for every } x, y \in \mathcal{G}.$$

Let $\mathfrak{g}$ be an even function from $\mathcal{G}$ to $\mathcal{X}$ that satisfies (1.6) and $\mathfrak{g}(0) = 0$.
Then, uniqueness of quartic function Q_4 from $\mathcal{G} \to \mathcal{X}$ exists and

$$\|\mathfrak{g}(2x) - 4\mathfrak{g}(x) - Q_4(x)\| \le |2|^4 \psi_{q_4}(x) \textit{ for each } x \in \mathcal{G}.$$

where

$$\psi_{q_4}(x) = \lim_{\eta\to\infty} \max\left\{ |2|^{4\jmath} \bar{\varphi}\left(\frac{x}{2^\jmath}\right) : 0 \le \jmath < \eta \right\}$$

and

$$\bar{\varphi}(x) = \max\left\{|4|\varphi(x,x), \varphi(x,2x)\right\} \textit{ exists for each } x \in \mathcal{G}.$$

Proof. The proof is similar to Theorem (3.3). □

Corollary 3.2. *Let r, s and δ are positive real numbers. Define a function $\mathfrak{g}$ from $\mathcal{G}$ to $\mathcal{X}$ and if a quartic mapping satisfies the inequality*

$$\|\mathcal{Q}\mathfrak{g}(x,y)\| \leq \delta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s) \textit{ for all } x, y \in \mathcal{G}.$$

$$\|\mathfrak{g}(2x) - 4\mathfrak{g}(x) - \mathcal{Q}_4(x)\| \leq \frac{1}{|2|^4}\psi_{q_4}(x) \textit{ for each } x \in \mathcal{G}.$$

where

$$\psi_{q_4}(x) = \delta(1 + |2|^{r+s} + |2|^s)\|x\|^{r+s}$$

Then,

(i) For $r + s > 4$, there is a unique quartic mapping $\mathcal{Q}_4(x) : \mathcal{G} \to \mathcal{X}$ such that

$$\|\mathfrak{g}(2x) - 4\mathfrak{g}(x) - \mathcal{Q}_4(x)\| \leq \frac{\delta}{|2|^4}(1 + |2|^{r+s} + |2|^s)\|x\|^{r+s}$$

(ii) For $r + s < 4$, there is a unique quartic mapping $\mathcal{Q}_4(x) : \mathcal{G} \to \mathcal{X}$ such that

$$\|\mathfrak{g}(2x) - 4\mathfrak{g}(x) - \mathcal{Q}_4(x)\| \leq |2|^4\delta(1 + |2|^{r+s} + |2|^s)\|x\|^{r+s} \textit{ for each } x \in \mathcal{G}.$$

For the case $r + s = 4$, we have the following counter example.

Example 3.2. *Let $p > 2$, be a prime number and $\mathfrak{g} : \mathcal{Q}_p \to \mathcal{Q}_p$ be defined by $\mathfrak{g}(x) = x^4 + 1$. Since $|2^\eta| = 1$ for all $\eta \in \mathcal{N}$. Then for all $\delta > 0$,*

$$\|\mathcal{Q}\mathfrak{g}(x,y)\| = 6 \leq \delta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s) \textit{ for all } x, y \in \mathcal{G}.$$

and

$$\left\|\frac{h_2(2^\eta x)}{2^{4\eta}} - \frac{h_2(2^{\eta-1}x)}{2^{4(\eta-1)}}\right\| = |3|^2.|5| \neq 0.$$

Hence $\left\{2^{-2\eta}h_1(2^\eta x)\right\}$ is not a Cauchy sequence. Where $h_2(x) = \mathfrak{g}(2x) - 4\mathfrak{g}(x)$.

Theorem 3.5. *Let a function φ from $\mathcal{G}^2 \to [0, \infty)$ be such that*

$$\lim_{\eta\to\infty} \frac{1}{|2|^{2\eta}}\varphi(2^{\eta+1}x, 2^{\eta+1}y) = 0 = \lim_{\eta\to\infty} \frac{1}{|2|^{2\eta}}\bar{\varphi}(2^\eta x) \textit{ for every } x, y \in \mathcal{G}. \tag{3.22}$$

$$\lim_{\eta\to\infty} \frac{1}{|2|^{4\eta}}\varphi(2^{\eta+1}x, 2^{\eta+1}y) = 0 = \lim_{\eta\to\infty} \frac{1}{|2|^{4\eta}}\bar{\varphi}(2^\eta x) \textit{ for every } x, y \in \mathcal{G}. \tag{3.23}$$

Let $\mathfrak{g}$ *be an even function from* $\mathcal{G}$ *to* $\mathcal{X}$ *that satisfies (1.6) and* $\mathfrak{g}(0) = 0$.
Then, uniqueness of quadratic function $\mathcal{Q}_2 : \mathcal{G} \to \mathcal{X}$ *and uniqueness of quartic function* $\mathcal{Q}_4 : \mathcal{G} \to \mathcal{X}$ *exists and*

$$\|\mathfrak{g}(x) - \mathcal{Q}_2(x) - \mathcal{Q}_4(x)\| \leq \frac{1}{|192|} \max\left\{\psi_{q_4}(x), |4|\psi_{q_2}\right\} \textit{ for each } x \in \mathcal{G}. \tag{3.24}$$

where $\psi_{q_2}(x)$ *is defined in (3.3) and* $\psi_{q_4}(x)$ *is defined in (3.15).*

Proof. By Theorems (3.1) and (3.3), there exists a quadratic function from $\mathcal{G}$ into $\mathcal{X}$ and quartic function from $\mathcal{G}$ into $\mathcal{X}$ such that

$$\|\mathfrak{g}(2x) - 16\mathfrak{g}(x) - \mathcal{Q}_2(x)\| \leq \frac{1}{|2|^2}\psi q_2(x) \tag{3.25}$$

$$\|\mathfrak{g}(2x) - 4\mathfrak{g}(x) - \mathcal{Q}_4(x)\| \leq \frac{1}{|2|^4}\psi_{q_4}(x) \tag{3.26}$$

$$\|\mathfrak{g}(x) - \mathcal{Q}_4(x) - \mathcal{Q}_2(x)\| \leq \tfrac{1}{|192|} \max\left\{\psi_{q_4}(x), |4|\psi_{q_2}(x)\right\} \textit{ for each } x \in \mathcal{G}.$$

So we get (3.24) by setting $\mathcal{Q}_4(x) = \frac{q_4(x)}{12}$ and $\mathcal{Q}_2(x) = \frac{-q_2(x)}{12}$ *for each* $x \in \mathcal{G}$.

To show that $\mathcal{Q}_2$ and $\mathcal{Q}_4$ are unique. Let $\mathcal{Q}_2'$, $\mathcal{Q}_4'$ be another quadratic and quartic functions respectively satisfying (3.24). Let $\bar{\mathcal{Q}}_2 = \mathcal{Q}_2 - \mathcal{Q}_2{}'$ and $\bar{\mathcal{Q}}_4 = \mathcal{Q}_4 - \mathcal{Q}_4{}'$.

Hence

$$\begin{aligned}\|\bar{\mathcal{Q}}_2(x) + \bar{\mathcal{Q}}_4(x)\| &\leq \max\left\{\|\mathfrak{g}(x) - \mathcal{Q}_2'(x) - \mathcal{Q}_4'(x)\|, \|\mathfrak{g}(x) - \mathcal{Q}_2(x) - \mathcal{Q}_4(x)\|\right\} \\ &\leq \max\left\{\tfrac{1}{|192|}\max\left\{\psi_4(x), |4|\psi_{q_2}(x)\right\}, \tfrac{1}{|192|}\max\left\{\psi_4(x), |4|\psi_{q_2}(x)\right\}\right\} \\ &\leq \tfrac{1}{|192|}\max\left\{\psi_{q_4}(x), |4|\psi_{q_2}(x)\right\} \textit{ for each } x \in \mathcal{G}.\end{aligned}$$

Since

$$\begin{aligned}&\lim_{\iota\to\infty}\lim_{\eta\to\infty}\max\left\{\frac{1}{|2|^{2(\jmath+1)}}\bar{\varphi}(2^\jmath x) : \iota \leq \jmath < \eta + \iota\right\} = 0 \textit{ for each } x \in \mathcal{G}. \\ &\lim_{\iota\to\infty}\lim_{\eta\to\infty}\max\left\{\frac{1}{|2|^{4(\jmath+1)}}\bar{\varphi}(2^\jmath x) : \iota \leq \jmath < \eta + \iota\right\} = 0 \textit{ for each } x \in \mathcal{G}. \\ &\lim_{\eta\to\infty}\frac{1}{|2|^{4\eta}}\|\bar{\mathcal{Q}}_2(2^\eta x) + \bar{\mathcal{Q}}_4(2^\eta x)\| = 0 \textit{ for every } x \in \mathcal{G}.\end{aligned} \tag{3.27}$$

Hence, we get $\bar{\mathcal{Q}}_4 = 0$ and $\bar{\mathcal{Q}}_2 = 0$ and the proof is complete. □

4. Conclusion

Many authors discussed the HUS of mixed type functional equation in NAN space in recent years. In this article, we have proved HUS for quadratic-quartic functional equation (1.4) in NAN space with some suitable counter examples.

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