International Journal of Analysis and Applications

Bipolar Fuzzy Filters of Gamma-Near Rings

V. P. Vineela Korada[1], S. Ragamayi[1], Aiyared Iampan[2,*]

[1] *Department of Engineering Mathematics, Koneru Lakshmaiah Education Foundation, Guntur-522502, Andhra Pradesh, India*

[2] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

[*] *Corresponding author:* aiyared.ia@up.ac.th

Abstract. The main objective of this paper is to present the notation of bipolar fuzzy filters of Γ-near rings and ordered Γ-near rings. As a consequence, we deal with bipolar fuzzy prime ideals of Γ-near rings and ordered Γ-near rings. Also, we examine the one-to-one correspondence of bipolar fuzzy filters and crisp filters of Γ-near rings. Later, we define and study the homomorphism of ordered Γ-near rings.

1. Introduction

The near-ring theory was introduced by Pilz [6]. The concept of Γ-rings, a generalization of a ring, was introduced by Nobusawa [5]. Γ-near rings (GNRs) were defined by Satyanarayana [16], and the ideal theory in GNRs was studied by Satyanarayana [16] and Booth [1]. Further, several authors studied various algebraic structures on GNRs, like ideals, weak ideals, bi-ideals, quasi-ideals, and normal ideals on GNRs. The idea of bipolar-valued fuzzy sets (BFSs) was given by Zhang [20], which is the extension of the theory of Zadeh's fuzzy sets (FSs) [19] to BFSs. Later, taking into consideration, many authors applied fuzzification on crisp sets, like Satyanarayana studied and invented the idea of fuzzy ideals, prime ideals of GNRs. Some results and properties on fuzzy ideals of GNRs are discussed by Jun [2]. In order to study uncertainty, the application of bipolar fuzzification, which is a generalization of FSs, has been developed by Jun and Lee [3]. Several researchers like Ragamayi [7–13, 17, 18] and Rao [14, 15] did their research on the development of the BFS theory on different algebraic structures like semigroups, groups, semirings, rings, etc.

As a continuity of all these, we introduced bipolar fuzzy ideals, bi-ideals, and weak bi-deals on GNRs in 2023. Now, we are studying bipolar fuzzy filters and prime ideals of GNRs.

Received: Oct. 25, 2023.

2010 *Mathematics Subject Classification.* 03E72, 16Y30, 16Y80.

Key words and phrases. Γ-near ring; bipolar fuzzy filter; bipolar fuzzy prime ideal; homomorphism.

https://doi.org/10.28924/2291-8639-22-2024-2
ISSN: 2291-8639

2. Preliminaries

This section reviews important definitions for research in this paper.

Definition 2.1. *[6] A near ring is a nonempty set R equipped with two binary operations $+$ and $\cdot$ such that*
(i) $(R, +)$ *is a group,*
(ii) $(R, \cdot)$ *is a semigroup,*
(iii) $(a+b)c = ac + bc, \forall a, b, c \in R$ *obeying only right distributive law over addition.*

Definition 2.2. *[16] A Γ-near ring (GNR) is a triple $(M_R, +, \Gamma)$ where*
(i) $(M_R, +)$ *is a group,*
(ii) Γ *is a nonempty set of binary operators on M_R such that for each $\alpha \in \Gamma, (M_R, +, \alpha)$ is a near ring,*
(iii) $\psi\alpha(\omega\beta\kappa) = (\psi\alpha\omega)\beta\kappa, \forall \psi, \omega, \kappa \in M_R, \alpha, \beta \in \Gamma.$

Definition 2.3. *[4] A GNR M_R is said to be zero-symmetric if $\psi\alpha 0 = 0, \forall \psi \in M_R, \alpha \in \Gamma.$*

Definition 2.4. *[2] An FS ξ in a GNR M_R is a fuzzy sub Γ-near ring of M_R if*
(i) $\xi(\psi - \omega) \geq \min\{\xi(\psi), \xi(\omega)\}, \forall \psi, \omega \in M_R,$
(ii) $\xi(\psi\alpha\omega) \geq \min\{\xi(\psi), \xi(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma.$

Definition 2.5. *[3,20] Let M_R be a GNR and B_R be a BFS of M_R. We say that $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ is a bipolar fuzzy sub Γ-near ring (BFSGNR) of M_R if*
(i) $\xi^+_{B_R}(\psi - \omega) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R,$
(ii) $\xi^-_{B_R}(\psi - \omega) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R,$
(iii) $\xi^+_{B_R}(\psi\alpha\omega) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma,$
(iv) $\xi^-_{B_R}(\psi\alpha\omega) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma.$
If $B = (\xi^+_{B_R}, \xi^-_{B_R})$ satisfies the conditions (i) and (ii), then it is called a bipolar fuzzy subgroup (BFSG) of M_R.

Definition 2.6. *[16] Let M_R be a GNR and A_R be a nonempty subset of M_R. Then A_R is said to be an ideal of M_R if*
(i) $\psi - \omega \in A_R, \forall \psi, \omega \in A_R,$
(ii) $\omega + \psi - \omega \in A_R, \forall \psi \in I_R, \omega \in M_R,$
(iii) $a\alpha(\psi + b) - a\alpha b \in A_R, \forall \psi \in A_R, a, b \in M_R, \alpha \in \Gamma,$
(iv) $\psi\alpha a \in A_R, \forall \psi \in A_R, a, b \in M_R, \alpha \in \Gamma.$

Definition 2.7. *[16] An ideal A_R of a GNR M_R is said to be a prime ideal of M_R if $\psi\alpha\omega \in A_R \Rightarrow \psi \in A_R$ or $\omega \in A_R, \forall \psi, \omega \in M_R, \alpha \in \Gamma.$*

Definition 2.8. *[14] Let M_R be a GNR and A_R be a nonempty subset of M_R. Then A_R is said to be a filter of M_R if*
(i) $\psi\alpha\omega \in A_R, \forall \psi \in A_R, \omega \in M_R, \alpha \in \Gamma,$
(ii) $\psi \leq \omega \Rightarrow \omega \in A_R, \forall \psi \in A_R, \omega \in M_R.$

Definition 2.9. *[2] An FS ξ in a GNR M_R is called a fuzzy ideal of M_R if*
(i) $\xi(\psi-\omega) \geq \min\{\xi(\psi), \xi(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi(\omega+\psi-\omega) \geq \xi(\psi), \forall \psi, \omega \in M_R$,
(iii) $\xi(a\alpha(\psi+b)-a\alpha b) \geq \xi(\psi), \forall \psi, a, b \in M_R, \alpha \in \Gamma$,
(iv) $\xi(\psi\alpha a) \geq \xi(\psi), \forall \psi, a, b \in M_R, \alpha \in \Gamma$.

Definition 2.10. *[2] A BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of a GNR M_R is called a bipolar fuzzy ideal (BFI) of M_R if*
(i) $\xi^+_{B_R}(\psi-\omega) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi^+_{B_R}(\omega+\psi-\omega) \geq \xi^+_{B_R}(\psi), \forall \psi, \omega \in M_R$,
(iii) $\xi^+_{B_R}(a\alpha(\psi+b)-a\alpha b) \geq \xi^+_{B_R}(\psi), \forall \psi, a, b \in M_R, \alpha \in \Gamma$,
(iv) $\xi^+_{B_R}(\psi\alpha a) \geq \xi^+_{B_R}(\psi), \forall \psi, a \in M_R, \alpha \in \Gamma$,
(v) $\xi^-_{B_R}(\psi-\omega) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(vi) $\xi^-_{B_R}(\omega+\psi-\omega) \leq \xi^-_{B_R}(\psi), \forall \psi, \omega \in M_R$,
(vii) $\xi^-_{B_R}(a\alpha(\psi+b)-a\alpha b) \leq \xi^-_{B_R}(\psi), \forall \psi, a, b \in M_R, \alpha \in \Gamma$,
(viii) $\xi^-_{B_R}(\psi\alpha a) \leq \xi^-_{B_R}(\psi), \forall \psi, a \in M_R, \alpha \in \Gamma$.

Definition 2.11. *[9] A BFI $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of a GNR M_R is said to be a bipolar fuzzy prime ideal (BFPI) of M_R if*
(i) $\xi^+_{B_R}(\psi\alpha\omega) = \max\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma$,
(ii) $\xi^-_{B_R}(\psi\alpha\omega) = \min\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma$.

Definition 2.12. *The (t,s) cut of an BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ is a crisp set defined by $B_{R(t,s)} = \{\psi \mid \xi^+_{B_R}(\psi) \geq t, \xi^-_{B_R}(\psi) \leq s\}$ for $t \in [0,1], s \in [-1,0]$.*

3. Bipolar Fuzzy Filters of GNRs

This section introduces and studies the notion of bipolar fuzzy filters of GNRs and their properties.

Definition 3.1. *A BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of a GNR M_R is called a bipolar fuzzy filter (BFF) of M_R if*
(i) $\xi^+_{B_R}(\psi-\omega) \leq \max\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi^-_{B_R}(\psi-\omega) \geq \min\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(iii) $\xi^+_{B_R}(\psi\alpha\omega) = \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma$,
(iv) $\xi^-_{B_R}(\psi\alpha\omega) = \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma$.

Example 3.1. *Let M_R be the real numbered set and $\Gamma = M_R$. Then M_R and Γ are additive commutative groups. Define the mapping $M_R * \Gamma * M_R \to M_R$ by $\psi\alpha\omega$ the usual product of $\psi, \alpha, \omega, \forall \psi, \omega \in M_R, \alpha \in \Gamma$. Then M_R is a GNR with zero symmetric. Let a BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of M_R defined by*

$$\xi^+_{B_R}(\psi) = \begin{cases} 0.55, & \text{if } \psi = 0 \\ 0.83, & \text{otherwise} \end{cases}$$

$$\xi^-_{B_R}(\psi) = \begin{cases} -0.12, & \text{if } \psi = 0 \\ -0.51, & \text{otherwise} \end{cases}$$

Then, by routine check, B_R is a BFF of M_R.

Definition 3.2. *A BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of an OGNR M_R is called a bipolar fuzzy filter (BFF) of M_R if*
(i) $\xi^+_{B_R}(\psi - \omega) \leq \max\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi^-_{B_R}(\psi - \omega) \geq \min\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(iii) $\xi^+_{B_R}(\psi\alpha\omega) = \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma$,
(iv) $\xi^-_{B_R}(\psi\alpha\omega) = \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma$,
(v) $\psi \leq \omega \Rightarrow \xi^+_{B_R}(\psi) \leq \xi^+_{B_R}(\omega), \forall \psi, \omega \in M_R$,
(vi) $\psi \leq \omega \Rightarrow \xi^-_{B_R}(\psi) \geq \xi^-_{B_R}(\omega), \forall \psi, \omega \in M_R$.

Theorem 3.1. *A BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of a GNR M_R is a BFF of M_R if and only if its level subset $B_{R(t_l,s_l)} \neq \emptyset$ is a filter of M_R for any $t_l \in [0,1], s_l \in [-1,0]$.*

Proof. Suppose B_R is a BFF of M_R and let $t_l \in [0,1]$ and $s_l \in [-1,0]$. Let $\psi \in B_{R(t_l,s_l)}, \omega \in M_R, \alpha \in \Gamma$. Then $\xi^+_{B_R}(\psi) \geq t_l$ and $\xi^+_{B_R}(\omega) \geq t_l$. Thus $\min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\} \geq t_l$, so $\xi^+_{B_R}(\psi\alpha\omega) \geq t_l$. Similarly, $\xi^-_{B_R}(\psi) \leq s_l$ and $\xi^-_{B_R}(\omega) \leq s_l$. Thus $\max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\} \leq s_l$, so $\xi^-_{B_R}(\psi\alpha\omega) \leq s_l$. Therefore, $\psi\alpha\omega \in B_{R(t_l,s_l)}$. Let $\psi \in B_{R(t_l,s_l)}, \omega \in M_R, \alpha \in \Gamma$ be such that $\psi \leq \omega$. Then $\xi^+_{B_R}(\psi) \geq t_l$ and $\xi^+_{B_R}(\psi) \leq \xi^+_{B_R}(\omega)$. Thus $t_l \leq \xi^+_{B_R}(\psi) \leq \xi^+_{B_R}(\omega)$, so $\xi^+_{B_R}(\omega) \geq t_l$. Similarly, $\xi^-_{B_R}(\psi) \leq s_l$ and $\xi^-_{B_R}(\psi) \geq \xi^-_{B_R}(\omega)$. Thus $s_l \geq \xi^-_{B_R}(\psi) \geq \xi^-_{B_R}(\omega)$, so $\xi^-_{B_R}(\omega) \leq s_l$. Therefore, $\omega \in B_{R(t_l,s_l)}$. Hence, $B_{R(t_l,s_l)}$ is a filter of M_R.

Conversely, suppose $B_{R(t_l,s_l)} \neq \emptyset$ is a filter of M_R for any $t_l \in [0,1], s_l \in [-1,0]$. Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$. Suppose $t_l = \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}$. Then $\xi^+_{B_R}(\psi) \geq t_l$ and $\xi^+_{B_R}(\omega) \geq t_l$. Similarly, if $s_l = \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}$, then $\xi^-_{B_R}(\psi) \leq s_l$ and $\xi^-_{B_R}(\omega) \leq s_l$. Therefore, $\psi, \omega \in B_{R(t_l,s_l)}$. Since $B_{R(t_l,s_l)}$ is a filter of M_R, $\psi\alpha\omega \in B_{R(t_l,s_l)}$. Thus $\xi^+_{B_R}(\psi\alpha\omega) \geq t_l = \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}$ and $\xi^-_{B_R}(\psi\alpha\omega) \leq s_l = \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}$. Now, $\psi, \omega \in M_R, \alpha \in \Gamma, t_l = \xi^+_{B_R}(\psi\alpha\omega)$ and $s_l = \xi^-_{B_R}(\psi\alpha\omega)$. Then $\psi\alpha\omega \in B_{R(t_l,s_l)}$, so $\psi, \omega \in B_{R(t_l,s_l)}$. Thus $\xi^+_{B_R}(\psi) \geq t_l, \xi^+_{B_R}(\omega) \geq t_l$ and $\xi^-_{B_R}(\psi) \leq s_l, \xi^-_{B_R}(\omega) \leq s_l$. So, $\min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\} \geq t_l = \xi^+_{B_R}(\psi\alpha\omega)$ and $\max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\} \leq s_l = \xi^-_{B_R}(\psi\alpha\omega)$. Therefore, $\xi^+_{B_R}(\psi\alpha\omega) = \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}$ and $\xi^-_{B_R}(\psi\alpha\omega) = \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}$. Hence, B_R is a BFF of M_R. □

Theorem 3.2. *A BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of a GNR M_R is a BFPI of M_R if and only if its level subset $B_{R(t_l,s_l)} \neq \emptyset$ is a prime ideal of M_R for any $t_l \in [0,1], s_l \in [-1,0]$.*

Proof. The proof is in the same way as Theorem 3.1. □

Theorem 3.3. *Let F be a non-empty subset of a GNR M_R. Then the characteristic set of F, $B_F = (\chi^+_{B_F}, \chi^-_{B_F})$ is a BFF of M_R if and only if F is a filter of M_R.*

Proof. Let $B_F = (\chi^+_{B_F}, \chi^-_{B_F})$ is a BFF of M_R.

Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$ be such that $\psi\alpha\omega \in F$. Since B_F is a BFF of M_R, we have $\chi^+_{B_F}(\psi\alpha\omega) = \min\{\chi^+_{B_F}(\psi), \chi^+_{B_F}(\omega)\}$ and $\chi^+_{B_F}(\psi\alpha\omega) = 1$. Thus $\chi^+_{B_F}(\psi) = 1$ and $\chi^+_{B_F}(\omega) = 1$. Also, $\chi^-_{B_F}(\psi\alpha\omega) = \max\{\chi^-_{B_F}(\psi), \chi^-_{B_F}(\omega)\}$ and $\chi^-_{B_F}(\psi\alpha\omega) = -1$. Thus $\chi^-_{B_F}(\psi) = -1$ and $\chi^-_{B_F}(\omega) = -1$. Therefore, $\psi, \omega \in F$. Hence F is a filter of M_R.

Conversely, suppose that F is a filter of M_R. Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$. (i) Let $\psi\alpha\omega \notin F$. Then $\psi \notin F$ or $\omega \notin F$. Thus $\chi^+_{B_F}(\psi\alpha\omega) = 0$ and $\chi^+_{B_F}(\psi) = 0$ or $\chi^+_{B_F}(\omega) = 0$. So, $\chi^+_{B_F}(\psi\alpha\omega) = \min\{\chi^+_{B_F}(\psi), \chi^+_{B_F}(\omega)\}$. Also, $\chi^-_{B_F}(\psi\alpha\omega) = 0$ and $\chi^-_{B_F}(\psi) = 0$ or $\chi^-_{B_F}(\omega) = 0$. Thus $\chi^-_{B_F}(\psi\alpha\omega) = \max\{\chi^-_{B_F}(\psi), \chi^-_{B_F}(\omega)\}$. (ii) Let $\psi\alpha\omega \in F$. Then $\psi, \omega \in F$. Thus $\chi^+_{B_F}(\psi\alpha\omega) = 1, \chi^+_{B_F}(\psi) = 1, \chi^+_{B_F}(\omega) = 1$. So, $\chi^+_{B_F}(\psi\alpha\omega) = \min\{\chi^+_{B_F}(\psi), \chi^+_{B_F}(\omega)\}$. Also, $\chi^-_{B_F}(\psi\alpha\omega) = -1, \chi^-_{B_F}(\psi) = -1, \chi^-_{B_F}(\omega) = -1$. Thus $\chi^-_{B_F}(\psi\alpha\omega) = \max\{\chi^-_{B_F}(\psi), \chi^-_{B_F}(\omega)\}$. Hence, $B_F = (\chi^+_{B_F}, \chi^-_{B_F})$ is a BFF of M_R. □

Theorem 3.4. *Let S be a non-empty subset of a GNR M_R. Then the characteristic set of S, $B_S = (\chi^+_{B_S}, \chi^-_{B_S})$ is a BFPI of M_R if and only if S is a prime ideal of M_R.*

Proof. The proof is in the same way as Theorem 3.3. □

Theorem 3.5. *Let $A_R = (\xi^+_{A_R}, \xi^-_{A_R})$ and $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be BFFs of an OGNR M_R. Then $A_R \cap B_R$ is a BFF of M_R.*

Proof. Let $A_R = (\xi^+_{A_R}, \xi^-_{A_R})$ and $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be BFFs of M_R. Let Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}(\xi^+_{A_R} \cap \xi^+_{B_R})(\psi - \omega) &= \min\{\xi^+_{A_R}(\psi - \omega), \xi^+_{B_R}(\psi - \omega)\}\\ &\leq \min\{\max\{\xi^+_{A_R}(\psi), \xi^+_{A_R}(\omega)\}, \max\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}\}\\ &= \max\{\min\{\xi^+_{A_R}(\psi), \xi^+_{B_R}(\psi)\}, \min\{\xi^+_{A_R}(\omega), \xi^+_{B_R}(\omega)\}\}\\ &= \max\{(\xi^+_{A_R} \cap \xi^+_{B_R})(\psi), (\xi^+_{A_R} \cap \xi^+_{B_R})(\omega)\},\end{aligned}$$

$$\begin{aligned}(\xi^+_{A_R} \cap \xi^+_{B_R})(\psi\alpha\omega) &= \min\{\xi^+_{A_R}(\psi\alpha\omega), \xi^+_{B_R}(\psi\alpha\omega)\}\\ &= \min\{\min\{\xi^+_{A_R}(\psi), \xi^+_{A_R}(\omega)\}, \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}\}\end{aligned}$$

$$\begin{aligned}&= \min\{\min\{\xi^+_{A_R}(\psi), \xi^+_{B_R}(\psi)\}, \min\{\xi^+_{A_R}(\omega), \xi^+_{B_R}(\omega)\}\}\\ &= \min\{(\xi^+_{A_R} \cap \xi^+_{B_R})(\psi), (\xi^+_{A_R} \cap \xi^+_{B_R})(\omega)\},\end{aligned}$$

if $\psi \leq \omega$, then $\xi^+_{A_R}(\psi) \leq \xi^+_{A_R}(\omega)$ and $\xi^+_{B_R}(\psi) \leq \xi^+_{B_R}(\omega)$. Thus

$$\begin{aligned}(\xi^+_{A_R} \cap \xi^+_{B_R})(\psi) &= \min\{\xi^+_{A_R}(\psi), \xi^+_{B_R}(\psi)\}\\ &\leq \min\{\xi^+_{A_R}(\omega), \xi^+_{B_R}(\omega)\}\\ &= (\xi^+_{A_R} \cap \xi^+_{B_R})(\omega).\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}(\xi^-_{A_R} \cap \xi^-_{B_R})(\psi - \omega) &\geq \min\{(\xi^-_{A_R} \cap \xi^-_{B_R})(\psi), (\xi^-_{A_R} \cap \xi^-_{B_R})(\omega)\},\\ (\xi^-_{A_R} \cap \xi^-_{B_R})(\psi\alpha\omega) &= \max\{(\xi^-_{A_R} \cap \xi^-_{B_R})(\psi), (\xi^-_{A_R} \cap \xi^-_{B_R})(\omega)\},\\ \psi \leq \omega \Rightarrow (\xi^-_{A_R} \cap \xi^-_{B_R})(\psi) &\geq (\xi^-_{A_R} \cap \xi^-_{B_R})(\omega).\end{aligned}$$

Hence, $A_R \cap B_R$ is a BFF of M_R. □

Theorem 3.6. *A BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of an OGNR M_R is a BFF of M_R if and only if its level subset $B_{R(t_l, s_l)} \neq \emptyset$ is a filter of M_R for any $t_l \in [0, 1], s_l \in [-1, 0]$.*

Proof. The proof is in the same way as Theorem 3.1. □

Let M_R be an OGNR, $a_l \in M_R$, and $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be a BFF of M_R. Then the set $\{\psi \in M_R \mid \xi^+_{B_R}(a_l) \leq \xi^+_{B_R}(\psi) \text{ and } \xi^-_{B_R}(a_l) \geq \xi^-_{B_R}(\psi)\}$ is denoted by $F_{B(a_l)}$.

Theorem 3.7. *Let $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be a BFF of an OGNR M_R. Then the set $F_{B(a_l)}$ is a filter of M_R.*

Proof. Let $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be a BFF of M_R. Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$ be such that $\psi\alpha\omega \in F_{B(a_l)}$. Then

$$\begin{aligned}\xi^+_{B_R}(a_l) \leq \xi^+_{B_R}(\psi\alpha\omega) &\Rightarrow \xi^+_{B_R}(a_l) \leq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}\\ &\Rightarrow \xi^+_{B_R}(a_l) \leq \xi^+_{B_R}(\psi), \xi^+_{B_R}(a_l) \leq \xi^+_{B_R}(\omega),\end{aligned}$$

$$\begin{aligned}\xi^-_{B_R}(a_l) \geq \xi^-_{B_R}(\psi\alpha\omega) &\Rightarrow \xi^-_{B_R}(a_l) \geq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}\\ &\Rightarrow \xi^-_{B_R}(a_l) \geq \xi^-_{B_R}(\psi), \xi^-_{B_R}(a_l) \geq \xi^-_{B_R}(\omega).\end{aligned}$$

Thus $\psi, \omega \in F_{B(a_l)}$. Let $\omega \in M_R, \psi \in F_{B(a_l)}$ be such that $\psi \leq \omega$. Then

$$\begin{aligned}&\xi^+_{B_R}(a_l) \leq \xi^+_{B_R}(\psi), \xi^+_{B_R}(\psi) \leq \xi^+_{B_R}(\omega) \Rightarrow \xi^+_{B_R}(a_l) \leq \xi^+_{B_R}(\omega),\\ &\xi^-_{B_R}(a_l) \geq \xi^-_{B_R}(\psi), \xi^-_{B_R}(\psi) \geq \xi^-_{B_R}(\omega) \Rightarrow \xi^-_{B_R}(a_l) \geq \xi^-_{B_R}(\omega).\end{aligned}$$

Thus $\omega \in F_{B(a_l)}$. Hence, $F_{B(a_l)}$ is a filter of M_R. □

Definition 3.3. *Let a function $\phi : M_R \to N_R$ be a homomorphism of OGNRs M_R and N_R. A BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of M_R is said to be a ϕ homomorphism invariant if $\phi(\psi) \leq \phi(\omega)$, then $\xi^+_{B_R}(\psi) \leq \xi^+_{B_R}(\omega)$ and $\xi^-_{B_R}(\psi) \geq \xi^-_{B_R}(\omega)$ for any $\psi, \omega \in M_R$.*

Definition 3.4. *Let M_R and N_R be two OGNRs and $\phi : M_R \to N_R$ be an onto homomorphism. If $f = (\xi^+_f, \xi^-_f)$ is a BFS of M_R, then the image of f under ϕ, denoted by $\phi(f) = (\phi(\xi^+_f), \phi(\xi^-_f))$, is the BFS of N_R, defined by $\phi(\xi^+_f)(\psi) = \inf\limits_{t_l \in \phi^{-1}(\psi)} \xi^+_f(t_l)$ and $\phi(\xi^-_f)(\psi) = \sup\limits_{t_l \in \phi^{-1}(\psi)} \xi^-_f(t_l)$ for all $\psi \in N_R$.*

Theorem 3.8. *Let $\phi : M_R \to N_R$ be an onto homomorphism of OGNRs M_R and N_R. If $f = (\xi^+_f, \xi^-_f)$ is a ϕ homomorphism invariant BFF of M_R, then $\phi(f)$ is a BFF of N_R.*

Proof. Let $f = (\xi^+_f, \xi^-_f)$ be a ϕ homomorphism invariant BFF of M_R. Suppose $\psi \in N_R, t_l \in \phi^{-1}(\psi)$, and $\psi = \phi(a_l)$. Then $a_l \in \phi^{-1}(\psi)$, so $\phi(t_l) = \psi = \phi(a_l)$. Thus $\phi(a_l\alpha b_l) = \phi(a_l)\alpha\phi(b_l) = \psi\alpha\omega$. So, $\phi(\xi^+_f)(\psi\alpha\omega) = \xi^+_f(a_l\alpha b_l) = \min\{\xi^+_f(a_l), \xi^+_f(b_l)\} = \min\{\phi(\xi^+_f)(\psi), \phi(\xi^+_f)(\omega)\}$ and $\phi(\xi^-_f)(\psi\alpha\omega) = \xi^-_f(a_l\alpha b_l) = \max\{\xi^-_f(a_l), \xi^-_f(b_l)\} = \max\{\phi(\xi^-_f)(\psi), \phi(\xi^-_f)(\omega)\}$. Since f is a ϕ homomorphism invariant, we have

$$\begin{aligned}\xi^+_f(t_l) = \xi^+_f(a_l) &\Rightarrow \phi(\xi^+_f)(\psi) = \inf_{t_l \in \phi^{-1}(\psi)} \xi^+_f(t_l) = \xi^+_f(a_l)\\ &\Rightarrow \phi(\xi^+_f)(\psi) = \xi^+_f(a_l),\end{aligned}$$

$$\begin{aligned}\xi^-_f(t_l) = \xi^-_f(a_l) &\Rightarrow \phi(\xi^-_f)(\psi) = \sup_{t_l \in \phi^{-1}(\psi)} \xi^-_f(t_l) = \xi^-_f(a_l)\\ &\Rightarrow \phi(\xi^-_f)(\psi) = \xi_f(a_l).\end{aligned}$$

Let $\psi, \omega \in N_R$. Then there exist $a_l, b_l \in M_R$ such that $\phi(a_l) = \psi, \phi(b_l) = \omega$, so $\phi(a_l - b_l) = \psi - \omega$. Thus

$$\begin{aligned}\phi(\xi_f^+)(\psi - \omega) &= \xi_f^+(a_l - b_l)\\ &\leq \max\{\xi_f^+(a_l), \xi_f^+(b_l)\}\\ &= \max\{\phi(\xi_f^+)(\psi), \phi(\xi_f^+)(\omega)\},\end{aligned}$$

$$\begin{aligned}\phi(\xi_f^-)(\psi - \omega) &= \xi_f^-(a_l - b_l)\\ &\geq \min\{\xi_f^-(a_l), \xi_f^-(b_l)\}\\ &= \min\{\phi(\xi_f^-)(\psi), \phi(\xi_f^-)(\omega)\}.\end{aligned}$$

Let $\psi, \omega \in N_R$ and $\psi \leq \omega$. Then there exist $a_l, b_l \in M_R$ such that $\phi(a_l) = \psi, \phi(b_l) = \omega$. Thus $\phi(\xi_f^+)(\psi) = \xi_f^+(a_l), \phi(\xi_f^+)(\omega) = \xi_f^+(b_l), \phi(\xi_f^-)(\psi) = \xi_f^-(a_l)$, and $\phi(\xi_f^-)(\omega) = \xi_f^-(b_l)$. Thus

$$\begin{aligned}\psi \leq \omega &\Rightarrow \phi(\psi) \leq \phi(\omega)\\ &\Rightarrow \xi_f^+(a_l) \leq \xi_f^+(b_l)\\ &\Rightarrow \phi(\xi_f^+)(\psi) \leq \phi(\xi_f^+)(\omega),\end{aligned}$$

$$\begin{aligned}\psi \leq \omega &\Rightarrow \phi(\psi) \leq \phi(\omega)\\ &\Rightarrow \xi_f^-(a_l) \geq \xi_f^-(b_l)\\ &\Rightarrow \phi(\xi_f^-)(\psi) \geq \phi(\xi_f^-)(\omega).\end{aligned}$$

Hence, $\phi(f)$ is a BFF of N_R. $\square$

Theorem 3.9. *Let $f : M_R \to N_R$ be a homomorphism of OGNRs M_R and N_R and $A_R = (\nu_{A_R}^+, \nu_{A_R}^-)$ be a BFF of N_R. If $A_R \circ f = B_R$ such that $B_R = (\xi_{B_R}^+, \xi_{B_R}^-)$, then B_R is a BFF of M_R.*

Proof. Assume that $A_R \circ f = B_R$ such that $B_R = (\xi_{B_R}^+, \xi_{B_R}^-)$. Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}\xi_{B_R}^+(\psi - \omega) &= \nu_{A_R}^+(f(\psi - \omega))\\ &= \nu_{A_R}^+(f(\psi) - f(\omega))\\ &\leq \max\{\nu_{A_R}^+(f(\psi)), \nu_{A_R}^+(f(\omega))\}\\ &= \max\{\xi_{B_R}^+(\psi), \xi_{B_R}^+(\omega)\},\end{aligned}$$

$$\begin{aligned}\xi_{B_R}^-(\psi - \omega) &= \nu_{A_R}^-(f(\psi - \omega))\\ &= \nu_{A_R}^-(f(\psi) - f(\omega))\\ &\geq \min\{\nu_{A_R}^-(f(\psi)), \nu_{A_R}^-(f(\omega))\}\\ &= \min\{\xi_{B_R}^-(\psi), \xi_{B_R}^-(\omega)\},\end{aligned}$$

$$\begin{aligned}\xi^+_{B_R}(\psi\alpha\omega) &= \nu^+_{A_R}(f(\psi\alpha\omega))\\ &= \nu^+_{A_R}(f(\psi)\alpha f(\omega))\\ &= \min\{\nu^+_{A_R}(f(\psi)), \nu^+_{A_R}(f(\omega))\}\\ &= \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\},\end{aligned}$$

$$\begin{aligned}\xi^-_{B_R}(\psi\alpha\omega) &= \nu^-_{A_R}(f(\psi\alpha\omega))\\ &= \nu^-_{A_R}(f(\psi)\alpha f(\omega))\\ &= \max\{\nu^-_{A_R}(f(\psi)), \nu^-_{A_R}(f(\omega))\}\\ &= \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}.\end{aligned}$$

Let $\psi, \omega \in M_R$ be such that $\psi \leq \omega$. Since $f : M_R \to N_R$ is a homomorphism, we have $f(\psi) \leq f(\omega)$. Thus $\nu^+_{A_R}(f(\psi)) \leq \nu^+_{A_R}(f(\omega))$, so $\xi^+_{B_R}(f(\psi)) \leq \xi^+_{B_R}(f(\omega))$. Also, $\nu^-_{A_R}(f(\psi)) \geq \nu^-_{A_R}(f(\omega))$, so $\xi^-_{B_R}(f(\psi)) \geq \xi^-_{B_R}(f(\omega))$. Hence, B_R is a BFF of M_R. □

Definition 3.5. *Let M_R and N_R be two OGNRs and $f : M_R \to N_R$ be a function. If $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ is a BFS of N_R, then the pre-image of B_R under f, denoted by $f^{-1}(B_R) = (f^{-1}(\xi^+_{B_R}), f^{-1}(\xi^-_{B_R}))$, is the BFS of M_R, defined by $f^{-1}(\xi^+_{B_R})(\psi) = \xi^+_{B_R}(f(\psi))$ and $f^{-1}(\xi^-_{B_R})(\psi) = \xi^-_{B_R}(f(\psi))$ for all $\psi \in M_R$.*

Theorem 3.10. *Let $f : M_R \to N_R$ be an onto homomorphism of OGNRs M_R and N_R. If $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ is a BFF of N_R, then $f^{-1}(B_R)$ is a BFF of M_R.*

Proof. Let $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be a BFF of N_R. Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}f^{-1}(\xi^+_{B_R})(\psi - \omega) &= \xi^+_{B_R}(f(\psi - \omega))\\ &= \xi^+_{B_R}(f(\psi) - f(\omega))\\ &\leq \max\{\xi^+_{B_R}(f(\psi)), \xi^+_{B_R}(f(\omega))\}\\ &= \max\{f^{-1}(\xi^+_{B_R})(\psi), f^{-1}(\xi^+_{B_R})(\omega)\},\end{aligned}$$

$$\begin{aligned}f^{-1}(\xi^-_{B_R})(\psi - \omega) &= \xi^-_{B_R}(f(\psi - \omega))\\ &= \xi^-_{B_R}(f(\psi) - f(\omega))\\ &\geq \min\{\xi^-_{B_R}(f(\psi)), \xi^-_{B_R}(f(\omega))\}\\ &= \min\{f^{-1}(\xi^-_{B_R})(\psi), f^{-1}(\xi^-_{B_R})(\omega)\},\end{aligned}$$

$$\begin{aligned}f^{-1}(\xi^+_{B_R})(\psi\alpha\omega) &= \xi^+_{B_R}(f(\psi\alpha\omega))\\ &= \xi^+_{B_R}(f(\psi)\alpha f(\omega))\\ &= \min\{\xi^+_{B_R}(f(\psi)), \xi^+_{B_R}(f(\omega))\}\\ &= \min\{f^{-1}(\xi^+_{B_R})(\psi), f^{-1}(\xi^+_{B_R})(f(\omega)\},\end{aligned}$$

$$\begin{aligned} f^{-1}(\xi_{B_R}^-)(\psi\alpha\omega) &= \xi_{B_R}^-(f(\psi\alpha\omega)) \\ &= \xi_{B_R}^-(f(\psi)\alpha f(\omega)) \\ &= \max\{\xi_{B_R}^-(f(\psi)), \xi_{B_R}^-(f(\omega))\} \\ &= \max\{f^{-1}(\xi_{B_R}^-)(\psi), f^{-1}(\xi_{B_R}^-)(\omega)\}. \end{aligned}$$

Let $\psi, \omega \in M_R$ be such that $\psi \leq \omega$. Since $f : M_R \to N_R$ is a homomorphism, we have $f(\psi) \leq f(\omega)$. Thus $\xi_{B_R}^+(f(\psi)) \leq \xi_{B_R}^+(f(\omega))$, so $f^{-1}(\xi_{B_R}^+(\psi)) \leq f^{-1}(\xi_{B_R}^+)(\omega)$. Also, $\xi_{B_R}^-(f(\psi)) \geq \xi_{B_R}^-(f(\omega))$, so $f^{-1}(\xi_{B_R}^-(\psi)) \geq f^{-1}(\xi_{B_R}^-(\omega))$. Hence, $f^{-1}(B_R)$ is a BFF of M_R. □

4. Conclusion

In this paper, we inspected the notations of BFFs and BFPIs of GNRs and OGNRs and studied their properties and relations among them. Further, we extended our study to the homomorphic image and pre-image of BFFs of OGNRs.

Acknowledgment: This research project was supported by the Thailand Science Research and Innovation Fund and the University of Phayao (Grant No. FF67-UoE-Aiyared-Iampan).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] G.L. Booth, A Note on Γ-Near Rings, Stud. Sci. Math. Hung. 23 (1988), 471–475.
[2] Y.B. Jun, M. Sapanci, M.A. Öztürk, Fuzzy Ideals in Gamma Near-Rings, Turk. J. Math. 22 (1998), 449–549.
[3] K.J. Lee, Bipolar Fuzzy Subalgebras and Bipolar Fuzzy Ideals of BCK/BCI-Algebras, Bull. Malays. Math. Sci. Soc. 32 (2009), 361–373.
[4] N. Meenakumari, T.T. Chelvam, Fuzzy Bi-Ideals in Gamma Near-Rings, J. Algebra Discr. Struct. 9 (2011), 43–52.
[5] N. Nobusawa, On a Generalization of the Ring Theory, Osaka J. Math. 1 (1964), 81–89. https://doi.org/10.18910/12354.
[6] G. Pilz, Near-Rings: The Theory and Its Applications, North-Holland, Amsterdam, (1983).
[7] S. Ragamayi, Y. Bhargavi, A Study of Vague Gamma-Nearrings, Int. J. Sci. Technol. Res. 9 (2020), 3960–3963.
[8] S. Ragamayi, Y. Bhargavi, Some Results on Homomorphism of Vague Ideal of a Gamma-Near Ring, Int. J. Sci. Technol. Res. 9 (2020), 3972–3975.
[9] S. Ragamayi, Y. Bhargavi, C. Krishnaveni, P. Bindu, Lattice-Fuzzy Prime-Ideal of a Gamma-Nearring, J. Crit. Rev. 7 (2020), 1–5. https://doi.org/10.31838/jcr.07.13.01.
[10] S. Ragamayi, G. Jaya Lalitha, P. Bindu, On L-Vague Maximal-Ideal of a Γ-Near Ring, AIP Conf. Proc. 2375 (2021), 020017. https://doi.org/10.1063/5.0066395.
[11] S. Ragamayi, N. Konda Reddy, P. Bindu, On L-Vague Prime-Ideal of a Γ-Near Ring and Its Characteristic Properties, AIP Conf. Proc., 2375, (2021), 020020. https://doi.org/10.1063/5.0066396.
[12] S. Ragamayi, N. Konda Reddy, G.J. Lalitha, P. Bindu, A Study of L-Fuzzy Cosets of a Γ-Near Ring, AIP Conf. Proc., 2375, (2021), 020010. https://doi.org/10.1063/5.0066393
[13] S. Ragamayi, N. Konda Reddy, B.S. Kumar, Results on L-Vague Ideal of a Γ-Near Ring, AIP Conf. Proc. 2375 (2021), 020014. https://doi.org/10.1063/5.0066388.
[14] M.M.K. Rao, Fuzzy Filters in Ordered Γ-Semirings, Caspian J. Math. Sci. 8 (2019), 18–34. https://doi.org/10.22080/cjms.2018.12048.1314.

[15] M.M.K. Rao, B. Venkateswarlu, Fuzzy Filters in Γ-Semirings, Malaya J. Mat. 3 (2015), 93–98. https://doi.org/10.26637/mjm301/009.

[16] B. Satyanarayana, Contributions to Near-Rings Theory, Doctoral Thesis, Nagarjuna University, (1984).

[17] V.P.V. Korada, S. Ragamayi, Application of Bi-Polar Fuzzy Theory to Ideals in Gamma-Near Rings and It's Characteristics, AIP Conf. Proc. 2707 (2023), 020015. https://doi.org/10.1063/5.0143352.

[18] V.P.V. Korada, S. Ragamayi, G.J. Lalitha, Some Results on Bi-Polar Fuzzy Bi-Ideals of Γ-Near Rings, AIP Conf. Proc. 2707 (2023), 020014. https://doi.org/10.1063/5.0143353.

[19] L.A. Zadeh, Fuzzy Sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.

[20] W.R. Zhang, Bipolar Fuzzy Sets and Relations: A Computational Framework for Cognitive Modeling and Multiagent Decision Analysis, in: Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference, IEEE, San Antonio, TX, USA, 1994: pp. 305–309. https://doi.org/10.1109/IJCF.1994.375115.