Int. J. Anal. Appl. (2023), 21:139

International Journal of Analysis and Applications

The Continuous Wavelet Transform in Sobolev Spaces Over Locally Compact Abelian Group and Its Approximation Properties

C. P. Pandey[1,*], M. M. Dixit[1], Mopi Ado[1], Sunil Kumar Singh[2]

[1] *Department of Mathematics, North Eastern Regional Institute of Science and Technology, Nirjuli, Itanagar, Arunachal Pradesh, 791109, India*

[2] *Department of Mathematics, Babasaheb Bhimrao Ambedkar University, Lucknow, 226025, India*

[*] *Corresponding author:* drcppandey@gemail.com

Abstract. The Sobolev space over locally compact abelian group $H^s(G)$ is defined and we extend the continuous wavelet transform to Sobolev space $H^s(G)$ for arbitrary real s. This generalisation of the wavelet transform naturally leads to a unitary operator between these spaces. Further, the asymptotic behaviour of the transforms of the L_2 function for small scaling parameters is examined. In special cases, the wavelet transform converges to a generalized derivative of its argument. We also discuss the consequences for the discrete wavelet transform arising from this property. Numerical examples illustrate the main result.

1. Introduction

Most of the space that we are interested in end up bring topological groups. In this section we define the terms topology and group so that we can work with them. We introduce the Haar measure, which is a translation-invariant measure.

A set S becomes a group [1] if an operator, say $+$, can be defined such that

- $x + (y + z) = (x + y) + z, \quad \forall \quad x, y, z \in S$
- There exist an element 0, such that $x + 0 = 0 + x = x \quad \forall \quad x \in S$
- For each $x \in S$ there $\exists$ an inverse element $x^{-1} = x$ such that $x+(-x) = (-x)+x = 0$.

In addition, S is a commutative group if it is also true that $x + y = y + x \quad \forall \quad x, y \in S$. Given a set S, a topology T is a set of subsets on S that

Received: Oct. 26, 2023.

2010 *Mathematics Subject Classification.* 65T60, 43A25, 11F85.

Key words and phrases. Sobolev space; wavelet; Fourier transform; Schwartz space.

https://doi.org/10.28924/2291-8639-21-2023-139

ISSN: 2291-8639

- contains S and the empty set ϕ.
- Is closed under finite intersections and finite unions of subsets X.

S is a topological group if it is a group operation and a topology such that the maps $\alpha : G \times G \to G$ are continuous, where $\alpha(x, y) = x + y$ and $\beta(x) = x^{-1}$.
If S is locally compact, that is, every point in S is contained in a compact neighbourhood, and its group operation is commutative, then we call it a Locally Compact Abelian (LCA) Group.
Given a topological space X, we define the Borel set as a set of subset of X such that:

- contains all subsets of the topology on X.
- Is closed under complements, countable unions and countable intersections of subsets.
- Is the smallest set of subsets that meets these condition.

A measure μ on X is a function on the Borel sets where

- $\mu(E) = \sum \mu(E_i)$ if $E \subset X$ and $E = \bigcup_{i\in 1} E_i$ where E_i is a countable pairwise disjoint set.
- $\mu(E)$ is finite for all $E \subset X$ where the closure of E is compact.

A measure μ is regular if for all Borel sets E, we have

$$\mu(E) = inf_{k \supset E}\mu(K) = sup_{k \subset E}\mu(K).$$

μ is invariant if $\mu(x + E) = \mu(E) \quad \forall x \in X$.
Let $M(X)$ be the space of all complex-valued regular measures on X where $\|\mu\| = |\mu(S)|$ is finite.
A haar measure [10] is a measure that is non-valued, regular, and invariant. In fact, Haar measures are unique up to a scalar, so we can call them Haar measure. That is, if $m - 1$ and m_2 are both non-negative, regular, translation-invariant measures on S, then there exists $\lambda \geq 0$ such that $m_1 = m_2$. The corresponding integral is called the Haar integral, which is translation invariant. That is, integrals over a set E and $x + E$ are equivalent.
Given a LCA group [9] G, we define an $L^p(G)$ space to be the space of all complex valued functions f on G such that the integral $\int |f|^p d\mu$ exists with respect to the Haar measure. $L^p(G)$ becomes an algebra under convolution, which is an important characteristic later on.

Definition 1.1. *A complex function γ on a LCA group [2] G is called a character of G if $|\gamma(x)| = 1$ for all $x \in G$ and if the function equation $\gamma(x + y) = \gamma(x)\gamma(y)$ for all $(x, y) \in G$ is satisfied. The set of all continuous characters of G form a group Γ, the dual group of G. Now it is customary to write $(x, \gamma) = \gamma(x)$ satisfy the following properties:([7], [2])*

- $(0, \gamma) = (x, 0) = 1$
- $(-x, \gamma) = (x, -\gamma) = (x, \gamma)^{-1} = \overline{(x, \gamma)}$
- $(x + y, \gamma) = (x, \gamma)(y, \gamma)$
- $(x, \gamma_1 + \gamma_2) = (x, \gamma_1)(x, \gamma_2)$

Definition 1.2. *The Fourier transform of $f \in L^1(G)$ is denoted by $\widehat{f}(\gamma)$ defined by [1]*

$$\widehat{f}(\gamma) = \int_G f(x)(-x,\gamma)dx, \tag{1.1}$$

and the inverse Fourier transform is defined by

$$f(x) = \int_G \widehat{f}(\gamma)(x,\gamma)d\gamma, x \in G. \tag{1.2}$$

Some important properties of the Fourier transform can be proved easily: [6]

(i) $\|\widehat{f}\|_{L^\infty(G)} \leq \|f\|_{L^1(G)}$.

(ii) If $f \in L^1(G)$, then $\widehat{f}$ is uniformly continuous.

(iii) Parseval formula: If $f \in L^1(G) \bigcap L^2(G)$, then $\|\widehat{f}\|_{L^2(G)} = \|f\|_{L^2(G)}$.

(iv) If the convolution of f and g is defined as

$$(f * g)(x) = \int_G f(x-y)g(y)dy, \tag{1.3}$$

then

$$(\widehat{f * g})(\gamma) = \widehat{f}(\gamma)\widehat{\gamma}. \tag{1.4}$$

Definition 1.3. *For $k, 0 \leq k < q, k = a_0 + a_1 p + ... + a_{c-1}p^{c-1}, 0 \leq a_i < p, i = 0, 1, 2, 3...c-1$, we define*

$$v(k) = (a_0 + a_1\epsilon_1 + ... + a_{c-1}\epsilon_{c-1})p^{-1}(0 \leq k < q)$$

for $k = b_0 + b_1 q + +... + a_{c-1}q^s, 0 \leq b_i < q, k \geq 0$,we get

$$v(k) = v(b_0) + p^{-1}v(b_1) + ... + p^{-s}v(b_s)$$

Note that for $k, l \geq 0, v(k+l) \neq v(k)+v(l)$. However, it is true that for all $r, s \geq 0, v(rq^s) = p^{-1}v(r)$, and for $r, s \geq 0, 0 \leq t < q^s$,
$v(rq^s + t) = v(rq^s) + v(t) = p^{-1}v(r) + v(t)$.

We denote $\mathcal{X}_{v(n)}$ by $\mathcal{X}_n (n \geq 0)$ and use the notation $N_0 = 0, 1, 2, 3, ...$ throughout this paper.

Distribution over LCA Group

We denote $\mathscr{S}(G)$ the space of all finite linear combinations of characteristic functions of the ball of G. The Fourier transform is a homeomorphism of $\mathscr{S}(G)$ onto $\mathscr{S}(G)$. The distribution space of $\mathscr{S}(G)$ is denoted by $\mathscr{S}'(G)$.

The Fourier transform of $g \in \mathscr{S}(G)$ is denoted by $\widehat{f}(\omega)$ and is defined by

$$\widehat{f} = \int_G f(x)(-x,\omega)dx, \omega \in G$$

and the inverse Fourier transform is defined by

$$f(x) = \int_G \widehat{f}(\omega)(x,\omega)d\omega, x \in G.$$

The Fourier transform and inverse Fourier transform of a distribution $f \in \mathscr{S}'(G)$ is defined by

$$\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle, \langle f^v, \varphi \rangle = \langle f, \varphi^v \rangle, for \quad all \quad \varphi \in \mathscr{S}(G).$$

Definition 1.4. Sobolev space over LCA groups

Let $s \in G$, Sobolev space over LCA group denoted by $H^s_\gamma(G)$, and defined by the space of all $f \in \mathscr{S}(G)$ such that

$$\int_{G^\wedge} \left(1+\gamma(\xi)^2\right)^s |\widehat{f}(\xi)|^2 d\xi \quad is \quad finite$$

where $f \in L^2(G), \xi \in \Gamma$. We denote Γ by the set

$$\Gamma = \{\gamma : G^\wedge \to [0,\infty) : \exists_{c_\gamma} \forall_\alpha, \beta \in G^\wedge \gamma(\alpha\beta) \leq c_\gamma[\gamma(\alpha)+\gamma(\beta)]\}$$

Moreover, for $f \in H^s_\gamma(G)$; its norm $\|f\|_{H^s_\gamma(G)}$ is defined as follows:

$$\|f\|^2_{H^s_\gamma(G)} = \int_{G^\wedge} (1+\gamma(\xi)^2)^s |\widehat{f}(\xi)|^2 d\xi$$

2. The Continuous Wavelet Transform

The wavelet transform is a tool for analyzing and synthesizing signals, with many applications in geophysics and acoustics [4]. It has a lot of advantages compared to the Fourier transform, e.g., the high-frequency components are studied with sharper time resolution than the low-frequency components.

The transformed signal is composed of its inner product with shifted and scaled versions of a fixed function called analyzing, or a basic wavelet.

Let $f \in L_2(G)$ be the signal and $\psi \in L_2(G)$ the analyzing wavelet. The mapping

$$f(\cdot) \longmapsto |a|^{-1/2} \left\langle f, \psi\left(\frac{\cdot - b}{a}\right)\right\rangle_0, \quad b \in G \quad a \in G \backslash \{0\}, \tag{2.1}$$

describes the analysis of f (up to constant factor), where $< \cdot, \cdot >_0$ denotes the inner product in $L_2(G)$. With an admissibility condition on ψ the right-hand side of (2.1) is an element in $L_2\left((G \times G \setminus \{0\}), \frac{dbda}{a^2}\right)$ and it is possible to synthesize f by these moments.

In the literature, one often finds the definition of the wavelet transform via an irreducible unitary representation of the group of affine-linear transformations of the real axis ('$ax+b$'-group). Hence, its essential properties are abstractly proved with the help of group theory (orthogonality relations). In the next section, some known results will be verified without group-theoretical arguments in such a way that the extension of the wavelet transform to Sobolev spaces becomes obvious. It will be seen that the signal and the wavelet transform share the same Sobolev order. The preponderant part of the paper deals with the asymptotic behaviour of (2.1) for small a. Without a heuristic frequency analysis, our inquiry explains the basics of the widespread use of wavelet techniques in edge detection and pattern recognition. It turns out that the right-hand side of (2.1) converges to a derivative of f, as already observed for a very special example in [5], for a great class of basic wavelets ψ.

We define it with the help of the shift operator as follows:

$$(T^b g)(x) = g(x-b), \quad b \in G \tag{2.2}$$

and the dilation-operator

$$(D^a g)(x) = |a|^{1/2} g\left(\frac{x}{a}\right), \quad a \in G_0 := G\backslash\{0\} \tag{2.3}$$

a unitary transform $U(b,a) : L_2(G, dt) \to L_2(G, dt)$, where dt denotes the Lebesgue-measure, by

$$(U(b,a)g)(x) = (T^b D^a g)(x) = |a|^{1/2} g\left(\frac{x-b}{a}\right), \quad (b,a) \in G \times G_0 \tag{2.4}$$

To simplify further calculations, we introduce the Fourier transform on $L^{'}(G)$

$$(Ff)(\omega) = \widehat{f}(\omega) = \int_G f(x)(-x,\omega)dx, \quad \omega \in G, \tag{2.5}$$

leading to

$$FT^b = (-b,\omega)F \tag{2.6}$$

$$FD^a = D^{1/a}F. \tag{2.7}$$

Hence we get

$$(U(b,a)g)^\wedge(\omega) = F(T^a D^b g)(\omega) = (-b,\omega)|a|^{1/2}\widehat{g}(a\omega) \tag{2.8}$$

In the sequel we describe the wavelet transform, based on function ψ.

Definition 2.1. *A function $\psi \in L_2(G, dt)$ is admissible if and only if ψ is not identical to zero and $\langle U(\cdot,\cdot)\psi, \psi\rangle_0$ lies in $L_2\left(G \times G_0, \frac{dbda}{a^2}\right)$,*

$$\int_{G_0}\int_G |\langle U(b,a)\psi,\psi\rangle_0|^2 \frac{dbda}{a^2} < \infty. \tag{2.9}$$

With '*' denoting the convolution [8], we reformulate the admissibility condition (2.9) as

$$\begin{aligned}
\int_{G_0}\int_G |\langle T^b D^a \psi,\psi\rangle_0|^2 \frac{dbda}{a^2} &= \int_{G_0}\int_G |(D^{-a}\psi * \overline{\psi})(b)|^2 \frac{dbda}{a^2} \\
&= \int_{G_0}\int_G |(D^{-a}\psi)^\wedge(\varrho) \cdot \widehat{\overline{\psi}}(\varrho)|^2 d\varrho \frac{da}{a^2} \\
&= \int_{G_0}\int_G |a||\hat{\psi}(-a\varrho|^2 |\widehat{\overline{\psi}}(\varrho)|^2 d\varrho \frac{da}{a^2} \\
&= \|\psi\|_0^2 \cdot \int_G \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega.
\end{aligned} \tag{2.10}$$

In the last step we substituted $\omega = -a\varrho$ an changed the order of integration. As a consequence we can characterize the admissible functions.

Lemma 2.1. *$\psi \in L_2(G, dt)\backslash\{0\}$ is admissible if and only if the integral*

$$\int_{G_0} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega \quad \textit{exists}.$$

Remark. As a necessary condition on the admissibility of an element $\psi \in L_2(G, dt)$ we derive

$$\hat{\psi}(0) = \int_G \psi(t)dt = 0; \tag{2.11}$$

i.e. the mean value of ψ has to be zero, if the integral exists (e.g. if ψ is in addition integrable). We call an admissible function also analyzing resp. basic wavelet or wavelet in short.

Theorem 2.1. *Let ψ be admissible and $f \in L_2(G, dt)$, Let $C_\psi = \int_G \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega$. The integral*

$$\begin{aligned} L_\psi f(b, a) &= \frac{1}{\sqrt{C_\psi}} \left\langle f, U(b, a)\psi \right\rangle_0 \\ &= \frac{1}{\sqrt{C_\psi}} \frac{1}{\sqrt{|a|}} \int_G \overline{\psi}\left(\frac{t-b}{a}\right) f(t)dt \end{aligned} \tag{2.12}$$

defines an element of $L_2\left(G \times G_0, \frac{dbda}{a^2}\right)$.

Moreover $L_\psi : L_2(G, dt) \to L_2\left(G \times G_0, \frac{dbda}{a^2}\right)$ is an isometry.

Proof. $L_\psi f(b, a)$ exists for any $(b, a) \in G \times G_0$ because f and $T^b D^a \psi$ are in $L_2(G, dt)$. A similar calculation to (2.10) results in

$$\begin{aligned} \|L_\psi f\|^2 &= \int_{G_0} \int_G |L_\psi f(b, a)|^2 \frac{dbda}{a^2} \\ &= \frac{1}{C_\psi} \int_{G_0} \int_G |\big(U(b, a)\psi, f\big)|^2 \frac{dbda}{a^2} \\ &= \frac{1}{C_\psi} \cdot \|f\|_0^2 \cdot \int_{G_0} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega = \|f|_0^2. \end{aligned} \tag{2.13}$$

□

Definition 2.2. *The operator $L_\psi : L_2(G, dt) \to L_2\left(G \times G_0, \frac{dbda}{a^2}\right)$ (ψ admissible) is called wavelet transform with analyzing (basic) wavelet ψ.*

3. Extension to Sobolev Spaces

In this section we extend the wavelet transform, which we defined on $L_2(G, dt)$, to Sobolev spaces $H^\alpha(G)$ and interpret its images as elements of the fiber space $L_2\left(\left(G_0, \frac{da}{a^2}\right), H^\alpha(G)\right)$ abbreviated by $\mathscr{F}^\alpha$ which is isomorphic to tensor product $L_2\left(G_0, \frac{da}{a^2}\right) \widehat{\otimes} H^\alpha(G)$ as well as to the Sobolev space with two variables $H^{0,\alpha}\left(G^2, \frac{dadb}{a^2}\right)$.

If μ is a measure on G_0 and $(B, \|\cdot\|)$ an arbitrary normed space, then $L_2\big((G_0, d\mu(x)), (B, \|\cdot\|)\big)$ consists of those $\phi \in B$ which depend on a real variable and for which holds

$$\int_{G_0} \|\phi(x)\|^2 d\mu(x) < \infty$$

$H^{\alpha}(G), \alpha \in G$, denotes the sobolev space of those tempered distributions γ having a regular and with respect to the weight $(1+\gamma(\xi)^2)^{\alpha}$ square integrable Fourier transform $\hat{\gamma}$. We sometimes call elements of $H^{\alpha}(G)$ signals.

From now on we assume ψ to be admissible and integrable. If ψ and f are real then $L_{\psi}f$ is real. Without loss of generity we assume ψ and f to be real. Under the assumption above we have

$$\begin{aligned} L_{\psi}f(b,a) &= \frac{1}{\sqrt{C_{\psi}}}\langle T^b D^a \psi, f\rangle_0 \\ &= \frac{1}{\sqrt{C_{\psi}}}\big(D^{-a}\psi * f\big)(b). \end{aligned} \tag{3.1}$$

From (2.7) we obtain the Fourier transform of L_{ψ} with respect to its shift argument

$$\big(L_{\psi}f(\cdot,a)\big)^{\wedge}(\omega) = \sqrt{\frac{1}{C_{\psi}}}|a|^{1/2}\hat{\psi}(-a\omega)\hat{f}(\omega). \tag{3.2}$$

Fix $a \in G_0$ and let $f \in \mathscr{S}(G)$, the Schwartz space on G. Let us now determine the $H^{\alpha}(G)$-norm of $L_{\psi}f(\cdot,a)$. For that we need an inequality from Fourier analysis

$$|\hat{\psi}(\omega)| \leqq \|\psi\|_{L_1}$$

leading to

$$\begin{aligned} \|L_{\psi}f(\cdot,a)\|^2_{H^s(G)} &= \int_G (1+\gamma(\xi)^2)^s |(L_{\psi}f(\cdot,a))^{\wedge}(\omega)|^2 d\omega \\ &= \frac{1}{C_{\psi}}\int_G (1+\gamma(\xi)^2)^s |(D^{-a}\psi)^{\wedge}(\omega)|^2 |\hat{f}(\omega)|^2 d\omega \\ &\leqq \frac{1}{C_{\psi}}\|D^{-a}\psi\|^2_{L_1}\int_G (1+\gamma(\xi)^2)^s |\hat{f}(\omega)|^2 d\omega \\ &= K(a,\psi)\cdot\|f\|^2_{H^s(G)}, \end{aligned} \tag{3.3}$$

where $K(a,\psi) = \frac{1}{C_{\psi}}|a|\|\psi\|^2_{H^s(G)}$.

The Schwartz space is dense in $H^{\alpha}(G)$. Therefore we are in a position to extend $L_{\psi}f(\cdot,a)$ uniquely for fixed a to a continuous mapping from $H^{\alpha}(G)$ to itself.

Lemma 3.1. *The integral operator L_{ψ} with an integrable and admissible ψ is an isometry from $H^{\alpha}(G), \alpha \in G$ to the fiber space $\mathscr{F}^{\alpha}$, i.e.*

$$\|L_{\psi}f\|_{\mathscr{F}^{\alpha}} = \left(\int_G \|L_{\psi}f(\cdot,a)\|^2_{\alpha}\frac{da}{a^2}\right)^{\frac{1}{2}} = \|f\|_{\alpha}.$$

Proof. It suffices to consider $f \in \mathscr{S}(G)$. The result is shown by a straight forward computation.

$$\begin{aligned} \|L_{\psi}f\|^2_{\mathscr{F}^{\alpha}} &= \int_{G_0}\int_G (1+\gamma(\xi)^2)^s |(L_{\psi}f(\cdot,a))^{\wedge}(\omega)|^2 d\omega \frac{da}{a^2} \\ &= \frac{1}{C_{\psi}}\int_{G_0}\int_G (1+\gamma(\xi)^2)^s |a||\hat{\psi}(-a\omega)|^2|\hat{f}(\omega)|^2 d\omega\frac{da}{a^2} \end{aligned} \tag{3.4}$$

Substituting $-a\omega = \xi$ and treating $\omega > 0$ and $\omega < 0$ separately leads to

$$\|L_\psi f\|^2_{\mathscr{F}^\alpha} = \frac{1}{C_\psi}\int_{G_0}\frac{|\hat{\psi}(\xi)|^2}{|\xi|}d\xi \cdot \int_G (1+\gamma(\xi)^2)^s|\hat{f}(\omega)|^2 d\omega = \|f\|^2_\alpha.$$

□

The signal f and its wavelet transform $L_\psi f$ share the same Sobolev order. For a linear isometry U between Hilbert spaces we have that see [12],

$$\begin{gathered} U*U = id \quad and \quad UU^* is \quad the \quad orthogonal \quad projection \\ onto \quad range(U) \quad (which \quad is \quad closed), \end{gathered} \tag{3.5}$$

where U^* is the adjoint operator of U. From statement (3.5) it follows immediately that the transform L_ψ is inverted, on its range, by its adjoint L^*_ψ and that an element $g \in \mathscr{F}^\alpha$ lies in range (L_ψ) if and only if $L_\psi L^*_\psi g = g$.

Next we figure out an explicit expression for $L^*_\psi : \mathscr{F}^\alpha \to H^\alpha$. In what follows we use $f \in \mathscr{S}(G), g(x,a) = g_1(x)\cdot g_2(a)$ with $g_1 \in \mathscr{S}(G)$, $g_2 \in C_0^\infty(G_0)$ and $\Lambda(a,\omega,\alpha) = (1 + \gamma(\xi)^2)^\alpha (L_\psi f(\cdot,a))^\wedge(\omega)\cdot(g(\cdot,a))\wedge(\omega)$. Setting up a scalar product on $\mathscr{F}^\alpha$ in a canonical manner,

$$(\varrho,\gamma)_\alpha = \int_{G_0}\langle\varrho(\cdot,a),\gamma(\cdot,a)\rangle_\alpha \frac{da}{a^2},$$

we get

$$(L_\psi f, g)_\alpha = \int_{G_0}\int_G \Lambda(a,\omega,\alpha)d\omega\frac{da}{a^2}. \tag{3.6}$$

Applying two times the Cauchy-Schwartz (C.S) inequality leads to

$$\int_{G_0}\int_G \Lambda(a,\omega,\alpha)d\omega\frac{da}{a^2} \leq \int_{G_0}\|L_\psi f(\cdot,a)\|_\alpha\|g(\cdot,a)\|_\alpha\frac{da}{a^2} \leq \|L_\psi f\|_{\mathscr{F}^\alpha}\|g\|_{\mathscr{F}^\alpha}$$

which allows to change the order of integration in (3.6).

$$(L_\psi f, g)_\alpha = \int_G (1+\gamma(\xi)^2)^s\hat{f}(\omega)\int_{G_0}\sqrt{\frac{1}{C_\psi}}(D^{-a}\psi)^\wedge(\omega)(g(\cdot,a))^\wedge(\omega)\frac{da}{a^2}d\omega. \tag{3.7}$$

We abbreviate the inner integral by $(\tilde{A}g)(\omega)$ and estimate $|\tilde{A}g|$ to conclude that $\tilde{A}g \in L_2(G,dt)$:

$$|\tilde{A}g(\omega)|^2 \leqq \int_{G_0}|(g(\cdot,a))^\wedge(\omega)|^2\frac{da}{a^2}. \tag{3.8}$$

Again, we used the C.S. inequality and get

$$\int_G|\tilde{A}g(\omega)|^2 d\omega \leqq \int_G\int_{G_0}|(g(\cdot,a))^\wedge(\omega)|^2\frac{da}{a^2}d\omega = \|g\|^2_{\mathscr{F}^0}. \tag{3.9}$$

Consequently, there exists a $Ag \in L_2(G,dt)$ with

$$(Ag)^\wedge(\omega) = \tilde{A}g(\omega) \tag{3.10}$$

and now the equation (3.7) reads as

$$(L_\psi f, g)_\alpha = \int_G (1+\gamma(\xi)^2)^s\hat{f}(\omega)(Ag)^\wedge(\omega)d\omega = \langle f, Ag\rangle_\alpha. \tag{3.11}$$

In the last step, we determine $Ag(x)$ using the fact that the integral

$$\int_{G_0}\int_G |(D^{-a}\psi)^\wedge(\omega)(g(\cdot,a))^\wedge(\omega)|d\omega\frac{da}{a^2} \tag{3.12}$$

exists.

$$\begin{aligned} Ag(x) &= \int_G (Ag)^\wedge(\omega)(-x,\omega)d\omega \\ &= \frac{1}{\sqrt{C_\psi}}\int_{G_0}\frac{1}{\sqrt{2\pi}}\int_G (D^{-a}\psi * g(\cdot,a))^\wedge(\omega)(-x,\omega)d\omega\frac{da}{a^2} \\ &= \frac{1}{\sqrt{C_\psi}}\int_{G_0}(D^{-a}\psi * g(.,a))(x)\frac{da}{a^2} \\ &= \frac{1}{\sqrt{C_\psi}}\int_{G_0}\int_G \frac{1}{\sqrt{|a|}}\psi\left(\frac{b-x}{a}\right)g(b,a)\frac{dbda}{a^2} \end{aligned} \tag{3.13}$$

We showed that the operators L_ψ and A are disjoints of each other on pre-hilbert spaces of H^α resp. $\mathscr{F}^\alpha$. This property is inherited by their extensions. Accordingly the extension of A on $\mathscr{F}$ is identical to L_ψ^*. The abstract characterization of range (L_ψ) results in

Lemma 3.2. *Range $(L_\psi) \subset \mathscr{F}^\alpha$ is a Hilbert space with reproducing kernel*

$$P(\tilde{b},\tilde{a},b,a) = \frac{1}{\sqrt{C_\psi}}(L_\psi\psi)\left(\frac{\tilde{b}-b}{a},\frac{\tilde{a}}{a}\right).$$

$$g \in range(L_\psi) \Leftrightarrow g(\tilde{b},\tilde{a}) = \int_{G_0}\int_G P(\tilde{b},\tilde{a},b,a)g(b,a)\frac{dbda}{a^2}$$

Proof. A direct calculation of $L_\psi L_\psi^*$ proves the lemma. □

We will now determine the H^s- distance of two wavelet transforms with different basic wavelets and different argument functions to study the dependence of the transform on its wavelet and its argument.

Lemma 3.3. *For admissible and integrable ψ,γ and $f,g \in H^s(G), s \in G$, holds:*

$$\|L_\psi f(\cdot,a) - L_\gamma g(\cdot,a)\|_s \leqq \sqrt{|a|}\left(\|\frac{\psi}{\sqrt{C_\psi}} - \frac{\gamma}{\sqrt{C_\psi}}\|_{L_1}\|f\|_s + \|\frac{\gamma}{\sqrt{C_\psi}}\|_{L_1}\|f-g\|_s.\right.$$

Proof.

$$\begin{aligned} \|L_\psi f(\cdot,a) - L_\gamma g(\cdot,a)\|_s &\leqq \|L_\psi f(\cdot,a) - L_\gamma f(\cdot,a)\|_s\|L_\psi f(\cdot,a) - L_\gamma g(\cdot,a)\|_s \\ = \left(\int_G (1+\gamma(\xi)^2)^s|\hat{f}(\omega)|^2\right.&\left.|\frac{(D^{-a}\psi)\wedge(\omega)}{\sqrt{C_\psi}} - \frac{(D^{-a}\gamma)\wedge(\omega)}{\sqrt{C_\psi}}|^2 d\omega\right)^{1/2} \\ +\left(\int_G (1+\gamma(\xi)^2)^s|\hat{f}(\omega)-\hat{g}(\omega)|^2\right.&\left.|\frac{(D^{-a}\gamma)\wedge(\omega)}{\sqrt{C_\psi}}|^2 d\omega\right)^{1/2} \end{aligned}$$

Performing the same steps as in (3.3) to each term of the sum yields the lemma. □

A direct application of lemma 3.3 gives

Corollary 3.1. *Let ψ and f be as in the preceding lemma. Then*

$$\|L_\psi f(\cdot, a)\|_s = 0(\sqrt{|a|}).$$

4. Asymptotic Behaviour for Small Dilation Parameters

We adopt the assumptions on f and ψ from the last paragraph. In addition we assume without loss of generality $\hat{\psi}$ to be real because the admissibility condition is valid not only for the real but also for the imaginary part of $\hat{\psi}$. Then

$$\begin{aligned} L_\psi f(b,a) &= \frac{1}{\sqrt{C_\psi}}\frac{1}{\sqrt{|a|}}\int_G \psi\left(\frac{t-b}{a}\right) f(t)dt \\ &= \frac{1}{\sqrt{C_\psi}}\sqrt{|a|}\int_G \hat{\psi}(a\omega)\hat{f}(\omega)(-b,\omega)d\omega \end{aligned} \tag{4.1}$$

is even in the second variable because $\hat{\psi}$ is. We restrict ourselves to the half-plane $a > 0$.
Considering (4.1) we realize that the integral expression looks like the ψ-average $\psi_a * f$ of f with $\psi_a(x) = a^{-1}\cdot\psi(a^{-1}x)$.
Indeed, we have

$$(\psi_a * f)(b) = \sqrt{\frac{C_\psi}{a}} L_\psi f(b,-a) = \sqrt{\frac{C_\psi}{a}} L_\psi f(b,a). \tag{4.2}$$

For $\psi \in L_1(G)$ (i.e. ψ is integrable) with$\int_G \psi(t)dt = 1$ the ψ-average of f converges to f in the L_2-norm, which means that

$$\lim_{a\to 0}\|\psi_a * f - f\|_0 = 0. \tag{4.3}$$

Unfortunately, a basic wavelet has a zero mean, and therefore (4.5) does not hold for the WT. Now we are interested in whether an asymptotic behaviour like (4.5) is possible under certain assumptions on the analyzing wavelet.
For the ψ-average of f we write $\Lambda_\psi f(\cdot,\cdot)$, i.e.

$$\Lambda_\psi f(b,a) = (\psi_a f)(b) = \frac{1}{a}\int_G \psi\left(\frac{b-t}{a}\right) f(t)dt. \tag{4.4}$$

Lemma 4.1. *Let $f \in H^s(G), s \in G$. Let $\psi \in L_1(G)$ with $\int_G \psi(t)dt = 1$. Then we have*

(i) $\Lambda_\psi f(\cdot, a) \to f(\cdot)$ in $H^s(G)$ as $a \to 0$,
(ii) $d^k(\Lambda_\psi f)(\cdot, a) = \Lambda_\psi(d^k f)(\cdot, a) = a^{-k}(\Lambda_{d^k\psi} f)(\cdot, a)$, if $d^k\psi \in L_1(K)$.

(d^k denoting k-th generalized derivative).

Proof. (i) $\|\psi_a * f - f\|_s^2 = \int_G I(a,\omega)d\omega$
where $I(a,\omega) = (1+\gamma(\xi)^2)^s|\hat{f}(\omega)|^2|1-\hat{\psi}(a\omega)|^2$.
With $M = sup_{\omega\in G}|1-\hat{\psi}(a\omega)|^2$ which exists by the lemma of Riemann-Lebesgue and is independent of a we find

$$I(a,\omega) \leqq M\cdot(1+\gamma(\xi)^2)^s|\hat{f}(\omega)|^2$$

as well as

$$\lim_{a\to 0} I(a,\omega) = 0 \quad a.e.$$

Applying the dominated convergence theorem yields the assertion.

(ii) Let $\{f_n\}_{n\in N} \subset \mathscr{S}(G)$ converge to f in $H^s(G)$. The equality $d^k(\Lambda_\psi f_n) = \Lambda_\psi d^k f_n = A^{-k}\Lambda_{d^k\psi} f_n$ is valid in $H^{s-k}(G)$. Since the operators Λ_ψ and d^k are continuous, the limits of the three terms are equal.

□

Lemma 4.2. *Let $0 \not\equiv \varrho \in H^\beta(G), \beta \geqq 1$. Then $d^k\varrho$ is admissible for $1 \leqq k \leqq \beta$.*

Proof. First , it is easy to see that $d^k\varrho$ is equal to zero if and only if $\varrho \equiv 0$ because zero is the only constant in $H^s(G), s \in G$. Therefore we have $d^k\varrho \neq 0$. second, $\beta - k \geqq 0$ implies $d^k\varrho \in L_2(G)$. Third, we use the relation

$$(d^k\varrho)^\wedge(\cdot) = i^k(\cdot)^k\hat{\varrho}(\cdot)$$

to estimate

$$\begin{aligned}\int_{G_0} \frac{|(d^k\varrho)^\wedge(\omega)|^2}{|\omega|} d\omega &= \int_{G_0} |\omega|^{2k-1}|\hat{\varrho}(\omega)|^2 d\omega \\ &\leqq \int_G (1+\gamma(\xi)^2)^{k-1/2}|\hat{\varrho}(\omega)|^2 d\omega \leqq \|\varrho\|_\beta^2.\end{aligned}$$

The result follows from Lemma 2.2. □

Our investigations now focus on the WT with analyzing wavelet $d^k\psi \in L_1(G)$ with $\psi \in H^\beta(G)\bigcap L_1(G), \beta \geqq 1$, and $\int_G \psi(t)dt = 1$ (thus ψ itself is not admissible).

Theorem 4.1. *Let $f \in H^s(G), s \in G$, and $\psi \in H^\beta(G)\bigcap L_1(G), \beta \in N$, with $\int_G \psi(t)dt = 1$ and $d^k\psi \in L_1(G)$ at least for one $k \in \{1, \ldots.., \beta\}$. Then*

$$\lim_{a\to 0} \| \frac{1}{a^{k+1/2}} L_{d^k\psi} f(\cdot) - \frac{1}{\sqrt{C_k}} d^k f(\cdot)\|_{s-k} = 0,$$

where C_k abbreviates $C_{d^k\psi}$.

Proof. ψ is not identical to zero . According to lemma 4.2 $d^k\psi$ is admissible . with an application of lemma 4.1(ii) we restate

$$\begin{aligned}L_{d^k\psi} f(b,a) &= \sqrt{\frac{a}{C_k}}\Lambda_{d^k\psi} f(b,a) \\ &= \frac{a^{k+1/2}}{\sqrt{C_k}}\Lambda_\psi d^k f(b,a) \\ &= \frac{a^{k+1/2}}{\sqrt{C_k}} d_b^k(\Lambda_\psi f)(b,a).\end{aligned}$$

Now we estimate

$$\|\frac{1}{a^{k+1/2}}L_{d^k\psi}f(\cdot,a)-\frac{1}{\sqrt{C_k}}d^kf(\cdot)\|_{s-k}=\frac{1}{\sqrt{C_k}}\|d^k[\Lambda_\psi f(.,a)-f(.)]\|_{s-k}$$
$$\leqq\frac{1}{\sqrt{C_k}}\|\Lambda_\psi f(\cdot,a)-f(\cdot)\|_s$$

using the boundedness of the differential operator from H^s to H^{s-k}. The term on the right tends to zero in part (i) of Lemma 4.1. This ends the proof. □

Remarks.

(i) For $s>\frac{1}{2}+k$ we have uniform convergence. This results immediately by Sobolev's Imbedding Theorem [11].

(ii) For compactly supported ψ we know that $L_{d^k\psi}f(\cdot,a)\in H^{s-k+\beta}(G)$ if $f\in H^s(G),\psi\in H^\beta(G)$, and $1\leqq k\leqq\beta$. Hence, in accordance with the theorem above, $\frac{1}{a^{k+1/2}}L_{d^k\psi}f(\cdot,a)$ is an approximation of $\frac{1}{\sqrt{C_k}}d^kf$ which is at least β levels smoother than its limit.

4.1. **Local Convergences.** In practical applications of wavelet transfrom, i.e., the analysis and synthesis of time-dependent signals, the signal f is compactly supported. Even if this signal possesses a high order of smoothness within its support, under a global viewpoint, we can only deduce that f is square integrable over the real line, which means $f\in H^0(G)$.
By theorem 4.1, $\frac{\sqrt{C_k}}{a^{k+1/2}}L_{d^k}f$ approximates the $k-th$ derivative of f only in $H^{-k}(R)$ although f is local element of the Sobolev space $H^s(G)$ with $s>0$ and therefore we would expect a kind of local convergence in the stronger norm of $H^{s-k}(G)$.
We specify the concept of local convergence. Therefore, we define the local Sobolev spaces [3].

Definition 4.1. *Let $\Omega\subset G$ be open.*

$$H^s_{loc}(\Omega):=\left\{f\quad is\quad a\quad distribution|\quad\forall\Omega'\subset\Omega,\Omega' compact,\quad\exists g_{\Omega'}\in H^s(G):f\equiv g_{\Omega'}on\Omega'\right\}$$

is called local Sobolev space of order s.

Lemma 4.3. *$f\in H^s_{loc}\Omega\Leftrightarrow f\cdot\Phi\in H^s(G)\quad\forall\Phi\in C_0^\infty(\Omega)$ suggests a concept of convergence in $H^s_{loc}(\Omega)$. ($C_0^\infty(\Omega)$ denotes the space of the test functions with compact support in Ω)*

Definition 4.2. *Let $\{F_n\}_{n\in N}$ be a sequence in $H^s_{loc}(\Omega)$ and $f\in H^s_{loc}(\Omega)$. $\{f_n\}_{n\in N}$ converges to f in H^s_{loc} (local convergence) if and only if $\|\Phi f_n-\Phi f\|_s$ converges to zero for any $\Phi\in C_0^\infty(\Omega)$.*

Remark. This concept of local convergence is well defined because the limit is uniquely determined. Without loss of generality, we assume that

$$supp(f)=[-T,T]=I.\tag{4.5}$$

Further we consider

$$f\in H^s_{loc}(I^0)\quad with\quad I^0=]-T,T[\quad and\quad s\in G.\tag{4.6}$$

For $0 < \epsilon < T$ let J_ϵ be the compact interval $[-\epsilon, \epsilon]$. We know from real analysis that there is a $\Gamma_\epsilon \in C_0^\infty(I^0)$ which is identical to 1 on J_ϵ.

Lemma 4.4. *(i)* $\Gamma_\epsilon(\cdot)f(\cdot) \in H^s(G)$

(ii) $\Gamma_\epsilon(\cdot)f(\cdot)$ *converges to* f *in* $H^s_{loc}(I^0)$ *as* ϵ *tends to* T.

(iii) $\Gamma_\epsilon(\cdot)f(\cdot) \equiv f(\cdot) \quad on \quad H^s_{loc}(J^0_\epsilon)$.

Proof. (i) Is the statement of Lemma 4.3.

(ii) Let $\Phi \in C_0^\infty(I^0)$. For sufficiently large ϵ_Φ with $0 < \epsilon_\Phi < T$ we have $supp(\Phi) \subseteq J_\epsilon$ for all ϵ with $\epsilon_\Phi \leqq \epsilon < T$. This implies $\Phi\Gamma_\epsilon f = \Phi f$ in $H^s(G)$ for $\epsilon_\Phi \leqq \epsilon < T$ and thus the assertion.

(iii) It is clear that both $\Gamma_\epsilon f$ and f are in $H^s_{loc}(J^0_\epsilon)$. We still have to show equality. Let a test function $\Phi \in C_0^\infty(J^0_\epsilon)$ act on the distribution $\Gamma_\epsilon f$:

$$(\Gamma_\epsilon(t)f(t), \Phi(t)) = (f(t), \Gamma_\epsilon(t)\Phi(t)) = (f(t), \Phi(t)).$$

□

A local version of theorem 4.1 reads as

Theorem 4.2. *Let fulfill (4.5) and (4.6). Let* ψ *be defined as in theorem 4.1 and* Γ_ϵ *as above. Then* $\frac{1}{a^{k+1/2}} L_{d^k\psi}(\Gamma_\epsilon f)(\cdot, a)$ *converges to* $\frac{1}{\sqrt{C_k}} d^k f$ *in* $H^{s-k}_{loc}(J^0_\epsilon)$ *for any* $\epsilon \in]0, T[$ *as a tends to zero.*

Remark. Even locally, we can reach convergence in the strongest norm.

Proof. First, we conclude that $d^k f = d^k(\Gamma_\epsilon f)$ in $H^{s-k}_{loc}(J^0_\epsilon)$ with the Leibniz rule and the action of $\Phi \in C_0^\infty(J^0_\epsilon)$ on $d^k(\Gamma_\epsilon f)$:

$$(d^k(\Gamma_\epsilon f), \Phi) = \sum_{i=0}^{k} \binom{k}{i} (d^{k-i} f, (d^i \Gamma_\epsilon)\Phi) = (d^k f, \Phi).$$

The last equality holds true because $\Gamma_\epsilon|_{supp(\Phi)} \equiv 1$.
Theorem 4.1 yields

$$\frac{1}{a^{k+1/2}} L_{d^k\psi}(\Gamma_\epsilon f)(\cdot, a) \to \frac{1}{\sqrt{C_k}} d^k(\Gamma_\epsilon f)(\cdot) \in H^{s-k}(G).$$

To continue the proof, we need the boundness of the multiplication operator on H^α. Let $\prod \in \mathscr{S}(G)$ and let $T_\prod : H^\alpha \to H^\alpha$ be defined by $T_\prod f = \prod .f$. Then $T_\prod$ is continuous for all $\alpha \in G$.We are now able to prove the desired convergence in $H^{s-k}_{loc}(J^0_\epsilon)$.
Let $\Phi \in C_0^\infty(J^0_\epsilon) \subset \mathscr{S}(G)$:

$$\begin{aligned} &\|\Phi(\cdot)\frac{1}{a^{k+1/2}} L_{d^k\pi}(\Gamma_\epsilon f)(\cdot, a) - \Phi(\cdot)\frac{1}{\sqrt{C_k}} d^k f(\cdot)\|_{s-k} \\ &\leqq \|T_\Phi\| \cdot \|\frac{1}{a^{k+1/2}} L_{d^k\pi}(\Gamma_\epsilon f)(\cdot, a) - \frac{1}{\sqrt{C_k}} d^k(\Gamma_\epsilon f)(\cdot)\|_{s-k}. \end{aligned}$$

□

Under local conditions of smoothness on the signal, statements can be made about the order of convergence.

Lemma 4.5. *Let f be two times continuously differentiable in a neighbourhood of $b \in G$(e.g.$f \in H^s_{loc}(N)$, where N is the nbd of a topological group) Let $\psi \in H^\beta(K) \bigcap L_1(G), \beta \geqq 1$,with $\int_G \psi(t)dt = 1$ and supp$\pi = M$, where M is the subgroup of G. For $a > 0$ sufficiently small holds*

$$a^{-3/2} L_{d\psi} f(b,a) = \frac{1}{\sqrt{C_1}} f'(b) + 0(a)$$

(prime indicates the first classical derivative).

Proof. Using the facts that for sufficiently small a df is equal to f' in M and that $M(a) = sup\xi \in M|f''(\xi)|$exists, we obtain by the mean value theorem

$$\begin{aligned}\left|a^{-3/2} L_{d\psi} f(b,a) - \frac{1}{\sqrt{C_1}} f'(b)\right| &= \frac{1}{\sqrt{C_1}} |\Lambda_\psi df(b,a) - f'(b)| \\ &\leqq \frac{1}{\sqrt{C_1}} \cdot \frac{1}{a} \int_M \left|\psi(\frac{t-b}{a})\right| |f'(t) - f'(b) dt| \\ &\leqq \frac{1}{\sqrt{C_1}} \cdot \frac{1}{a} \cdot M(a) \int_M \left|\psi\left(\frac{t-b}{a}\right)\right| |t-b| dt \\ &= K(a,b,f) \cdot a\end{aligned}$$

with $K(a,b,f) = \frac{1}{\sqrt{C_1}} \cdot M(a) \int_M |\psi(y)||y|dy$. □

Acknowledgement: This work is supported by CSIR grant No. 09/725(014)/2019-EMR-1.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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