Int. J. Anal. Appl. (2024), 22:63

International Journal of Analysis and Applications

A Study on Degree Based Topological Indices of Harary Subdivision Graphs With Application

Mukhtar Ahmad[1,*], Ather Qayyum[2], Gulnaz Atta[3], Siti Suzlin Supadi[2], Muhammad Saleem[4], Usman Ali[5]

[1]*Department of Mathematics, Khawaja Fareed University of Engineering and Information Technology Rahim Yar Khan, Pakistan*

[2]*Institute of Mathematical Sciences, Faculty of Science, Universiti Malaya, Malaysia*

[3]*Department of Mathematics, University of Education Lahore DGK Campus, Pakistan*

[4]*Department of Mathematics, National College of Business Administration and Economics Lahore, Multan Sub-Campus, Pakistan*

[5]*Department of Mathematics, Institute of Southern Punjab Multan, Pakistan*

[*]*Corresponding author:* itxmemuktar@gmail.com

Abstract. Combinatorial design theory and graph decompositions play a critical role in the exploration of combinatorial design theory and are essential in mathematical sciences. The process of graph decomposition involves partitioning the set of edges in a graph G. An n-sun graph, characterized by a cycle with an edge connecting each vertex to a terminating vertex of degree one, is introduced in this study. The concept of n-sun decomposition is applied to certain even-order graphs. The indices covered in this study include the general connectivity index of the harary graphs, Zagreb indices, symmetric division degree indices and randic indices.

1. Introduction and Preliminaries

We mean a simple undirected linked graphs with the graph $G = (V, E)$. In G, the cycle length n is characterized by C_n. The n-sun graph seems to be a C_n cycle including on the edges ending with $C_n[1]$ from each vertex. Therefore, each the n-sun graph comprises precisely one cycle of vertices of the n and n pendants in length. The graphic decomposition it's a set $G_1, G_2, ..., G_n$ of G with edge-disjoint subsequences so which each G's edge is that's approximately one G_i. Since the mid nineteenth century, graph decompositions, recognized for their applications in the

Received: Oct. 25, 2023.

2020 *Mathematics Subject Classification.* 05C09.

Key words and phrases. topological indices; hyper Zagreb index; atom-bond connectivity index; first and second Zagreb polynomial; first multiple Zagreb indics; second multiple Zagreb indics; harary graphs.

https://doi.org/10.28924/2291-8639-22-2024-63

ISSN: 2291-8639

theory of combinatorial design, have been studied. Walecki had a loan from the building The decomposition of complete graphs by the Hamilton cycle many decades after its launch[2] − [4]. We have decomposed even-order complete graphs, K_{2n} into with n-suns. In complete graphs, The basis of the decomposition is the The design of cycles for Hamilton by Walecki.

A systematic technique with a method of labeling is given to the decomposition. We demonstrated a decomposition of the spanning tree of K_{2n} by orderly extracting edges from the cycles in n-suns. The speciality of any such spanning tree is that it contains a perfect matching of K_{2n}. Equally essential $2n$ are the complete bipartite graphs K_{2n}. Also, their decomposition of the n-sun is provided. The graphs of $2n$-vertex, k-connected are consequently next sort in graphs, labeled graphs on Harary, with the smallest possible number of edges . In interconnection network topology, such graphs are commonly used. The n-sun decomposition is investigated for various kinds of harary graphs.

A G graph is called a complete graph, K_n, in which all two separate points are adjacent. The whole the $K_{m,n}$ bipartite graph is a graph with the vertices that could be divided U and W in two sets, so each of the edges of $k_{m,n}$ is each side of U and W on the other end, respectively. Frank Harary created the harary graphs are a class of charts named $H_{k,n}$,starting with a n-cycle graph with numbered vertices around its perimeter consecutively by $0, 1, 2, ..., n-1$ towards the clock. If k and n are all the same, by joining each vertex to the nearest vertex, form $H_{k,n}$ vertices of $\frac{k}{2}$ around the circle in both directions. Unless k becomes different as well as n is even now, each vertex forms $H_{k,n}$ is joined together in each direction towards the closest one $\frac{k-1}{2}$ vertex and the The $H_{k,n}$ is k-regular, totally different the vertex, k-linked graph of n-vertex in both cases. The situation of all the uncommon n in $H_{k,n}$ is removed, because the n-sun is just that, specified order graphs for even.

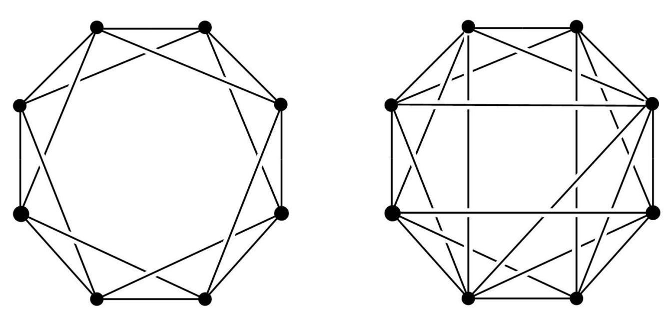

Figure 1. Harary graph are $H_{8,4}$ and $H_{8,5}$

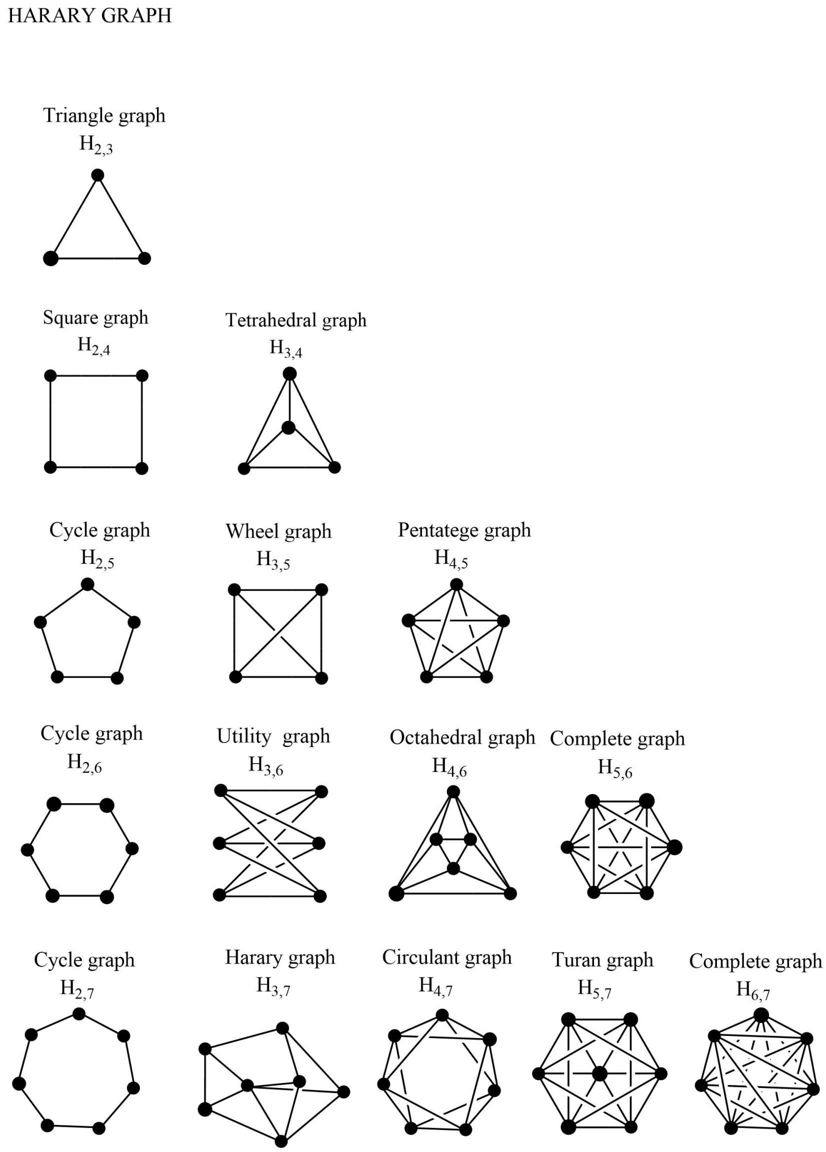

FIGURE 2. An array harary graph

1.1. **Definition.** A specific example of a k-connected graph is the harary graph $H_{k,n}$ with graph vertices of n having the least number of edges possible.

1.2. **Definition.** A harary graph $H_{n,k}$ is a graph on n the vertices $v_1, v_2, ., ., ., v_n$ are defined as follows construction:

Case = 1 If k is even, then every vertex v_i is adjacent to $v_{i\pm1}, v_{i\pm2}, ., ., ., v_{i\pm\frac{k}{2}}$, where the indices follow a cyclic convention that

$v_i = v_{i\pm n}(e.g. v_{n\pm3}$ represents $v_3)$.

Case = 2 If k is odd and n is even, then $H_{n,k}$ equals $H_{n,k-1}$ with additional adjacencies between each v_i and $v_{i+\frac{n}{2}}$ for each i.

Case = 3 If k and n are not distinct either, then $H_{n,k}$ is $H_{n,k-1}$ with additional adjacencies $v_1v_{1+\frac{n-1}{2}}, v_1v_{1+\frac{n-1}{2}}, v_2v_{2+\frac{n-1}{2}}, v_3v_{3+\frac{n-1}{2}}, v\frac{n-1}{2}, v_n$.

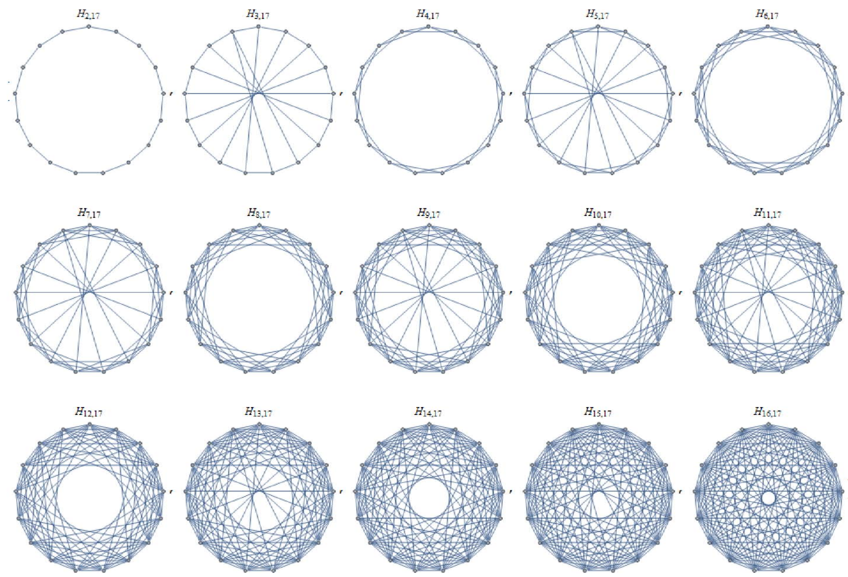

FIGURE 3. An array harary graph

Note that these graphs are (except when n and k are both odd) highly symmetrical, and that each contains $[\frac{nk}{2}]$ edges. We thus have an explicit construction of minimal k-connected graphs, following our proof of the harary graphs connectivity:

Harary theorem: The graph $H_{n,k}$ is k-connected.

Proof: We shall start by proving this in the symmetric cases. If $k = 2r$ for some r, then let us consider some subset S of $V(H_{n,k})$, with $|S| < k$; let us assume $v_1 \notin S$, and there is some other $v_i \notin S$. Let us consider the sets $v_2, ..., v_{i-1}$ and $v_{i+1}, ..., v_n$. S is drawn from these two sets, and $|S| < 2r$, so one or the other of these sets contains fewer than r elements of S; without loss of generality, let$v_2, ..., v_{i-1} \cap S$ have fewer than S elements. Now, either $i \leq k+1$, in which case $v_1 \sim v_i$, or there is some element v_{i_1} of $v_1, v_2..., v_{k+1}$ which is not in S, since $|S \cap v_2, v_3..., v_{k+1}| < k$. So $v_1 \sim v_{i_1}$ now either $i = k + i_1$ in which case v_{i_1} v_i, or there is some element v_{i_2} of $v_{i_1}v_{i_2}, ..., v_{i_1+k}$ which is not in S, since $|S \cap v_2, v_3..., v_{k+1}| < k$, so $v_{i_1} \sim v_{i_2}$. Continuing this procedure as far as necessary, we will construct a path from v_1 to v_i. Thus, since v_i was an arbitrary vertex of $V(H_{n,k})$, and by symmetry v_1 is equivalent to any other choice of vertex, we have shown connectivity between arbitrary vertices from $V(H_{n,k}) - S$, so $V(H_{n,k})$ is a k-connected graph on n-vertices with as few edges as possible.

Next we discuss the topological indices.

42 Balaban et al., included $M_1(G)$ and $M_2(G)$ among the topological indices in a review paper and called them "Zagreb group indices". In this regard, some explanation is required. First, in the early 1980s, only a few topological indices were recognized and the authors of the study wanted as many of them as possible. Secondly, both authors of the article at that time were representatives of the Department of theoretical chemistry of the Zagreb institute. The term "Zagreb group index" was soon shortened to "Zagreb index" and $M_1G)$ is now known as "Zagreb's first index" and $M_2(G)$ is known as "Zagreb's second index". In 1972 the first and second Zagreb indices were established by Gutmann and Trynaistik, which are topological graph indices based on past degrees. It is an important molecular descriptor closely related to many chemical properties. Thus, it attracted more and more attention of chemists and mathematicians.

The first Zagreb index, denoted as $M_1(G)$ represents the sum of squared degrees of the vertices in a given (molecular) graph resently M. K. Iqbal et.al[22]. Another perspective on this index is viewing it as the sum of the degrees of the edges within graph G. The expression for $M_1(G)$ is defined as shown [7].

$$M_1(G,x) = \sum_{\mu\omega \in E(G)} x^{d_\mu + d_\omega} \tag{1.1}$$

The second grabbing index, denoted as $M_2(G)$ corresponds to the summation of the degree products of adjacent vertices from a pair of vertices in the given (molecular) graph G. The definition of $M_2(G)$ can be found in reference [7].

$$M_2(G,x) = \sum_{\mu\omega \in E(G)} x^{d_\mu \times d_\omega} \tag{1.2}$$

In 1972, the inaugural Zagreb index was introduced as an ancient topological index. Subsequently, several variations of the Zagreb index were proposed. For instance, in 2013, Shirdel et al. presented a novel index known as the "hyper Zagreb index," which was later identified as [8].

$$HM(G) = \sum_{\mu\omega \in E(G)} (d_\mu + d_\omega)^2 \tag{1.3}$$

The graph G is called the geometric arithmetic Index (GA) resently A. Asghar et.al[18].

$$GA(G) = \sum_{\mu\omega \in E(G)} 2\frac{\sqrt{d_\mu d_\omega}}{d_\mu + d_\omega} \tag{1.4}$$

In 2015, E. Deutshi and S. Klavzar introduced a novel polynomial known as the m-polynomial, representing a new topological index. This polynomial was defined in the context of degree-based topological indices as follows, based on the degree of a vertex[9]: [Computational approach to the drug acetaminophen using topological indices based on powers and m-polynomials]

$$M_1(G,x,y) = \sum_{\mu\omega \in E(G)} x^{d_\mu} y^{d_\omega} \tag{1.5}$$

The first Zagreb index is defined in Shuxian in terms of two polynomials, which exhibit the following structure resently Zaib Hassan Niazi et.al[19]:

$$M_1^*(G,x) = \sum_{(\omega_i) \in V(G)} d(\omega_i).x^{d(\omega_i)} \tag{1.6}$$

$$M_0(G,x) = \sum_{(\omega_i) \in V(G)} x^{d(\omega_i)} \tag{1.7}$$

Two polynomials of Zagreb type are defined as follows resently Mukhtar Ahmad et.al[20]:

$$M_{a,b}(G,x) = \sum_{(\omega_i\omega_j) \in E(G)} x^{ad(\omega_i)+bd(\omega_i)} \tag{1.8}$$

$$M'_{a,b}(G,x) = \sum_{(\omega_i\omega_j) \in E(G)} x^{(a+d(\omega_i))+(b+d(\omega_i))} \tag{1.9}$$

Two updated models of the Zagreb index, the first multiplicative Zagreb index $PM_1(G)$ and the second multiplicative Zagreb index PM_2 were introduced by Todeshine et al. for molecular graph G certain characteristics of both the $PM_1(G)$ and $PM_2(G)$ indices of particular chemical structures have been investigated[10]. The first multiplicative Zagreb index for the molecular graph G is defined as follows resently Mukhtar Ahmad et.al[21].

$$M_1(G) = \sum_{\omega_i \in V(G)} [d(\omega_i)^2] \tag{1.10}$$

The second multiplicative Zagreb index for the molecular graph G is defined as follows.

$$M_2(G) = \sum_{(\omega_i\omega_j) \in E(G)} [d(\omega_i).d(\omega_j)] \tag{1.11}$$

The first multiplicative Zagreb polynomial for the molecular graph G is defined as follows.

$$M_1(G,x) = \sum_{(\omega_i\omega_j) \in E(G)} x^{d(\omega_i)+d(\omega_j)} \tag{1.12}$$

The second multiplicative Zagreb polynomial for the molecular graph G is defined as follows.

$$M_2(G,x) = \sum_{(\omega_i\omega_j) \in E(G)} x^{d(\omega_i)\times d(\omega_j)} \tag{1.13}$$

Historically, the graph invariants now known as Zagreb indices were the first vertex-degree-based structure descriptors. However, at first, they were meant to be used for something quite else, and they were only much later included to the list of topological indices. Milan randic proposed the first real degree-based topological index in his key study, On characterisation of molecular branching, published in 1775. His index was [11] as follows.

$$R_{\alpha}(G)=\sum_{\mu\omega\in E(G)}(d_{\mu}\times d_{\omega})^{\alpha} \tag{1.14}$$

Ernesto Estrada has described a new topological index, which is a new version of the random index. He called it the atom-bond connectivity index, which is conveniently reduced to (ABC). Various applications of graph invariants have been found and are currently being used in chemistry, environmental sciences, pharmacology, etc. The atom-bond connectivity index (ABC) is one of them. It is defined as follows[12]. [Estrada, Torres, Rodriguez, and Gutman, 1998b]

$$ABC(G)=\sum_{\mu\omega\in E(G)}\sqrt{\frac{d_{\mu}+d_{\omega}-2}{d_{\mu}d_{\omega}}} \tag{1.15}$$

The general random index (or product connectivity index) was proposed by Bolloba and Erdos and is defined as follows resently A. Asghar et.al[18].

$$\chi_{\alpha}(G)=\sum_{\mu\in V(G)}(d_{\omega})^{2} \tag{1.16}$$

General connection indexes come in two different varieties, depending on the value of the real number alpha. If alpha equals negative one-half, it corresponds to a random index. On the other hand, if alpha equals one, it corresponds to the second Zagreb index. Recently, Trinajstick and Zhou modified the concept of a random index and obtained a new index called the total connectedness index of the sum, which is defined as follows resently Zaib Hassan Niazi et.al[19].

$$\chi_{\alpha}(G)=\sum_{\mu,\omega\in E(G)}[d(\mu)+d(\omega)]^{\alpha} \tag{1.17}$$

The given Zagreb indices, namely the first, second and third, can be described as follows in a more professional manner resently Mukhtar Ahmad et.al[20].

$$MR_{1}(G)=\sum_{\mu\omega\in E(G)}|(d(\mu)-1)+(d(\omega)-1)| \tag{1.18}$$

$$MR_{2}(G)=\sum_{\mu\omega\in E(G)}[(d(\mu)-1)(d(\omega)-1)] \tag{1.19}$$

$$MR_{3}(G)=\sum_{\mu\omega\in E(G)}|(d(\mu)-1)-(d(\omega)-1)| \tag{1.20}$$

$$RR(G)=\sum_{\mu\omega\in E(G)}\sqrt{d(\mu)\times d(\omega)} \tag{1.21}$$

The reduced inverse random index is defined as [13].

$$RRR(G)=\sum_{\mu\omega\in E(G)}\sqrt{(d_{\mu}-1)(d_{\omega}-1)} \tag{1.22}$$

In 2015, Furtula and Gutman [14] introduced another topological index called the forgotten index or $F-index$, which can be described in a more professional manner. See [15, 16, 17] for details about the $F - index$. The forgotten index of the graph G can be defined in a more professional manner using synonyms.

$$F(G) = \sum_{(\mu\omega)\in E(G)} [(d_\mu)^2 + (d_\omega)^2] \tag{1.23}$$

The forgotten polynomial of the graph G can be defined in a more professional manner using synonyms.

$$F(G,x) = \sum_{(\mu\omega)\in E(G)} x^{[(d_\mu)^2+(d_\omega)^2]} \tag{1.24}$$

2. Main Results

In this part, we'll developed a number of conclusions regarding the harary graph's degree-based topological indices.

Theorem 2.1 Let $H_{k,n}$ be the harary subdivision graph. Then for $\eta \geq 2$, first Zagreb polynomial indices are:

$M_1(G) = \frac{\eta(\eta-1)}{2} x^{(2\eta-2)}$.

Proof: The harary graph $2, 3, 4, ..., \eta - 1$ appears in figure(graph). The harary graph $2, 3, 4, ..., \eta - 1$ contains η number of vertices and C_2^η number of edge. The degree of each vertex in the graph $H_{k,n}$ is equal to $\eta - 1$ and now we introduce the first general Zagreb topological index.

$M_1(G) = \sum_{\mu\omega\in E(G)} x^{d_\mu+d_\omega} \rightarrow eq.(1)$

Put $d_\mu = (\eta - 1)$ and $d_\omega = (\eta - 1)$ in eq.(1)

$M_1(G) = |E(R)| x^{(\eta-1)+(\eta-1)}$

$M_1(G) = |E(R)| x^{(2\eta-2)}$

Put $|E(R)| = \frac{\eta(\eta-1)}{2}$ in eq.(1)

$M_1(G) = \frac{\eta(\eta-1)}{2} x^{(2\eta-2)}$

Theorem 2.2 Let $H_{k,n}$ be the harary subdivision graph. Then for $\eta \geq 2$, second Zagreb Polynomial indices are,

$M_2(G) = \frac{\eta(\eta-1)}{2} x^{(\eta-1)^2}$.

Proof: The harary graph $2, 3, 4, ..., \eta - 1$ appears in figure(graph). The harary graph $2, 3, 4, ..., \eta - 1$ contains η number of vertices and C_2^η number of edges. The degree of each vertex in the graph $H_{k,n}$ is equal to $\eta - 1$ and now we introduce the second general Zagreb topological index.

$M_2(G) = \sum_{\mu\omega \in E(G)} x^{d_\mu \times d_\omega} \rightarrow (1)$

Put $d_\mu = (\eta - 1)$ and $d_\omega = (\eta - 1)$ in eq.(1)

$M_2(G) = |E(R)|x^{(\eta-1)(\eta-1)}$

$M_2(G) = |E(R)|x^{(\eta-1)^2}$

Put $|E(R)| = \frac{\eta(\eta-1)}{2}$ in eq.(1)

$M_2(G) = \frac{\eta(\eta-1)}{2} x^{(\eta-1)^2}$.

Theorem 2.3 Let $H_{k,n}$ be the harary subdivision graph. Then for $\eta \geq 2$, randic indices are,

$R_\alpha(G) = \frac{\eta(\eta-1)}{2}(\eta - 1)^{2\alpha}$.

Proof: The harary graph $2, 3, 4, ..., \eta - 1$ appears in figure(graph). The harary graph $2, 3, 4, ..., \eta - 1$ contains η number of vertices and C_2^η number of edges. The degree of each vertex in the graph $H_{k,n}$ is equal to $\eta - 1$ and now we introduce the randic indices.

$R_\alpha(G) = \sum_{\mu\omega \in E(G)} (d_\mu d_\omega)^\alpha \rightarrow eq.(1)$

Put $d_\mu = (\eta - 1)$ and $d_\omega = (\eta - 1)$ in eq.(1)

$R_\alpha(G) = |E(R)|((\eta - 1)(\eta - 1))^\alpha$

$R_\alpha(G) = |E(R)|(\eta - 1)^{2\alpha}$

Put $|E(R)| = \frac{\eta(\eta-1)}{2}$ in eq.(1)

$R_\alpha(G) = \frac{\eta(\eta-1)}{2}(\eta - 1)^{2\alpha}$.

Theorem 2.4 Let $H_{k,n}$ be the harary subdivision graph. Then for $\eta \geq 2$, general sum-connectivity indices are

$\chi_\alpha(G) = \frac{\eta(\eta-1)}{2}(2\eta - 2)^\alpha$.

Proof: The harary graph $2, 3, 4, ..., \eta - 1$ appears in figure(graph). The harary graph $2, 3, 4, ..., \eta - 1$ contains η number of vertices and C_2^η number of edges. The degree of each vertex in the graph $H_{k,n}$ is equal to $\eta - 1$ and now we introduce the general sum-connectivity topological index.

$\chi_\alpha(G) = \sum_{\mu\omega \in E(G)} (d_\mu d_\omega)^\alpha \rightarrow eq.(1)$

Put $d_\mu = (\eta - 1)$ and $d_\omega = (\eta - 1)$ in eq.(1)

$\chi_\alpha(G) = |E(R)|((\eta - 1) + (\eta - 1))^\alpha$

$\chi_\alpha(G) = |E(R)|(2\eta - 2)^\alpha$

Put $|E(R)| = \frac{\eta(\eta-1)}{2}$ in eq.(1)

$\chi_\alpha(G) = \frac{\eta(\eta-1)}{2}(2\eta - 2)^\alpha.$

Theorem 2.5 Let $H_{k,n}$ be the harary subdivision graph. Then for $\eta \geq 2$, atom bond connectivity indices are,

$ABC(G) = \frac{\eta(\eta-1)}{2} \frac{\sqrt{2\eta-3}}{(\eta-1)}.$

Proof: The Harary graph $2, 3, 4, ..., \eta - 1$ appears in figure(graph). The harary graph $2, 3, 4, ..., \eta - 1$ contains η number of vertices and C_2^η number of edges. The degree of each vertex in the graph $H_{k,n}$ is equal to $\eta - 1$ and now we introduce the atom bond connectivity topological index.

$ABC(G) = \sum_{\mu\omega \in E(G)} \sqrt{\frac{d_\mu + d_\omega - 2}{d_\mu d_\omega}} \rightarrow eq.(1)$

Put $d_\mu = (\eta - 1)$ and $d_\omega = (\eta - 1)$ in eq.(1)

$ABC(G) = |E(R)| \sqrt{\frac{(\eta-1)+(\eta-1)-2}{(\eta-1)(\eta-1)}}$

$ABC(G) = |E(R)| \sqrt{\frac{(2\eta-3)}{(\eta-1)^2}}$

Put $|E(R)| = \frac{\eta(\eta-1)}{2}$ in eq.(1)

$ABC(G) = \frac{\eta(\eta-1)}{2} \frac{\sqrt{2\eta-3}}{(\eta-1)}.$

Theorem 2.6 Let $H_{k,n}$ be the harary subdivision graph. Then for $\eta \geq 2$, first multiple Zagreb indices are,

$PM_1(G) = (2(\eta - 1))^{\frac{\eta(\eta-1)}{2}}.$

Proof: Since graph G and $(V(G), E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be hararay subdivision graph and now first multiple Zagreb index of topological index is,

$PM_1(G) = \prod_{\mu\omega E(R)}(d_\mu + d_\omega) \rightarrow eq.(1)$
Put $d_\mu = (\eta - 1)$ and $d_\omega = (\eta - 1)$ in eq.(1)

$PM_1(G) = \prod_{\mu\omega E(R)}((\eta - 1) + (\eta - 1))$

$PM_1(R) = (2\eta - 2)^{E(R)}$

Put $|E(R)| = \frac{\eta(\eta-1)}{2}$ in eq.(1)

$PM_1(G) = (2\eta - 2)^{\frac{\eta(\eta-1)}{2}}$

$PM_1(G) = (2(\eta - 1))^{\frac{\eta(\eta-1)}{2}}$.

Theorem 2.7 Let $H_{k,n}$ be the harary subdivision graph. Then for $\eta \geq 2$, second multiple Zagreb indices are,

$PM_2(G) = (\eta - 1)^{\eta(\eta-1)}$.

Proof: Since graph G and $(V(G), E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be hararay subdivision graph and now second multiple Zagreb index of topological index is,

$PM_2(G) = \prod_{\mu\omega E(G)}(d_\mu \times d_\omega) \rightarrow eq.(1)$

Put $d_\mu = (\eta - 1)$ and $d_\omega = (\eta - 1)$ in eq.(1)

$PM_2(G) = \prod_{\mu\omega\in E(R)}((\eta - 1) * (\eta - 1))$

$PM_2(G) = ((\eta - 1)^2)^{E(R)}$

Put $|E(R)| = \frac{\eta(\eta-1)}{2}$ in eq.(1)

$PM_2(G) = ((\eta - 1)^2)^{\frac{\eta(\eta-1)}{2}}$

$PM_2(G) = (\eta - 1)^{\eta(\eta-1)}$.

Theorem 2.8 Suppose graph G and $(V(G), E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be hararay subdivision graph. Then for $\eta \geq 2$, hyper Zagreb index is,

$HM(G) = \frac{\eta(\eta-1)}{2}(2\eta-2)^2$.

Proof: The harary graph $2,3,4,...,\eta-1$ appears in figure(graph). The harary graph $2,3,4,...,\eta-1$ contains η number of vertices and C_2^η number of edges. The degree of each vertex in the graph $H_{k,n}$ is equal to $\eta-1$ and now we introduce the hyper Zagreb topological index.

$HM(G) = \sum_{\mu\omega\in E(G)}(d_\mu + d_\omega)^2 \rightarrow (1)$

Put $d_\mu = (\eta-1)$ and $d_\omega = (\eta-1)$ in eq.(1)

$HM(G) = |E(R)|((\eta-1)+\eta-1))^2$

$HM(G) = |E(R)|(2\eta-2)^2$

Put $|E(R)| = \frac{\eta(\eta-1)}{2}$ in eq.(1)

$HM(G) = \frac{\eta(\eta-1)}{2}(2\eta-2)^2$.

Theorem 2.9 Suppose graph G and $(V(G),E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be hararay subdivision graph. Then for $\eta \geq 2$, geometric arithmetic Index (GA) Zagreb index is,

$GA(G) = \frac{\eta(\eta-1)}{2}$.

Proof: The harary graph $2,3,4,...,\eta-1$ appears in figure(graph). The harary graph $2,3,4,...,\eta-1$ contains η number of vertices and C_2^η number of edges. The degree of each vertex in the graph $H_{k,n}$ is equal to $\eta-1$ and now we introduce the geometric arithmetic index (GA) topological index.

$GA(G) = \sum_{\mu\omega\in E(G)} 2\frac{\sqrt{d_\mu d_\omega}}{d_\mu + d_\omega} \rightarrow (1)$

Put $d_\mu = (\eta-1)$ and $d_\omega = (\eta-1)$ in eq.(1)

$GA(G) = |E(R)|2\frac{\sqrt{(\eta-1)(\eta-1)}}{(\eta-1)+(\eta-1)}$

$GA(G) = |E(R)|2\frac{\sqrt{(\eta-1)^2}}{(2\eta-2)}$

Put $|E(R)| = \frac{\eta(\eta-1)}{2}$ in eq.(1)

$GA(G) = \frac{\eta(\eta-1)}{2}2\frac{\sqrt{(\eta-1)^2}}{2(\eta-1)}$

$GA(G) = \frac{\eta(\eta-1)}{2}2\frac{(\eta-1)}{2(\eta-1)}$

$GA(G) = \frac{\eta(\eta-1)}{2}$.

Theorem 2.10 Suppose graph G and $(V(G), E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be hararay subdivision graph. Then for $\eta \geq 2$, reduced reciprocal randic is,

$$RRR(G) = \frac{\eta(\eta-1)}{2}(\eta - 2).$$

Proof: Since graph G and $(V(G), E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be hararay subdivision graph and now reduced reciprocal randic of topological index is,

$$RRR(G) = \sum_{\mu\omega \in E(G)} \sqrt{(d_\mu - 1)(d_\omega - 1)} \rightarrow eq.(1)$$

Put $d_\mu = (\eta - 1)$ and $d_\omega = (\eta - 1)$ in eq.(1)

$$RRR(G) = \sum_{\mu\omega \in E(R)} \sqrt{((\eta - 1) - 1)((\eta - 1) - 1)}$$

$$RRR(G) = |E(R)| \sqrt{(\eta - 2)(\eta - 2)}$$

$$RRR(G) = |E(R)| \sqrt{(\eta - 2)^2}$$

$$|E(R)| = \frac{\eta(\eta-1)}{2}$$

$$RRR(G) = \frac{\eta(\eta-1)}{2} \sqrt{(\eta - 2)^2}$$

$$RRR(G) = \frac{\eta(\eta-1)}{2}(\eta - 2).$$

Theorem 2.11 Suppose graph G and $(V(G), E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be harary subdivision graph. Then for $\eta \geq 2$, The first Zagreb index is associated by two polynomials, which are,

(i) $M_1^*(R, x) = \eta(\eta - 1).x^\eta$

(ii) $M_0(R, x) = \eta x^{\eta-1}$.

Proof: Since graph G and $(V(G), E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be harary subdivision graph and now The first Zagreb index is associated by two polynomials, which are,

$$M_1^*(G, x) = \sum_{\omega_i \in V(G)} d(\omega_i).X^{\omega_i}$$

$M_1^*(G,x) = \sum_{\omega_i \in V(G)} d(\omega_i).x^{\omega_i}$

(i)

$M_1^*(G,x) = \sum_{\omega_i \in V(G)} d(\omega_i).x^{\omega_i}$

Edges $E(G) = \frac{\eta(\eta-1)}{2}$

Vertices $V(G) = \eta$

Each vertex in the harary graph has a degree that is

$H_{k,n} = (d_\mu, d_\omega) = (\eta - 1)$

so we have
Here

$|E(R)| = \frac{\eta(\eta-1)}{2}$
$d_\mu = (\eta - 1)$
$d_\omega = (\eta - 1)$

Two polynomials connected to the general form of the first Zagreb index,

$M_1^*(G,x) = \sum_{\omega_i \in V(G)} d(\omega_i).x^{\omega_i} \rightarrow eq.(1)$

Put $d_\mu = (\eta - 1)$ and $d_\omega = (\eta - 1)$ in eq.(1)

$M_1^*(R,x) = \sum_{\omega_i \in V(R))} (\eta - 1).x^{\omega_i}$

$M_1^*(R,x) = \sum_{\omega_i \in V(R)} (\eta - 1).x^{\eta}$

$M_1^*(R,x) = |V(R)|(\eta - 1).x^{\eta}$

$M_1^*(R,x) = \eta(\eta - 1).x^{\eta}.$

(ii)

The two polynomials referring to the first Zagreb index of the graph G are defined as:

$$M_0(G,x) = \sum_{\omega_i \in V(G)} x^{d(\omega_i)}$$

$M_0(G,x) = \sum_{\omega_i \in V(G)} x^{d(\omega_i)}$

Edges $E(G) = \frac{\eta(\eta-1)}{2}$

Vertices $V(G) = \eta$

Each vertex in the harary graph has a degree that is

$H_{k,n} = (d_\mu, d_\omega) = (\eta - 1)$

so we have
Here

$|E(R)| = \frac{\eta(\eta-1)}{2}$
$d_\mu = (\eta - 1)$
$d_\omega = (\eta - 1)$

Two polynomials connected to the general form of the first Zagreb index,

$M_0(R,x) = \sum_{\omega_i \in V(G)} x^{d(\omega_i)}$

$M_0(R,x) = \sum_{\omega_i \in V(R)} x^{\eta-1}$

$M_0(R,x) = |V(R)| x^{\eta-1}$

$|V(R)| = \eta$

$M_0(R,x) = \eta x^{\eta-1}$.

Theorem 2.12 Suppose graph G and $(V(G),E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be harary subdivision graph. Then for $\eta \geq 2$, the first Zagreb index is associated by two polynomials, which are,

(i) $M_{a,b}(R,x) = \frac{\eta(\eta-1)}{2} x^{[(a+b)(\eta-1)]}$

(ii) $M'_{a,b}(R,x) = \frac{\eta(\eta-1)}{2} x^{(\eta^2+(a+b-2)\eta-(a+b-1))}$.

Proof: Since graph G and $(V(G),E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be harary subdivision graph and now the first Zagreb index is associated by two polynomials, which are

$$M_{a,b}(G,x) = \sum_{(\omega_i\omega_j) \in E(G)} x^{ad(\omega_i)+bd(\omega_i)}$$

(i)

$M_{a,b}(G,x) = \sum_{(\omega_i\omega_j)\in E(G)} x^{ad(\omega_i)+bd(\omega_i)}$

Edges $E(G) = \frac{\eta(\eta-1)}{2}$

Vertices $V(G) = \eta$

Each vertex in the harary graph has a degree, that is,

$H_{k,n} = (d_\mu, d_\omega) = (\eta - 1)$

So we have

$|E(R)| = \frac{\eta(\eta-1)}{2}$

$d_\mu = (\eta - 1)$

$d_\omega = (\eta - 1)$

Zagreb type polynomials the general form,

$M_{a,b}(R,x) = \sum_{(\omega_i\omega_j)\in E(G)} x^{ad(\omega_i)+bd(\omega_i)} \rightarrow eq.(1)$

Put $d_\mu = (\eta - 1)$ and $d_\omega = (\eta - 1)$ in eq.(1)

$M_{a,b}(R,x) = \sum_{(\omega_i\omega_j)\in E(R)} x^{[a(\eta-1)+b(\eta-1)]}$

$M_{a,b}(R,x) = \sum_{(\omega_i\omega_j)\in E(R)} x^{[(a+b)(\eta-1)]}$

$M_{a,b}(R,x) = |E(R)| x^{[(a+b)(\eta-1)]}$

$|E(R)| = \frac{\eta(\eta-1)}{2}$

$M_{a,b}(R,x) = \frac{\eta(\eta-1)}{2} x^{[(a+b)(\eta-1)]}.$

(ii)

$$M'_{a,b}(G,x) = \sum_{(\omega_i\omega_j)\in E(G)} x^{(a+d(\omega_i))+(b+d(\omega_i))}$$

$M'_{a,b}(G,x) = \sum_{(\omega_i\omega_j)\in E(G)} x^{(a+d(\omega_i))+(b+d(\omega_i))}$

Edges $E(G) = \frac{\eta(\eta-1)}{2}$

Vertices $V(G) = \eta$

Each vertex in the harary graph has a degree that is

$H_{k,n} = (d_\mu, d_\omega) = (\eta-1)$

so we have

Here

$|E(R)| = \frac{\eta(\eta-1)}{2}$
$d_\mu = (\eta-1)$
$d_\omega = (\eta-1)$

Zagreb type polynomials the general form,

$M'_{a,b}(G,x) = \sum_{(\omega_i\omega_j)\in E(G)} x^{(a+d(\omega_i))+(b+d(\omega_i))} \rightarrow eq.(1)$

Put $d_\mu = (\eta-1)$ and $d_\omega = (\eta-1)$ in eq.(1)

$M'_{a,b}(R,x) = \sum_{(\omega_i\omega_j)\in E(R)} x^{(a+(\eta-1))(b+(\eta-1))}$

$M'_{a,b}(R,x) = |E(R)|x^{(a+(\eta-1))(b+(\eta-1))}$

$M'_{a,b}(R,x) = |E(R)|x^{(a+\eta-1)(b+\eta-1)}$

$M'_{a,b}(R,x) = |E(R)|x^{(\eta^2+\eta(a+b-2)-(a+b-1))}$

$|E(R)| = \frac{\eta(\eta-1)}{2}$

$M'_{a,b}(R,x) = \frac{\eta(\eta-1)}{2}x^{(\eta^2+(a+b-2)\eta-(a+b-1))}.$

Theorem 2.13 Suppose graph G and $(V(G), E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be harary subdivision graph. Then for $\eta \geq 2$, the forgotten polynomial index is,

$F(G) = \frac{\eta(\eta-1)}{2}x^{|2(\eta^2-2\eta+1)|}.$

Proof: Since graph G and $(V(G), E(G)) = (\eta, \frac{\eta(\eta-1)}{2})$ be harary subdivision graph and now the forgotten polynomial index is,

$F(G) = \sum_{(\mu\omega)\in E(x)} x^{[(d_\mu)^2+(d_\omega)^2]} \to eq.(1)$

Put $d_\mu = (\eta-1)$ and $d_\omega = (\eta-1)$ in eq.(1)

$F(R) = \sum_{(\mu\omega)\in E(R)} x^{|(\eta-1)^2+(\eta-1)^2|}$

$F(G) = \sum_{(\mu\omega)\in E(R)} x^{|\eta^2+1-2\eta+\eta^2+1-2\eta|}$

$F(G) = |E(R)|x^{|2\eta^2-4\eta+2|}$

$F(G) = |E(R)|x^{|2(\eta^2-2\eta+1)|}$

$|E(R)| = \frac{\eta(\eta-1)}{2}$

$F(G) = \frac{\eta(\eta-1)}{2}x^{|2(\eta^2-2\eta+1)|}$.

3. Numerical Examples

In this part, we'll developed a number of conclusions regarding the harary graph's degree-based topological indices.

Example 3.1. There must be η if this is a positive natural number $\eta = 2, 3, ...$, so $E(G)$ exists in the graph for vertices $V(G) = \eta = \frac{\eta(\eta-1)}{2}$ edges and the peculiarity of this graph is that approximately the degree of each vertex corresponds to $\eta - 1$.

$$\eta = 2, 3, ...$$
$$V(G) = \eta \; eq.(1)$$
$$E(G) = \frac{\eta(\eta-1)}{2} \; eq.(2)$$

Equations (1) and (2) become, when $\eta = 3$ is substituted in.

Vertex $V(G) = \eta$

Here

$\eta = 3$

Edges $E(G) = C_2^\eta = \frac{3(3-1)}{2} = 3$

Degree of harary graph $(d_\mu, d_\omega) = 3 - 1 = 2$.

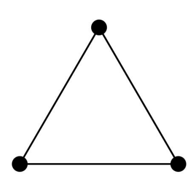

Figure 4. Harary graph are $H_{2,3}$

Example 3.2. There must be η if this is a positive natural number $\eta = 2, 3, ...,$ so $E(G)$ exists in the graph for vertices $V(G) = \eta = \frac{\eta(\eta-1)}{2}$ edges and the peculiarity of this graph is that approximately the degree of each vertex corresponds to $\eta - 1$.

$$\eta = 2, 3, ...$$

$$V(G) = \eta \; eq.(1)$$

$$E(G) = \frac{\eta(\eta-1)}{2} \; eq.(2)$$

Equations (1) and (2) become, when $\eta = 3$ is substituted in.

Vertex $V(G) = \eta$

Here

$\eta = 4$

Edges $E(G) = C_2^\eta = \frac{4(4-1)}{2} = 6$

Degree of harary graph $(d_\mu, d_\omega) = 4 - 1 = 3$

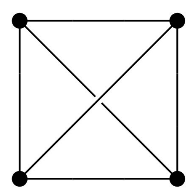

FIGURE 5. Harary graph are $H_{3,4}$

Example 3.3. There must be η if this is a positive natural number $\eta = 2, 3, ...,$ so $E(G)$ exists in the graph for vertices $V(G) = \eta = \frac{\eta(\eta-1)}{2}$ edges and the peculiarity of this graph is that approximately the degree of each vertex corresponds to $\eta - 1$.

$$\eta = 2, 3, ...$$

$$V(G) = \eta \; eq.(1)$$

$$E(G) = \frac{\eta(\eta-1)}{2} \; eq.(2)$$

Equations (1) and (2) become; when $\eta = 3$ is substituted in.

Vertex $V(G) = \eta$

Here

$\eta = 6$

Edges $E(G) = C_2^\eta = \frac{6(6-1)}{2} = 15$

Degree of harary graph $(d_\mu, d_\omega) = 6 - 1 = 5$

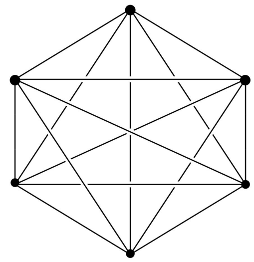

FIGURE 6. Harary graph are $H_{5,6}$

Example 3.4. There must be η if this is a positive natural number $\eta = 2, 3, ...$, so $E(G)$ exists in the graph for vertices $V(G) = \eta = \frac{\eta(\eta-1)}{2}$ edges and the peculiarity of this graph is that approximately the degree of each vertex corresponds to $\eta - 1$.

$$\eta = 2, 3, ...$$
$$V(G) = \eta \; eq.(1)$$
$$E(G) = \frac{\eta(\eta-1)}{2} \; eq.(2)$$

Equations (1) and (2) become, when $\eta = 3$ is substituted in.

Vertex $V(G) = \eta$

Here

$\eta = 8$

Edges $E(G) = C_2^{\eta} = \frac{8(8-1)}{2} = 28$

Degree of harary graph $(d_{\mu}, d_{\omega}) = 8 - 1 = 7$

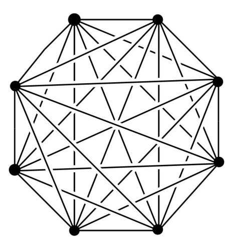

FIGURE 7. Harary graph are $H_{7,8}$

Example 3.5. There must be η if this is a positive natural number $\eta = 2, 3, ...$, so $E(G)$ exists in the graph for vertices $V(G) = \eta = \frac{\eta(\eta-1)}{2}$ edges and the peculiarity of this graph is that approximately the degree of each vertex corresponds to $\eta - 1$.

$$\eta = 2, 3, ...$$
$$V(G) = \eta \; eq.(1)$$
$$E(G) = \frac{\eta(\eta-1)}{2} \; eq.(2)$$

Equations (1) and (2) become, when $\eta = 3$ is substituted in.

Vertex $V(G) = \eta$

Here

$\eta = 25$

Edges $E(G) = C_2^{\eta} = \frac{25(25-1)}{2} = 300$

Degree of harary graph $(d_\mu, d_\omega) = 25 - 1 = 24$.

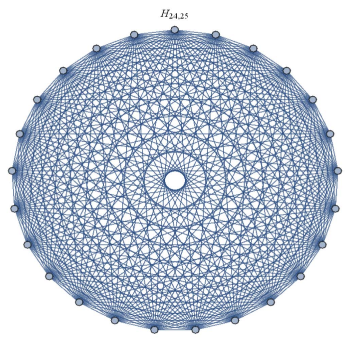

FIGURE 8. Harary graph are $H_{24,25}$

4. Conclusion

This article explores the decomposition of various graphs using the n-sun method. A topological index of harary graphs has been discovered specifically in the case of harary graphs. The extension or generalization of the n-sun decomposition for harary graphs is investigated. The identification of a necessary and sufficient condition for the existence and total decomposition of n-sun is deemed valuable. Research on n-sun harary graphs is recommended and the creation of graceful graphs for the presented n-sun graphs is proposed by carefully selecting the graphs.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] W.D. Wallis, Magic Graphs, Birkhäuser, Basel, (2001).

[2] B. Alspach, J.C. Bermond, D. Sotteau, Decomposition Into Cycles I: Hamilton Decompositions, in: G. Hahn, G. Sabidussi, R.E. Woodrow (Eds.), Cycles and Rays, Springer Netherlands, Dordrecht, 1990: pp. 9–18. https://doi.org/10.1007/978-94-009-0517-7_2.

[3] B. Alspach, The Wonderful Walecki Construction, Bull. Inst. Comb. Appl. 52 (2008), 7–20.

[4] B. Alspach, H. Gavlas, Cycle Decompositions of K_n and $K_n - I$, J. Comb. Theory, Ser. B. 81 (2001), 77–99. https://doi.org/10.1006/jctb.2000.1996.

[5] D.B. West, Introduction to Graph Theory, Prentice Hall, Upper Saddle River, (2001).

[6] J.L. Gross, J. Yellen, Handbook of Graph Theory, CRC Press, Boca Raton, (2004).

[7] M.H. Khalifeh, H. Yousefi-Azari, A.R. Ashrafi, The First and Second Zagreb Indices of Some Graph Operations, Discr. Appl. Math. 157 (2009), 804–811. https://doi.org/10.1016/j.dam.2008.06.015.

[8] G.H. Shirdel, H. Rezapour, A.M. Sayadi, The Hyper-Zagreb Index of Graph Operations, 4 (2013), 213–220.

[9] S.M. Sankarraman, A Computational Approach on Acetaminophen Drug Using Degree-Based Topological Indices and M-Polynomials, Biointerface Res. Appl. Chem. 12 (2021), 7249–7266. https://doi.org/10.33263/briac126.72497266.

[10] I. Gutman, Multiplicative Zagreb Indices of Trees, Bull. Soc. Math. Banja Luka. 18 (2011), 17–23.

[11] S. Hayat, M. Imran, On Degree Based Topological Indices of Certain Nanotubes, J. Comput. Theor. Nanosci. 12 (2015), 1599–1605. https://doi.org/10.1166/jctn.2015.3935.

[12] E. Estrada, L. Torres, L. Rodriguez, I. Gutman, An Atom-Bond Connectivity Index: Modelling the Enthalpy of Formation of Alkanes, Indian. J. Chem. 37A (1998), 849–855.

[13] X. Ren, X. Hu, B. Zhao, Proving a Conjecture Concerning Trees With Maximal Reduced Reciprocal Randic Index, MATCH Commun. Math. Comput. Chem. 76 (2016), 171–184.

[14] B. Furtula, I. Gutman, A Forgotten Topological Index, J. Math. Chem. 53 (2015), 1184–1190. https://doi.org/10.1007/s10910-015-0480-z.

[15] A.R. Ashrafi, M. Mirzargar, PI, Szeged and Edge Szeged of an Infinite Family of Nanostardendrimers, Indian J. Chem. 47A (2008), 538–541.

[16] Z. Chen, M. Dehmer, F. Emmert-Streib, Y. Shi, Entropy Bounds for Dendrimers, Appl. Math. Comput. 242 (2014), 462–472. https://doi.org/10.1016/j.amc.2014.05.105.

[17] M.V. Diudea, A.E. Vizitiu, M. Mirzagar, A.R. Ashrafi, Sadhana Polynomial in Nano-Dendrimers, Carpathian J. Math. 26 (2010), 59–66.

[18] Z. Hassan Niazi, M.A.T. Bhatti, M. Aslam, Y. Qayyum, M. Ibrahim, A. Qayyum, d-Lucky Labeling of Some Special Graphs, Amer. J. Math. Anal. 10 (2022), 3–11. https://doi.org/10.12691/ajma-10-1-2.

[19] A. Asghar, A. Qayyum, N. Muhammad, Different Types of Topological Structures by Graphs, Eur. J. Math. Anal. 3 (2022), 3.

[20] M. Ahmad, S. Hussain, U. Parveen, I. Zahid, M. Sultan, A. Qayyum, On Degree-Based Topological Indices of Petersen Subdivision Graph, Eur. J. Math. Anal. 3 (2023), 20.

[21] M. Ahmad, M.J. Hussain, G. Atta, S. Raza, I. Waheed, A. Qayyum, Topological Evaluation of Four Para-Line Graphs Absolute Pentacene Graphs Using Topological Indices, Int. J. Anal. Appl. 21 (2023), 66. https://doi.org/10.28924/2291-8639-21-2023-66.

[22] R.M.K. Iqbal, M. Ahmad, A. Qayyum, S.S. Supadi, M.J. Hussain, S. Raza, On Degree-Based Topological Indices of Toeplitz Graphs, Int. J. Anal. Appl. 21 (2023), 111. https://doi.org/10.28924/2291-8639-21-2023-111.