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Effects of Rotation and Magnetic Field on Rayliegh Benard Convection

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Abstract. In this paper, a numerical method based on the Chebyshev tau method is applied to analyze the effects of rotation and magnetic fields on Rayleigh-Bénard convection. The rotation and magnetic fields are assumed to be parallel to the vertical direction. The perturbation equations and boundary conditions are analyzed using normal mode analysis. The equations are then converted into a non-dimensional form and transformed into a generalized eigenvalue problem of the form $AX = RBX$, where R represents the eigenvalue corresponding to the Rayleigh number. The MATLAB software package is utilized to determine the relationship between the Rayleigh number and the Taylor number (rate of rotation), as well as the relationship between the Rayleigh number and the magnetic parameter (strength of the magnetic field) for different boundary conditions (free-free, rigid-rigid, or one free and the other rigid). The numerical and graphical results are presented and found to be in full agreement with the results obtained from previous analytical and numerical studies of the problem.

1. Introduction

Rayleigh-Bénard convection is a classical problem in fluid mechanics, it is a type of natural convection occurring in a planar horizontal layer of fluid heated from below. It has wide range of application in physics and engineering sciences. There are many analytical and numerical studies have been done for the problem. Lord Rayleigh (1916) [1] used the experimental results provided by Benard (1900) and gave the first stability analysis for the problem, he found a dimensionless

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number R called Rayleigh number which measures the ratio of buoyancy and viscosity forces multiplied by the ratio of momentum and thermal diffusivities given by

$$R = \frac{g\alpha\beta d^4}{\kappa\nu}$$

where α is coefficient of thermal expansion, g is gravity acceleration, β is temperature gradient, d is the vertical distance between the plates, κ is thermal diffusivity, ν is kinematic viscosity. This number increases as the temperature gradient β increased till exceed a certain critical value R_c, this value determines the stability of the fluid flow, when $R < R_c$ the flow is laminar, and it becomes turbulent at the high values $R > R_c$. Chandrasekhar [2]- [5] studied analytically instability of a layer of fluid heated from below and subject to simultaneous action of a magnetic field and rotation for three cases of boundaries free-free, rigid-rigid and rigid-free, he found the relation between the critical Rayleigh number and the critical wave number for various values of rotation and magnetic parameters. Wang [7] studied Linear instability analysis of Rayleigh Benard convection in a cylinder with traveling magnetic field, Yadav [8] gave a numerical investigation of the effect of magnetic field on the onset of nanofluid convection, A.Abasher [9] study the effect of rotation numerically by using Chebyshev tau method, Zimmermann [10] performed experimental and numerical Investigation of a Rayleigh-Bénard Convection Affected by Coriolis Force.
In this paper we study the effect of rotation and magnetic field numerically by using Chebyshev tau method [11]- [13]. The paper outlined as follows. In section 2 we introduced the perturbation equations and the boundary conditions, in section 3 we used normal mode analysis to analyze the system, in section 4 we obtained the non dimensional form of the equations, in section 5 we used the numerical method to convert the system to generalized eigen value problem, in section 7 we presented the graphical and numerical results and in the last section 7 we concluded the results.

2. The Perturbation Equations

Assume the fluid confined between two horizontal planes which are located at $z = 0$ and $z = d$, also assume the direction of the rotation and magnetic field coincides with the vertical. The perturbation equations and the boundary conditions were introduced by Chandrasekhar ([2], Page 199) as follows

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta + \beta w, \tag{2.1}$$

$$\frac{\partial h_z}{\partial t} = \eta \nabla^2 h_z + H \frac{\partial w}{\partial z}, \tag{2.2}$$

$$\frac{\partial \zeta}{\partial t} = \nu \nabla^2 \zeta + 2\Omega \frac{\partial w}{\partial z} + \frac{\mu H}{4\pi\rho} \frac{\partial \xi}{\partial z} \tag{2.3}$$

$$\frac{\partial \xi}{\partial t} = \eta \nabla^2 \xi + H \frac{\partial \zeta}{\partial z}, \tag{2.4}$$

$$\frac{\partial \nabla^2 w}{\partial t} = g\alpha\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + \nu\nabla^4 w - 2\Omega\frac{\partial \zeta}{\partial z} + \frac{\mu H}{4\pi\rho}\frac{\partial \nabla^2 h_z}{\partial z}, \tag{2.5}$$

subject to the boundary conditions

$$\begin{cases} \theta = 0, \quad w = 0, \quad \zeta = 0 \quad \text{and} \quad \dfrac{dw}{dz} = 0 \quad \text{on rigid boundary}, & (2.6a) \\ \theta = 0, \quad w = 0, \quad \dfrac{d\zeta}{dz} = 0 \quad \text{and} \quad \dfrac{d^2 w}{dz^2} = 0 \quad \text{on free boundary}. & (2.6b) \end{cases}$$

where θ, w, h_z, ζ and ξ are the perturbation of the velocity, magnetic field, vorticity and the current density respectively in the $z-$direction, ρ is the density, β is the temperature gradient where $\beta = \left|\frac{d\mathbf{T}}{dz}\right|$, g is the gravitational acceleration, α the coefficient of volume expansion, ν the kinematic viscosity, κ the thermal diffusivity.

3. Normal Modes Analysis

We can write the perturbations w, θ, ζ, h_z and ξ as a dependence on x, y, and t of the form

$$\begin{aligned} \theta &= \Theta(z)\exp\left(i\left(k_x x + k_y y\right) + \mathfrak{p}t\right) \\ \zeta &= Z(z)\exp\left(i\left(k_x x + k_y y\right) + \mathfrak{p}t\right) \\ w &= W(z)\exp\left(i\left(k_x x + k_y y\right) + \mathfrak{p}t\right) \\ \xi &= X(z)\exp\left(i\left(k_x x + k_y y\right) + \mathfrak{p}t\right) \\ h_z &= K(z)\exp\left(i\left(k_x x + k_y y\right) + \mathfrak{p}t\right) \end{aligned} \tag{3.1}$$

where $k = \sqrt{k_x^2 + k_y^2}$ is the wave number of the disturbance and $\mathfrak{p}$ is a constant. for functions with this dependence on x, y, and t, we find

$$\frac{\partial}{\partial t} = \mathfrak{p}, \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2 \quad \text{and} \quad \nabla^2 = \frac{d^2}{dz^2} - k^2 \tag{3.2}$$

From (3.1), the system (2.1)-(2.5) become

$$\mathfrak{p}\Theta = \beta w + \kappa\left(\frac{d^2}{dz^2} - k^2\right)\Theta, \tag{3.3}$$

$$\mathfrak{p}K = \eta\left(\frac{d^2}{dz^2} - k^2\right)K + H\frac{dW}{dz}, \tag{3.4}$$

$$\mathfrak{p}\left(\frac{d^2}{dz^2} - k^2\right)W = -g\alpha k^2\Theta + \nu\left(\frac{d^2}{dz^2} - k^2\right)^2 W - 2\Omega\frac{dZ}{dz} + \frac{\mu\mathbf{H}}{4\pi\rho}\frac{d}{dz}\left(\frac{d^2}{dz^2} - k^2\right)K, \tag{3.5}$$

$$\mathfrak{p}X = \eta\left(\frac{d^2}{dz^2} - k^2\right)X + H\frac{\mathrm{dZ}}{\mathrm{dz}}, \tag{3.6}$$

$$\mathfrak{p}Z = \nu\left(\frac{d^2}{dz^2} - k^2\right)Z + 2\Omega\frac{dW}{dz} + \frac{\mu\mathbf{H}}{4\pi\rho}\frac{dX}{dz}, \tag{3.7}$$

and the boundary conditions (2.6) become

$$\begin{aligned} &\Theta = 0, \quad W = 0 \quad \text{for} \quad z = 0 \text{ and } z = d, \\ &Z = 0, \quad \frac{\mathrm{dW}}{\partial z} = 0 \quad \text{on rigid surface} \\ &\frac{dZ}{dz} = 0 \quad \frac{d^2W}{dz^2} = 0 \text{ on free surface} \end{aligned} \tag{3.8}$$

4. Non-Dimensional Form of the Equations

To write the system (3.3)-(3.7) and the boundary conditions (3.8) in non-dimensional form, we have the distance between the two surfaces is d. If we define the following non-dimensional variables

$$a = kd, \quad \sigma = \frac{\mathrm{p}d^2}{v}, \quad z^* = \frac{z}{d}, \quad P_1 = \frac{v}{\kappa}, \text{ and } P_2 = \frac{v}{\eta},$$

the first and the second operators $\frac{d}{dz}$ and $\frac{d^2}{dz^2}$ are given as

$$\frac{d}{dz} = \frac{d}{dz^*}\frac{dz^*}{dz} = \frac{1}{d}\frac{d}{dz^*} = \frac{1}{d}D, \quad \text{and} \quad \frac{d^2}{dz^2} = \frac{1}{d^2}\frac{d^2}{dz^{*2}} = \frac{1}{d^2}D^2$$

Then the system (3.3)-(3.7) in non-dimensional form given as

$$\left(D^2 - a^2 - P_1\sigma\right)\Theta = -\left(\frac{\beta d^2}{\kappa}\right)W, \tag{4.1}$$

$$\left(D^2 - a^2 - P_2\sigma\right)K = -\left(\frac{Hd}{\eta}\right)DW, \tag{4.2}$$

$$\left(D^2 - a^2 - P_2\sigma\right)X = -\left(\frac{Hd}{\eta}\right)DZ, \tag{4.3}$$

$$\left(D - a^2 - \sigma\right)Z = -\left(\frac{2\Omega d}{v}\right)DW - \left(\frac{\mu Hd}{4\pi\rho\nu}\right)DX, \tag{4.4}$$

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \sigma\right)W + \left(\frac{\mu Hd}{4\pi\rho\nu}\right)D\left(D^2 - a^2\right)K - \left(\frac{2\Omega d^3}{\nu}\right)\mathrm{DZ} = \left(\frac{g\alpha d^2}{\nu}\right)a^2\Theta, \tag{4.5}$$

and the boundary conditions

$$\begin{aligned} &\Theta = 0, \quad W = 0 \quad \text{for } z = 0 \text{ and } z = 1, \\ &Z = 0, \quad DW = 0 \quad \text{on a rigid surface}, \\ &DZ = 0, \quad D^2W = 0 \quad \text{on a free surface.} \end{aligned} \tag{4.6}$$

at the stationary convection $(\sigma = 0)$, the system become

$$\left(D^2 - a^2\right)\Theta = -\left(\frac{\beta d^2}{\kappa}\right)W, \tag{4.7}$$

$$\left(D^2 - a^2\right)K = -\left(\frac{\mathrm{Hd}}{\eta}\right)DW, \tag{4.8}$$

$$\left(D^2 - a^2\right) X = -\left(\frac{Hd}{\eta}\right) DZ, \tag{4.9}$$

$$\left(D - a^2\right) Z = -\left(\frac{2\Omega d}{\nu}\right) DW - \left(\frac{\mu Hd}{4\pi\rho\nu}\right) DX, \tag{4.10}$$

$$\left(D^2 - a^2\right)^2 W + \left(\frac{\mu Hd}{4\pi\rho\nu}\right) D\left(D^2 - a^2\right) K - \left(\frac{2\Omega d^3}{\nu}\right) DZ = \left(\frac{g\alpha d^2}{\nu}\right) a^2\Theta. \tag{4.11}$$

Eliminate X between equations (4.9) and (4.10), we get

$$\left[\left(D^2 - a^2\right)^2 - QD^2\right] Z = -\left(\frac{2\Omega d}{\nu}\right) D\left(D^2 - a^2\right) W. \tag{4.12}$$

Also, eliminate K between (4.8) and (4.11) we obtain

$$\left[\left(D^2 - a^2\right)^2 - QD^2\right] W - \left(\frac{2\Omega d^3}{\nu}\right) DZ = \left(\frac{g\alpha d^2}{\nu}\right) a^2\Theta. \tag{4.13}$$

Eliminate Θ between (4.7) and (4.13), we get

$$\left(D^2 - a^2\right)\left\{\left[\left(D^2 - a^2\right)^2 - QD^2\right] W - d\sqrt{T} DZ\right\} = -Ra^2 W, \tag{4.14}$$

where,

$$R = \frac{g\alpha\beta d^4}{\kappa\nu} \quad \text{and} \quad T = \frac{4\Omega^2 d^4}{\nu^2}, \tag{4.15}$$

are the Rayleigh number and Taylor number respectively. (assume $d = 2$). Also (4.12) written as

$$\left[\left(D^2 - a^2\right)^2 - QD^2\right] Z + \sqrt{T} D\left(D^2 - a^2\right) W = 0. \tag{4.16}$$

The equations (4.14) and (4.16) must be solved subject to the boundary conditions (4.6).

5. Numerical Solution

Chebyshev Tau methods is a numerical method used to solve differential equations, It is based on Chebyshev polynomials to find approximate solution of the differential equation. Returning to (4.14) and (4.16) and the boundary conditions (4.6)

$$\left(D^2 - a^2\right)\left\{\left[\left(D^2 - a^2\right)^2 - QD^2\right] W - 2\sqrt{T} DZ\right\} = -Ra^2 W, \tag{5.1}$$

$$\left[\left(D^2 - a^2\right)^2 - QD^2\right] Z + \sqrt{T} D\left(D^2 - a^2\right) W = 0, \tag{5.2}$$

subject to

$$\begin{aligned} &W = 0, \quad \text{for} \quad z = 0 \text{ and } z = 1, \\ &Z = 0, \quad DW = 0 \quad \text{on a rigid surface} \\ &DZ = 0, \quad D^2 W = 0 \quad \text{on a free surface.} \end{aligned} \tag{5.3}$$

First convert the domain $[0,1]$ to domain of Chebyshev polynomials $[-1,1]$, use the relation $x = 2z-1$, when $z = 0 \Rightarrow x = -1$ and when $z = 1 \Rightarrow x = 1$, also the derivatives $D_z = 2D_x$ and $D_z^2 = 4D_x^2$, the system (5.1)-(5.3) become

$$\left(4D^2-a^2\right)\left\{\left[\left(4D^2-a^2\right)^2-4QD^2\right]W-4\sqrt{T}\mathrm{DZ}\right\}=-Ra^2W, \tag{5.4}$$

$$\left[\left(4D^2-a^2\right)^2-4QD^2\right]Z+\sqrt{T}D\left(4D^2-a^2\right)W=0, \tag{5.5}$$

$$\begin{gathered} W=0,\ for\ x=-1 \text{ and } x=1 \\ DZ=0,\ DW=0,\ (\text{on } a \text{ rigid surface}) \\ Z=0,\ \ D^2W=0,\ (\text{on } a \text{ free surface}) \end{gathered} \tag{5.6}$$

let $A=\left[\left(4D^2-a^2\right)^2-4QD^2\right]W-4\sqrt{T}DZ$ then we can write (5.4) - (5.6) as

$$\left[\left(4D^2-a^2\right)^2-4QD^2\right]W-4\sqrt{T}DZ-A=0, \tag{5.7}$$

$$\left(4D^2-a^2\right)A=-Ra^2W, \tag{5.8}$$

$$\left[\left(4D^2-a^2\right)^2-4QD^2\right]Z+\sqrt{T}D\left(4D^2-a^2\right)W=0 \tag{5.9}$$

subject to

$$\begin{gathered} W=A=0,\ for\ x=-1 \text{ and } x=1 \\ \mathrm{DW}=0,\ (\text{on } a \text{ rigid surface}) \\ D^2W=0,\ (\text{on } a \text{ free surface}) \end{gathered} \tag{5.10}$$

Now expand W, A and Z as Chebyshev polynomials

$$W(x)=\sum_{n=0}^{N}w_nT_n(x),\ A(x)=\sum_{n=0}^{N}a_nT_n(x) \text{ and } Z(x)=\sum_{n=0}^{N}z_nT_n(x) \tag{5.11}$$

where $-1 \le x \le 1$, the derivative D, D^2 and D^4 are given by

$$[]DW=\sum_{n=0}^{N}w_n^{(1)}T_n,\ \ D^2W=\sum_{n=0}^{N}w_n^{(2)}T_n,\ \ D^3W=\sum_{n=0}^{N}w_n^{(3)}T_n,\ \text{ and }\ D^4W=\sum_{n=0}^{N}w_n^{(4)}T_n, \tag{5.12}$$

$$D^2Z=\sum_{n=0}^{N}z_n^{(2)}T_n,\ \ D^4Z=\sum_{n=0}^{N}z_n^{(4)}T_n,\ \text{ and }\ D^2A=\sum_{n=0}^{N}a_n^{(2)}T_n, \tag{5.13}$$

where,
for all $n=0,..,N-1$
$w_n^{(1)},=\mathbf{D}\left[w_n\right]_{n=0,1...,N-1}^{T},\quad w_N^{(1)}=0,$

for all $n=0,..,N-2$

$w_n^{(2)} = \mathbf{D}^2 \left[w_n\right]^T_{n=0,1...,N-2}, \quad w^{(2)}_{N-1} = w^{(2)}_N = 0,$
$a_n^{(2)} = \mathbf{D}^2 \left[a_n\right]^T_{n=0,1...,N-2} \quad , a^{(2)}_{N-1} = a^{(2)}_N = 0,$
$Z_n^{(2)} = \mathbf{D}^2 \left[z_n\right]^T_{n=0,1...,N-2} \quad , z^{(2)}_{N-1} = z^{(2)}_N = 0$

for all $n = 0, .., N-3$
$w_n^{(3)} = \mathbf{D}^3 \left[w_n\right]^T_{n=0,1...,N-3}, \quad w^{(3)}_{N-2} = w^{(3)}_{N-1} = w^{(2)}_N = 0,$

for all $n = 0, .., N-4$
$w_n^{(4)} = \mathbf{D}^4 \left[w_n\right]^T_{n=0,1...,N-4}, \quad w^{(4)}_{N-3} = w^{(4)}_{N-2} = w^{(4)}_{N-1} = w^{(4)}_N = 0,$
$Z_n^{(4)} = \mathbf{D}^4 \left[z_n\right]^T_{n=0,1...,N-4} \quad , z^{(4)}_{N-3} = z^{(4)}_{N-2} = z^{(4)}_{N-1} = z^{(4)}_N = 0,$
Where $\mathbf{D}, \mathbf{D}^2, \mathbf{D^3}$ and $\mathbf{D^4}$ are chebyshev derivative matrices given as

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 4 & 0 & 0 & 0 & \cdots & 0 \\ 3 & 0 & 6 & 0 & 0 & \cdots & 0 \\ 0 & 8 & 0 & 8 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ N & 0 & 2N & 0 & 2N & \cdots & 0 \end{pmatrix}, \tag{5.14}$$

$$\mathbf{D}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 24 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 32 & 0 & 48 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 120 & 0 & 80 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & N(N^2-1) & 0 & N(N^2-9) & 0 & N(N^2-25) & \cdots & 0 \end{pmatrix}, \tag{5.15}$$

$\mathbf{D}^3 = (\mathbf{D})^3 = \mathbf{D}\mathbf{D}^2$ and $\mathbf{D}^4 = (\mathbf{D})^4 = \mathbf{D}^2\mathbf{D}^2$.
Substitute (5.11), (5.12) into the system (5.7)-(5.10), we get

$$\sum_0^N \left[16 w_n^{(4)} - \left(8a^2 + 4Q\right) w_n + a^4 w_n\right] T_n - \sum_0^N a_n T_n - 4\sqrt{T} \sum_0^N z_n^{(1)} T_n = 0, \tag{5.16}$$

$$\sum_0^N \left(4a_n^{(2)} - a^2 a_n\right) T_n = -Ra^2 \sum_0^N w_n T_n, \tag{5.17}$$

$$\sum_0^N \left[16 z_n^{(4)} - \left(8a^2 + 4Q\right) z_n + a^4 z_n\right] T_n + 4\sqrt{T} \sum_0^N \left(w_n^{(3)} - a^2 w_n^{(1)}\right) T_n = 0, \tag{5.18}$$

By taking the inner product with T_n for $n = 0, \ldots, N$ for (5.16)-(5.18) we get $3(N+1)$ equations for each $n = 0, 1, \ldots, N$ as follows

$$\left[16w_n^{(4)} - \left(8a^2 + 4Q\right)w_n + a^4 w_n\right] - a_n - 4\sqrt{T}z_n^{(1)} = 0 \tag{5.19}$$

$$4a_n^{(2)} - a^2 a_n = -Ra^2 w_n \tag{5.20}$$

$$16z_n^{(4)} - \left(8a^2 + 4Q\right)z_n + a^4 z_n + 4\sqrt{T}\left(w_n^{(3)} - a^2 w_n^{(1)}\right) = 0 \tag{5.21}$$

By substituting $w_n^{(1)}$, $w_n^{(2)}$, $a_n^{(2)}$, $z_n^2, w_n^{(4)}$ and $z_n^{(4)}$ we can write (5.19)-(5.21) as

$$\left[16\mathbf{D}^4 - \left(8a^2 + 4Q\right)\mathbf{D}^2 + a^4\mathbf{I}\right][w_n]^T - \mathbf{I}\,[a_n]^T - 4\sqrt{T}\mathbf{D}\,[z_n]^T = 0, \tag{5.22}$$

$$\left(4\mathbf{D}^2 - a^2\mathbf{I}\right)[a_n]^T = -Ra^2\mathbf{I}\,[w_n]^T, \tag{5.23}$$

$$\left[16\mathbf{D}^4 - \left(8a^2 + 4Q\right)\mathbf{D}^2 + a^4\mathbf{I}\right][z_n]^T + 4\sqrt{T}\left(\mathbf{D}^3 - a^2\mathbf{D}\right)[w_n]^T = 0 \tag{5.24}$$

These equations (5.22)-(5.24) represent a generalized eigenvalue problem in the form

$$\mathbf{A}X = R\mathbf{B}X, \tag{5.25}$$

where,

$$\mathbf{A} = \begin{pmatrix} \left[16\mathbf{D}^4 - \left(8a^2 + 4Q\right)\mathbf{D}^2 + a^4\mathbf{I}\right] & -\mathbf{I} & -4\sqrt{T}\mathbf{D} \\ \mathbf{0} & \left(4\mathbf{D}^2 - a^2\mathbf{I}\right) & \mathbf{0} \\ 4\sqrt{T}\left(\mathbf{D}^3 - a^2\mathbf{D}\right) & \mathbf{0} & \left[16\mathbf{D}^4 - \left(8a^2 + 4Q\right)\mathbf{D}^2 + a^4\mathbf{I}\right], \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -a^2\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

and,

$$\mathbf{X} = \left(w_0, \ldots, w_N, a_0, \ldots, a_N, z_0, \ldots, z_n\right)^T.$$

The boundary conditions (5.10), use the formula of calculating the $n-$derivatives of Chebyshev polynomial which is

$$\left.\frac{d^p T_n}{dx^p}\right|_{x=\pm 1} = (\pm 1)^{n+p} \prod_{k=0}^{p-1} \frac{n^2 - k^2}{2k+1}, \qquad T_n(\pm 1) = (\pm 1)^n \tag{5.26}$$

From this formula we can formulate the conditions of W as

$$W(\pm 1) = \sum_{n=0}^{N} w_n T_n(\pm 1) = \sum_{n=0}^{N} (\pm 1)^{n+1} w_n = \sum_{n=0}^{N} (\pm 1)^n w_n \tag{5.27}$$

$$\mathbf{D}W(\pm 1) = \sum_{n=0}^{N} w_n T_n'(\pm 1) = \sum_{n=0}^{N} (\pm 1)^n n^2 w_n \tag{5.28}$$

$$D^2W(\pm 1) = \sum_{n=0}^{N} w_n T_n''(\pm 1) = \sum_{n=0}^{N} (\pm 1)^{n+2} \left(\frac{n^2(n^2-1)}{3}\right) w_n \tag{5.29}$$

The conditions for A as

$$A(\pm 1) = \sum_{n=0}^{N} a_n T_n \ |_{x=\pm 1} = \sum_{n=0}^{N} (\pm 1)^{n+1} a_n = \sum_{n=0}^{N} (\pm 1)^{n} a_n \tag{5.30}$$

The conditions for Z,

$$Z(\pm 1) = \sum_{n=0}^{N} z_n T_n(\pm 1) = \sum_{n=0}^{N} (\pm 1)^{n+1} z_n = \sum_{n=0}^{N} (\pm 1)^{n} z_n \tag{5.31}$$

$$\mathbf{D}Z(\pm 1) = \sum_{n=0}^{N} z_n T_n'(\pm 1) = \sum_{n=0}^{N} (\pm 1)^{n} n^2 w_n \tag{5.32}$$

For Free-Free boundaries we have eight boundary conditions

$$W(\pm 1) = 0,\ D^2W(\pm 1) = 0,\ A(\pm 1) = 0 \text{ and } DZ(\pm 1) = 0$$

BC1 up to BC8 as follows

BC1 : $W(-1) = 0 \ \Rightarrow \left[1, -1, 1, -1, \ldots, (-1)^N\right] [w_n]^T = 0,$

BC2 : $W(1) = 0 \ \Rightarrow [1, 1, 1, 1, \ldots, 1] [w_n]^T = 0,$

BC3 :

$D^2W(-1) = 0 \Rightarrow \left[0,\ 0,\ 4,\ -24,\ 80, -200, \ldots, (-1)^{N+2} \frac{N^2(N^2-1)}{3}\right] [w_n]^T = 0,$

BC4 : $D^2W(1) = 0 \Rightarrow \left[0,\ 0,\ 4,\ 24,\ 80,\ 200, \ldots, \frac{N^2(N^2-1)}{3}\right] [w_n]^T = 0,$

$BC5$: $A(-1) = 0 \Rightarrow \left[1, -1, 1, \ldots, (-1)^N\right] [a_n]^T = 0,$

$BC6$: $A(1) = 0 \Rightarrow [1, 1, 1, \ldots, 1] [a_n]^T \ = 0,$

BC7 : $DZ(-1) = 0 \ \Rightarrow \left[0, 1,\ -4,\ 9, -16, 25, \ldots, (-1)^N N^2\right] [z_n]^T = 0,$

BC8 : $\mathrm{DZ}(1) = 0 \ \Rightarrow \left[0, 1,\ 4,\ 9, 16, 25, \ldots, N^2\right] [z_n]^T = 0,$

For Rigid-Rigid boundaries we have eight boundary conditions

$$W(\pm 1) = 0,\ DW(\pm 1) = 0,\ A(\pm 1) = 0 \textit{ and } Z(\pm 1) = 0$$

BC1 up to BC8 as follows

BC1 : $W(-1) = 0 \Rightarrow \left[1, -1, 1, -1, \ldots, (-1)^N\right] [w_n]^T = 0$

BC2 : $W(1) = 0 \quad \Rightarrow [1, 1, 1, 1, \ldots, 1] [w_n]^T = 0$

BC3 : $\mathrm{DW}(-1) = 0 \Rightarrow \left[0, 1,\ -4,\ 9, -16, 25, \ldots, (-1)^N N^2\right] [w_n]^T = 0$

BC4 : $\mathrm{DW}(1) = 0 \ \Rightarrow \left[0, 1,\ -4,\ 9, -16, 25, \ldots, (-1)^N N^2\right] [w_n]^T = 0$

$BC5$: $A(-1) = 0 \Rightarrow \left[1, -1, 1, \ldots, (-1)^N\right] [a_n]^T \ = 0$

BC6 : $A(1) = 0 \Rightarrow [1, 1, 1, \ldots, 1] [a_n]^T \ = 0$

BC7 : $Z(-1) = 0 \ \Rightarrow \left[1, -11, 1, \ldots, (-1)^N\right] [z_n]^T = 0$

BC8 : $Z(1) = 0 \ \Rightarrow [1, 1, 1, \ldots, 1] [z_n]^T = 0$

For Rigid-Free boundaries we have eight boundary conditions, the lower boundary is rigid and the upper boundary is free, so the conditions are

$$W(\pm 1) = 0,\ A(\pm 1) = 0,$$

$$DW(-1) = 0 \ \text{ and } \ Z(-1) = 0 \ (\text{Rigid at } x = -1)$$

$$D^2W(1) = 0 \ \text{ and } \ DZ(1) = 0 \ (\text{Free at } x = 1)$$

then BC1 up to BC8 written as

BC1 : $W(-1) = 0 \Rightarrow \left[1, -1, 1, -1, \ldots, (-1)^N\right][w_n]^T = 0,$

BC2 : $W(1) = 0 \quad \Rightarrow [1, 1, 1, 1, \ldots, 1]\,[w_n]^T = 0,$

BC3 : $\mathrm{DW}(-1) = 0 \Rightarrow \left[0, 1, \ -4, \ 9, -16, 25, \ldots, (-1)^N N^2\right][w_n]^T = 0,$

BC4 : $D^2W(1) = 0 \ \Rightarrow \left[0, \ 0, \ 4, \ 24, \ 80, \ 200, \ldots, \frac{N^2\left(N^2-1\right)}{3}\right][w_n]^T = 0,$

$BC5$: $A(-1) = \left[1, -1, 1, \ldots, (-1)^N\right][a_n]^T = 0,$

$BC6$: $A(1) = [1, 1, 1, \ldots, 1]\,[a_n]^T = 0,$

BC7 : $Z(-1) = 0 \ \Rightarrow \left[1, -11, 1, \ldots, (-1)^N\right][z_n]^T = 0,$

BC8 : $\mathrm{DZ}(1) = 0 \ \Rightarrow \left[0, 1, \ 4, \ 9, 16, 25, \ldots, N^2\right][z_n]^T = 0.$

For each case of the boundary conditions free-free , rigid-rigid and rigid-free, insert the boundary conditions BC1 up to BC4 into the matrix **A** in the system (5.25), the corresponding rows in the matrix **B** are zeros. The matrices **A** and **B** can written as

$$\mathbf{A} = \begin{pmatrix} \left[16\mathbf{D^4} - \left(8a^2 + 4Q\right)\mathbf{D^2} + a^4\mathbf{I}\right] & -\mathbf{I} & -4\sqrt{T}\mathbf{D} \\ \text{BC1} & 0\ldots0 & 0\ldots0 \\ \text{BC2} & 0\ldots0 & 0\ldots0 \\ \text{BC3} & 0\ldots0 & 0\ldots0 \\ \text{BC4} & 0\ldots0 & 0\ldots0 \\ \mathbf{0} & 4\mathbf{D}^2 - a^2\mathbf{I} & \mathbf{0} \\ 0\ldots0 & \text{BC5} & 0\ldots0 \\ 0\ldots0 & \text{BC6} & 0\ldots0 \\ 4\sqrt{T}\left(\mathbf{D^3} - a^2\mathbf{D}\right) & \mathbf{0} & \left[16\mathbf{D^4} - \left(8a^2 + 4Q\right)\mathbf{D^2} + a^4\mathbf{I}\right] \\ 0\ldots0 & 0\ldots0 & \text{BC7} \\ 0\ldots0 & 0\ldots0 & \text{BC8} \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0\ldots0 & 0\ldots0 & 0\ldots0 \\ 0\ldots0 & 0\ldots0 & 0\ldots0 \\ 0\ldots0 & 0\ldots0 & 0\ldots0 \\ 0\ldots0 & 0\ldots0 & 0\ldots0 \\ -a^2\mathbf{I} & \mathbf{0} & \mathbf{0} \\ 0\ldots0 & 0\ldots0 & 0\ldots0 \\ 0\ldots0 & 0\ldots0 & 0\ldots0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0\ldots0 & 0\ldots0 & 0\ldots0 \\ 0\ldots0 & 0\ldots0 & 0\ldots0 \end{pmatrix}$$

By using MATLAB, we can calculate the set of all eigenvalues R for each value of the wave number a for the system (5.25), and we can find the minimum eigenvalue R_c and the corresponding value a_c, The table (1) shows the critical Rayleigh number and the critical wave number in the absence of rotation and magnetic field ($T = 0$ and $Q = 0$) for the three cases of the boundaries free-free, rigid-rigid and rigid-free.

Free-Free		Rigid-Rigid		Rigid-Free	
a_c	R_c	a_c	R_c	a_c	R_c
2.2	657	3.17	1707	2.68	1100

Table 1. The critical Rayleigh number R_c and critical wave number a_c, for the three cases of the boundaries in the absence of rotation and magnetic field $T = 0, Q = 0$

6. MATLAB Code Implementation

Listing 1. Formulation of Chebyshev Matrix D.

```
function D=ChebD(N)
D=zeros(N+1);
for i=1:N+1
    for j=1:1:N+1
        if mod(j,2)==0
            D(1,j)=j-1;
        end
        if i+2*j-1<=(N+1) D(i,i+2*j-1)=2*(i+2*j-1)-2;
        end
    end
end
```

Listing 2. Formulation of Free-Free Boundary Conditions.

```
function [BC1 BC2 BC3 BC4 BC5 BC6 BC7 BC8]=Bcs_FF(N)
for m=1:N+1
    n=m-1;
    BC1(m)=(-1)^n;          BC2(m)=1;
    BC3(m)=4*(-1)^(n+2)*n^2*(n^2-1)/3;    BC4(m)=4*n^2*(n^2-1)/3;
    BC5(m)=(-1)^n;          BC6(m)=1;
    BC7(m)=(-1)^(n+1)*n^2;  BC8(m)=n^2;
end
```

Listing 3. Formulation of Rigid-Rigid Boundary Conditions.

```
function [BC1 BC2 BC3 BC4 BC5 BC6 BC7 BC8]=Bcs_RR(N)
for m=1:N+1
    n=m-1;
    BC1(m)=(-1)^n; BC2(m)=1;
    BC3(m)=(-1)^(n+1)*n^2; BC4(m)=n^2;
    BC5(m)=(-1)^n; BC6(m)=1;
    BC7(m)=(-1)^n; BC8(m)=1;
end
```

Listing 4. Formulation of Rigid-Free Boundary Conditions.

```
function [BC1 BC2 BC3 BC4 BC5 BC6 BC7 BC8]=Bcs_RF(N)
for m=1:N+1
    n=m-1;
    BC1(m)=(-1)^n; BC2(m)=1;
    BC3(m)=(-1)^(n+1)*n^2; BC4(m)=4*n^2*(n^2-1)/3;
    BC5(m)=(-1)^n; BC6(m)=1;
    BC7(m)=(-1)^n; BC8(m)=n^2;
end
```

Listing 5. Rayleigh-Bénard Convection (Effect of Rotation and Magnetic Field).

```
% Benard problem with effect of rotation and magnetic field
% for Free-Free Boundaries, Rigid-Rigid boundaries and Rigid-Free
% boundaries
clear all
N=30;
%formulation boundary conditions
fprintf(['Enter the type of the boundaries: 1 for free-free,
2 for Rigid-Rigid, 3 for Rigid-Free**\n']);
Type_of_boundaries=input('the type of the boundaries =\n ');
switch Type_of_boundaries
 case 1
 [BC1 BC2 BC3 BC4 BC5 BC6 BC7 BC8] = Bcs_FF(N);
 case 2
 [BC1 BC2 BC3 BC4 BC5 BC6 BC7 BC8] = Bcs_RR(N);
 case 3
```

```
 [BC1 BC2 BC3 BC4 BC5 BC6 BC7 BC8] = Bcs_RF(N);
 otherwise
 fprintf('Wrong selection, select 1,2 or 3\n');
end
%T1_values=[10,100,200,500,1000];
T1_values=700; % Taylor number
for T1=T1_values
    Q1=[200 300 400 500 600 700];
Q=Q1*pi^2; T=T1*pi^4; a=0.1:0.01:20;
format long
D=ChebD(N);D2=D*D; D3=D2*D; D4=D2*D2;  % formulation of D,D2,D4
I=eye(N+1); O=zeros(N+1);
for kk=1:length(T)
    fprintf('for T=%d \n',T1(kk));
    for jj=1:length(Q)
        for ii=1:length(a)
            A=[(4*D2-a(ii)^2*I)^2-4*Q(jj)*D2 -I -4*sqrt(T(kk))*D;...
                O (4*D2-a(ii)^2*I) O;...
                4*sqrt(T(kk))*D3-sqrt(T(kk))*a(ii)^2*D
                O (4*D2-a(ii)^2*I)^2-4*Q(jj)*D2];
            A(N+1,1:N+1)=BC4; A(N,1:N+1)=BC3;
            A(N-1,1:N+1)=BC2; A(N-2,1:N+1)=BC1;
            A(2*N+1,N+2:2*N+2)=BC5; A(2*N+2,N+2:2*N+2)=BC6;
            A(3*N+2,2*N+3:3*N+3)=BC7; A(3*N+3,2*N+3:3*N+3)=BC8;
            B=[O O O;-a(ii)^2*I O O;O O O];
            lambdas=eig(A,B); R(ii)=min(lambdas);
        end
        % find eigenvalue lambda
        [Rc acloc]=min(R); ac=a(acloc);
        fprintf('Q1=%d \t\t\t ac=%2.2f \t\t\t Rc=%5.2f\n',Q1(jj),ac,Rc);
        hold on
        myPlot=plot(a,R);%plot (ac,Rc, 'b.', 'MarkerSize', 7);
    end
end
title('Free-Free, T1=700'); xlabel('wave number a');
ylabel('Rayliegh number R');
```

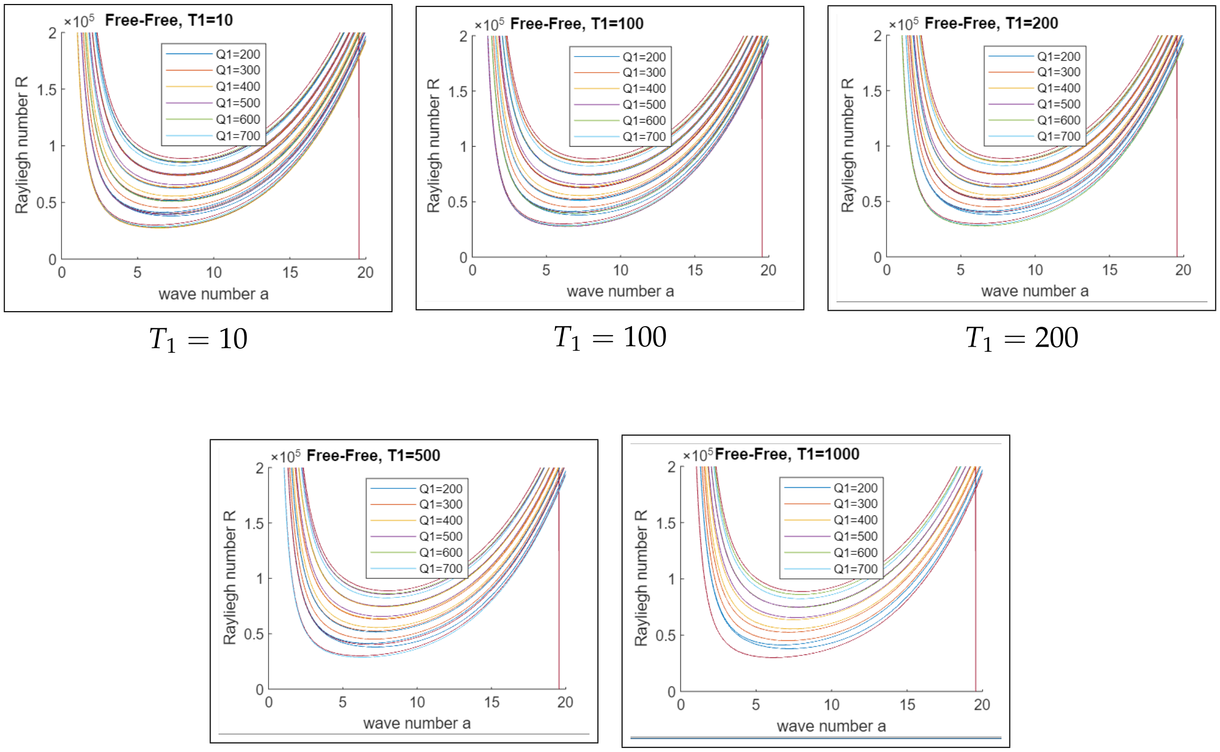
```
xlim([0 20]); ylim([0 200000]);
hold off
end
lgd=legend('Q1=200', 'Q1=300', 'Q1=400', 'Q1=500', 'Q1=600', 'Q1=700',
'Location', 'north');
lgd.FontSize = 8;
```

7. Numerical Results

The following table and graphs show the critical Rayleigh number R_c and the critical wave number a_c for free-free boundaries for various values of $Q_1 = \frac{Q}{\pi^2}$ and $T_1 = \frac{T}{\pi^4}$.

Free-Free Boundaries										
Q_1	$T_1 = 10$		$T_1 = 100$		$T_1 = 200$		$T_1 = 500$		$T_1 = 1000$	
	a_c	R_c	a_c	R_c	a_c	R_c	a_c	R_c	a_c	R_c
200	6.43	27409	6.41	27657	6.38	27931	6.29	28749	6.15	30090
300	6.93	39289	6.91	39474	6.89	39679	6.82	40291	6.72	41298
400	7.29	50924	7.28	51075	7.26	51241	7.22	51740	7.14	52562
500	7.59	62396	7.58	62524	7.57	62667	7.53	63092	7.46	63795
600	7.84	73748	7.83	73861	7.82	73986	7.79	74360	7.74	74979
700	8.06	85007	8.05	85108	8.04	85220	8.01	85556	7.97	86111

Table 2. Critical Rayleigh number R_c and critical wave number a_c for various values of $T_1 = 10, 100, 200, 500, 1000$ and $Q_1 = 200, 300, 400, 500, 600, 700$ when both boundaries are free.



$T_1 = 10$ $T_1 = 100$ $T_1 = 200$

$T_1 = 500$ $T_1 = 1000$

FIGURE 1. The variation of Rayleigh number R with wave number a for different values $T_1 = 10, 100, 200, 500, 1000$ and $Q_1 = 200, 300, 400, 500, 600, 700$ when both boundaries are free

The following table and graphs show the critical Rayleigh number R_c and the critical wave number a_c for rigid-rigid boundaries for various values of $Q_1 = \frac{Q}{\pi^2}$ and $T_1 = \frac{T}{\pi^4}$.

Rigid-Rigid Boundaries										
Q_1	$T_1 = 10$		$T_1 = 100$		$T_1 = 200$		$T_1 = 500$		$T_1 = 1000$	
	a_c	R_c	a_c	R_c	a_c	R_c	a_c	R_c	a_c	R_c
200	6.54	29838	6.56	30251	6.58	30749	6.69	32593	7.09	37994
300	7.02	42114	7.02	42353	7.03	42626	7.05	43496	7.10	45155
400	7.38	54087	7.38	54217	7.38	54362	7.38	54804	7.39	55562
500	7.67	65859	7.66	65845	7.66	65828	7.65	65769	7.64	65650
600	7.91	77491	7.90	77226	7.89	76933	7.86	76070	7.82	74688
700	8.11	89033	8.09	88343	8.07	87595	8.02	85476	7.94	82367

TABLE 3. Critical Rayleigh number R_c and critical wave number a_c for various values of $T_1 = 10, 100, 200, 500, 1000$ and $Q_1 = 200, 300, 400, 500, 600, 700$ when both boundaries are rigid.

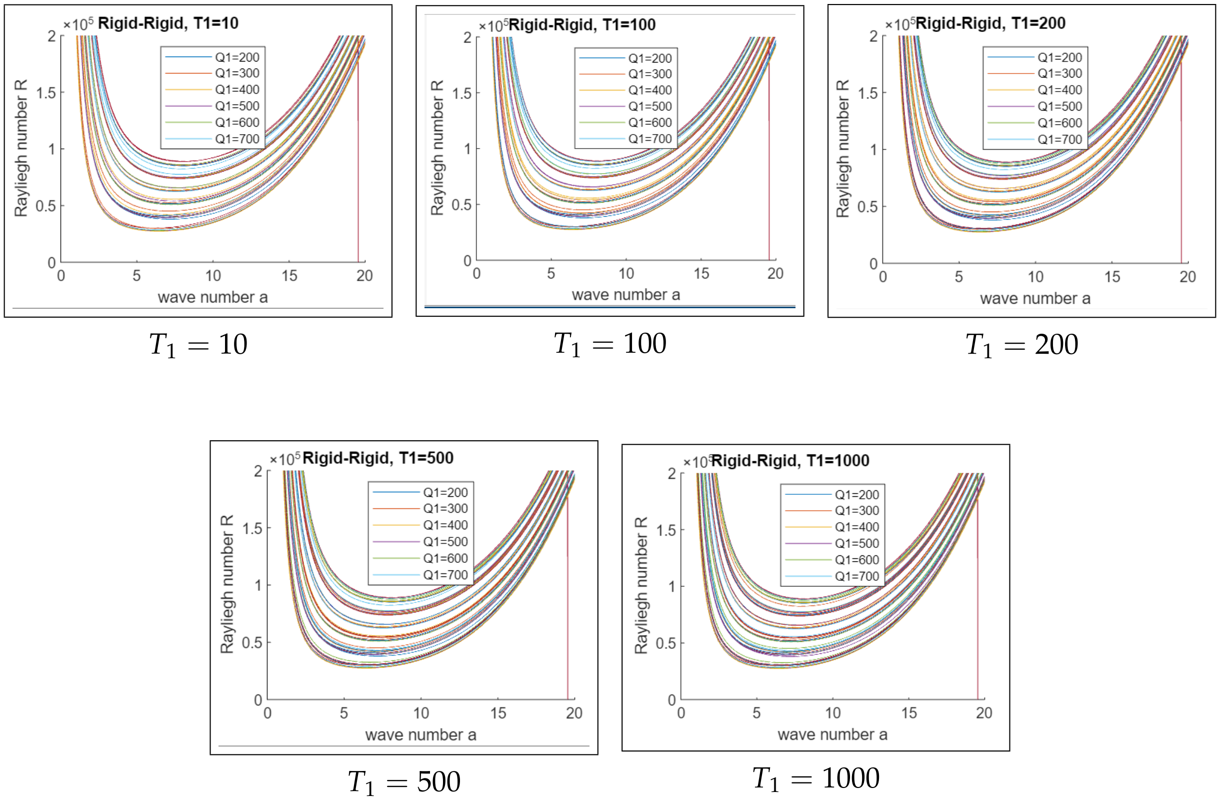
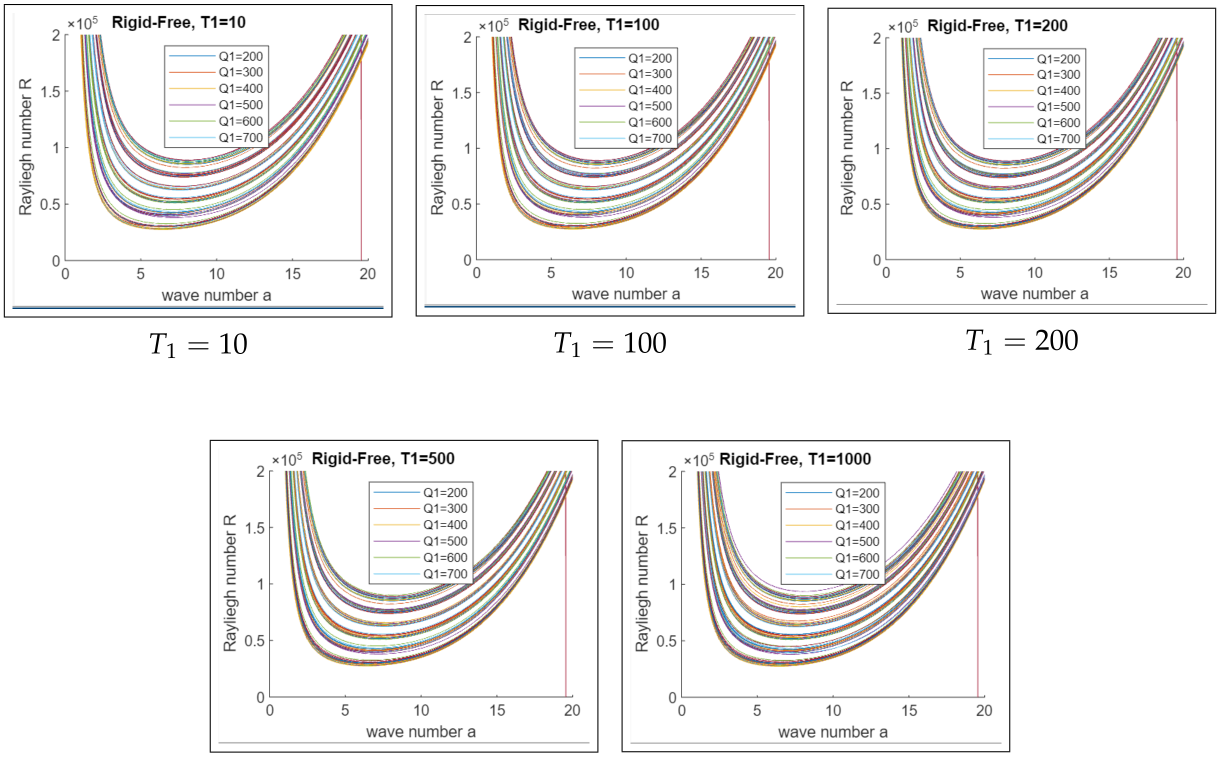


Figure 2. The variation of Rayleigh number R with wave number a for different values $T_1 = 10, 100, 200, 500, 1000$ and $Q_1 = 200, 300, 400, 500, 600, 700$ when both boundaries are rigid

The following table and graphs show the critical Rayleigh number R_c and the critical wave number a_c for rigid-free boundaries for various values of $Q_1 = \frac{Q}{\pi^2}$ and $T_1 = \frac{T}{\pi^4}$.

Rigid-Free Boundaries										
Q_1	$T_1 = 10$		$T_1 = 100$		$T_1 = 200$		$T_1 = 500$		$T_1 = 1000$	
	a_c	R_c	a_c	R_c	a_c	R_c	a_c	R_c	a_c	R_c
200	6.49	28600	6.49	29043	6.49	29592	6.59	31762	7.50	39117
300	6.97	40679	6.97	40995	6.96	41360	6.96	42546	7.00	44929
400	7.34	52484	7.33	52757	7.33	53068	7.32	54034	7.31	55783
500	7.63	64105	7.63	64398	7.62	64727	7.61	65742	7.60	67527
600	7.88	75588	7.88	75971	7.87	76402	7.86	77729	7.85	80066
700	8.10	86953	8.10	87514	8.10	88149	8.09	90122	8.08	93668

TABLE 4. Critical Rayleigh number R_c and critical wave number a_c for various values of $T_1 = 10, 100, 200, 500, 1000$ and $Q_1 = 200, 300, 400, 500, 600, 700$ when the lower boundary is rigid and the upper is free.



$T_1 = 10$ $T_1 = 100$ $T_1 = 200$

$T_1 = 500$ $T_1 = 1000$

FIGURE 3. The variation of Rayleigh number R with wave number a for different values $T_1 = 10, 100, 200, 500, 1000$ and $Q_1 = 200, 300, 400, 500, 600, 700$ when the lower boundary is rigid and the upper is free

8. Conclusion

The numerical results for the three cases of the boundary conditions are in full agreement with the results obtained by Chandrasekhar [2], we can conclude the following

- As the temperature gradient increases, the Rayleigh number R also increases. The convection motion begins at the critical Rayleigh number R_c, which represents the minimum value of R. For $R < R_c$, the flow is stable, while it becomes unstable for $R > R_c$.
- From the tables 2, 3 and 4, it is observed that for the three cases of the boundaries (free-free, rigid-rigid and rigid-free), an increase in the Taylor number T is associated with an increase in the critical Rayleigh number R_c. Additionally, an increase in the magnetic parameter Q is also linked to an increase in the critical Rayleigh number R_c. This implies that both rotation and the presence of a magnetic field tend to suppress convective motion due to the Coriolis force resulting from rotation and the Lorentz force resulting from the magnetic field. These forces act to reduce the vertical motion of the fluid.
- From the tables 2, 3, and 4, it can be observed that the critical Rayleigh number varies depending on the type of boundary conditions. For example, if we consider the values $(T = 10, Q = 200)$, we find $R_c = 27409$ for the free-free boundary case, $R_c = 28600$ for the rigid-free boundary case, and $R_c = 29838$ for the rigid-rigid boundary case. This difference in critical Rayleigh numbers can be attributed to the constraints imposed by the boundary conditions. In the case of free boundaries, there are no constraints on vertical motion, allowing for faster convection compared to the cases with rigid boundaries. On the other hand, the presence of constraints in the rigid boundary cases limits the convective motion, resulting in a higher critical Rayleigh number.

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