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On Null Vertex in Fuzzy Graphs

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Abstract. We introduce a new type of vertex in fuzzy graphs, namely null vertex, which is neither a boundary vertex nor an interior vertex. Here, we initiate a study on the null vertex in fuzzy graphs, explore its properties and establish its presence in various types of fuzzy graphs.

1. Introduction

Zadeh [11] put forth the idea of fuzzy set and this idea brought about revolutionary changes in the area of research. As Euler pioneered the concept of graph theory, Rosenfeld [7] developed fuzzy graph theory in 1975. G. Chartrand [2,3] developed boundary vertex and interior vertex in crisp graphs. Mini Tom [9] introduced boundary vertex and interior vertex in fuzzy graphs.

In fuzzy graphs, there exists some vertices which are distinct from boundary vertices and interior vertices. Here, we introduce the idea of null vertex in fuzzy graphs. Null vertex in a fuzzy graph is a vertex which is neither a boundary vertex nor an interior vertex. The null vertex, if it exists, need not be unique. We study the structural characteristics of the null vertices and establish its presence in various types of fuzzy graphs.

We prove that, a fuzzy end vertex in a fuzzy graph is a boundary vertex. In fuzzy path graphs and fuzzy star graphs, there exists only boundary vertices and interior vertices. We also prove that, there exists null vertices in fuzzy cycles, complete fuzzy graphs, fuzzy wheel graphs and fuzzy helm graphs. For terms and definitions of metric spaces, reader may refer [8]. For terms and definitions of graphs, refer [4].

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2. Preliminaries

Definition 2.1. *[5]* A fuzzy graph (FG) is $G : (V, \sigma, \mu)$, V is the vertex set, $\sigma : V \to [0, 1]$, $\mu : V \times V \to [0, 1]$ with

$$\mu(p, q) \leq \sigma(p) \wedge \sigma(q), \forall p, q \in V.$$

Definition 2.2. *[10]* In a FG G, a sequence of vertices $p_0, p_1, p_2 \dots p_n$, with $\mu(p_{i-1}, p_i) > 0, i = 1, 2, \dots n$ is a path P. Length of P is

$$L(P) = \sum_{i=1}^{n} \mu(p_{i-1}, p_i).$$

For two vertices p, q, let $P = \{P_i : P_i$ is a $p - q$ path $, i = 1, 2, ...\}$. Sum distance from p to q is

$$d_s(p, q) = Min\{L(P_i) : P_i \in P, i = 1, 2, ...\}$$

Definition 2.3. *[6]* An arc of a FG G with least membership degree is a weakest arc and the strength of path P is the membership degree of the weakest arc. P is a cycle if $p_0 = p_n$, $n \geq 3$, and a fuzzy cycle if the number of weakest arcs in P is more than one.

Definition 2.4. *[5]* The maximum of the strengths of all paths from the vertex p to the vertex q is the strength of connectedness, $CONN_G(p, q)$ between them.
If $\mu(p, q) \geq CONN_{G-(p,q)}(p, q)$, then (p, q) is a strong arc.

Definition 2.5. *[1]* The vertices p, q are neighbours in the FG G if $\mu(p, q) > 0$. If (p, q) is a strong arc, then q is a strong neighbour of p. If q has exactly one strong neighbour, then q is a fuzzy end vertex.

Definition 2.6. A FG G is strong if $\mu(u, v) = \min\{\sigma(u), \sigma(v)\}$, $\forall$ (u, v) in E

Definition 2.7. A FG G is complete if $\mu(u, v) = \min\{\sigma(u), \sigma(v)\}$, $\forall u, v$ in V

3. Main Results

Definition 3.1. *[9]* A vertex v in a FG G is a boundary vertex of a vertex u in G if

$$d_s(u, v) \geq d_s(u, w), \text{ for each neighbour } w \text{ of } v.$$

The boundary verticess of u is represented by u^b.
v is a boundary vertex of G if v is a boundary vertex of some vertex of G.

Definition 3.2. *[9]* A vertex w in a FG G is an interior vertex of G if for each vertex $u \neq w, \exists$ a vertex $v \neq w \neq u$ with

$$d_s(u, v) = d_s(u, w) + d_s(w, v).$$

Remark 3.1. *[9]* A boundary vertex of a FG G is not an interior vertex of G.

Proposition 3.1. In a connected FG, a fuzzy end vertex is a boundary vertex.

Proof. Consider a FG G with vertices $v_i, 1 \leq i \leq n$. Let v_1 be a fuzzy end vertex. v_1 has only one neighbour say, v_2. Clearly,

$$d_s(v_i, v_1) \geq d_s(v_i, v_2), \quad 1 < i \leq n.$$

Hence, v_1 is a boundary vertex of $v_i, i \neq 1$ in G. □

Theorem 3.1. In a fuzzy path graph P_n, all the vertices are either boundary vertices or interior vertices.

Proof. Consider P_n with vertices $v_1, v_2, \ldots..v_n$. As v_1, v_n are fuzzy end vertices, v_1, v_n are boundary vertices by proposition 3.1. The vertices $v_j, 2 \leq j \leq (n-1)$ are interior vertices of P_n, since for every vertex v_i, $\exists$ a vertex $v_k, i \neq j \neq k$ with

$$d_s(v_i, v_k) = d_s(v_i, v_j) + d_s(v_j, v_k), \quad 1 \leq i, k \leq n.$$

Thus, all the vertices are either boundary vertices or interior vertices. □

Theorem 3.2. In a fuzzy star graph $K_{1,n}$, all the vertices are either boundary vertices or interior vertices.

Proof. Consider the vertices $v, u_1, u_2, .., u_n$ in $K_{1,n}$ such that v is the central vertex and $u_1, u_2, .., u_n$ are fuzzy end vertices. $u_1, u_2, .., u_n$ are boundary vertices by proposition 3.1.

$$\text{Let} \quad \mu(u_i, v) = a, 0 < a \leq 1$$

$$\text{Since,} \quad d_s(u_i, v) = d_s(v, u_j) = a, 1 \leq i, j \leq n.$$

$$d_s(u_i, v) + d_s(v, u_j) = a + a = 2a$$

$$\text{Also, } d_s(u_i, u_j) = 2a$$

$$\text{Therefore, } d_s(u_i, u_j) = d_s(u_i, v) + d_s(v, u_j), \forall i \neq j$$

So, v is an interior vertex. □

Remark 3.2. In FGs, there exists vertices which are different from boundary vertices and interior vertices. These vertices are termed as null vertices.

Definition 3.3. A null vertex in a FG is a vertex which is neither a boundary vertex nor an interior vertex.

Theorem 3.3. A complete FG $G = K_n, n \geq 3$ with vertices $v_1, v_2, v_3, \ldots., v_n$ has exactly one null vertex v_1 when $\sigma(v_1)$ is the weakest vertex.

Proof. Let $G = K_n, n \geq 3$ be the complete FG with vertices $v_1, v_2, v_3, \ldots., v_n$.
Suppose v_1 is the weakest vertex with

$$\sigma(v_1) < \sigma(v_i) < 2\sigma(v_1), 1 < i \leq n.$$

The neighbours of v_1 are $v_2, v_3, \ldots., v_{n-1}$.
Since v_1 is the weakest vertex,

$$d_s(v_i, v_1) = \mu(v_i, v_1) = \sigma(v_i) \wedge \sigma(v_1) = \sigma(v_1).$$

$$d_s(v_i, v_j) = \mu(v_i, v_j) < 2\sigma(v_1), i \neq 1 \neq j$$

$$\text{So, } d_s(v_i, v_1) < d_s(v_i, v_j), \text{ for all neighbours } v_j \text{ of } v_1.$$

Hence v_1 is not a boundary vertex of v_i.

$$\text{Also, } d_s(v_i, v_1) = d_s(v_1, v_j) = \sigma(v_1).$$

$$d_s(v_i, v_1) + d_s(v_1, v_j) = 2\sigma(v_1).$$

But, $d_s(v_i, v_j) < 2\sigma(v_1)$.

$$\text{Therfore, } d_s(v_i, v_j) \neq d_s(v_i, v_1) + d_s(v_1, v_j), \forall\ i, j, \text{ where } i \neq j \neq 1.$$

Hence, v_1 is not an interior vertex.
Thus, v_1 is a null vertex. □

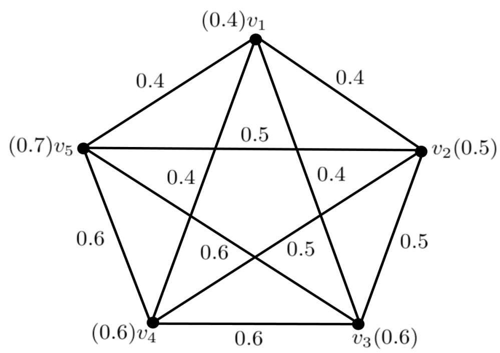

Figure 1. Complete fuzzy graph K_5

Example 3.1. Figure 1 is the complete FG K_5 with vertices $v_i, 1 \leq i \leq 5$, where v_1 is the weakest vertex,

$$\sigma(v_1) = 0.4, \sigma(v_2) = 0.5, \sigma(v_3) = \sigma(v_4) = 0.6, \sigma(v_5) = 0.7,$$
$$\sigma(v_1) < \sigma(v_i) < 2\sigma(v_1), 1 < i \leq 5.$$

Since K_4 is complete,

$$\mu(v_1, v_i) = 0.4, 1 \leq i \leq 5,$$
$$\mu(v_2, v_3) = \mu(v_2, v_4) = \mu(v_2, v_5) = 0.5, \mu(v_3, v_4) = \mu(v_3, v_5) = \mu(v_4, v_5) = 0.6.$$

Consider the vertex v_1.
The neighbours of v_1 are $v_i, \quad 2 \leq i \leq 5$,

$$\text{Here, } d_s(v_2, v_1) = 0.4, d_s(v_2, v_3) = d_s(v_2, v_4) = d_s(v_2, v_5) = 0.5$$

v_1 is not a boundary vertex of v_2 since, for the neighbours v_3, v_4, v_5 of v_1

$$d_s(v_2, v_1) < \begin{cases} d_s(v_2, v_3) \\ d_s(v_2, v_4) \\ d_s(v_2, v_5) \end{cases}$$

Similarly, v_1 is not a boundary vertex of the other vertices and hence not a boundary vertex of K_5.

$$\text{Since, } d_s(v_i, v_1) + d_s(v_1, v_j) = 0.4 + 0.4 = 0.8,$$
$$d_s(v_i, v_j) < 0.8,$$
$$d_s(v_i, v_j) \neq d_s(v_i, v_1) + d_s(v_1, v_j), \quad i \neq 1 \neq j.$$

i.e., v_1 is not an interior vertex.
Hence, v_1 is the null vertex.

Theorem 3.4. The strong fuzzy cycle $C_n, n \geq 3$ with vertices $v_i, 1 \leq i \leq n$ has exactly two null vertices v_1 and v_{n-2} when $\sigma(v_n) = \sigma(v_{n-1}) = b$ and $\sigma(v_i) = a$, otherwise, where

$$(n-2)a < b < (n-1)a.$$

Proof. Consider the strong fuzzy cycle $C_n, n \geq 3$ with vertices $v_i, 1 \leq i \leq n$, taken in order,

$$\sigma(v_n) = \sigma(v_{n-1)} = b, \quad \sigma(v_i) = a, \quad 1 \leq i \leq (n-2).$$
$$\text{where } (n-2)a < b < (n-1)a.$$

Since C_n is strong,

$$\mu(v_n, v_{n-1}) = min\{\sigma(v_n), \sigma(v_{n-1})\} = b,$$
$$\mu(v_i, v_j) = min\{\sigma(v_i), \sigma(v_j)\} = a, \text{ for all other edges } (v_i, v_j).$$

Consider the vertex v_1 in C_n.
The neighbours of v_1 are v_2 and v_n.
Here, v_1 is not a boundary vertex of $v_i, i = 2 \ldots n-1, \quad i \neq 1, n$, since

$$d_s(v_i, v_1) < d_s(v_i, v_n), \text{ for the neighbour } v_n \text{ of } v_1.$$

v_1 is not a boundary vertex of v_n, since

$$d_s(v_n, v_1) < d_s(v_n, v_2), \text{ for the neighbour } v_2 \text{ of } v_1.$$

Thus, v_1 is not a boundary vertex of $v_i, = 2 \ldots n, i \neq 1$

Consider the vertex v_{n-2} in C_n.
The neighbours of v_{n-2} are v_{n-1} and v_{n-3}.
v_{n-2} is not a boundary vertex of $v_i, i = 1, 2, \ldots (n-3), n, \quad i \neq (n-1), (n-2)$, since

$$d_s(v_i, v_{n-2}) < d_s(v_i, v_{n-1}), \text{ for the neighbour } v_{n-1} \text{ of } v_{n-2}.$$

v_{n-2} is not a boundary vertex of v_{n-1}, since

$$d_s(v_{n-1}, v_{n-2}) < d_s(v_{n-1}, v_{n-3}), \text{ for the neighbour } v_{n-3} \text{ of } v_{n-2}.$$

Thus, v_{n-2} is not a boundary vertex of $v_i, = 1, 2, \ldots (n-3), (n-1), n, \quad i \neq (n-2)$

Consider the vertex v_{n-1}.
For the vertex v_{n-1} in C_n, there does not exist a vertex v_j in C_n such that

$$d_s(v_{n-1}, v_j) = d_s(v_{n-1}, v_1) + d_s(v_1, v_j), \quad 1 \neq j \neq (n-1)$$

So, v_1 is not an interior vertex.

Similarly, Consider the vertex v_n.

For the vertex v_n in C_n, there does not exist a vertex v_k in C_n such that

$$d_s(v_n, v_k) = d_s(v_n, v_{n-2}) + d_s(v_{n-2}, v_k), \quad n \neq k \neq (n-2)$$

So, v_{n-2} is not an interior vertex.

Thus v_1 and v_{n-2} are null vertices. □

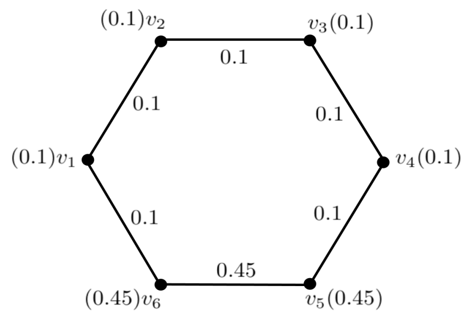

Figure 2. Fuzzy Cycle C_6

Example 3.2. Consider the strong fuzzy cycle C_6 in figure 2 with vertices $v_i, 1 \leq i \leq 6$.

Let $\sigma(v_i) = 0.1, 1 \leq i \leq 4, \quad \sigma(v_5) = \sigma(v_6) = 0.45$

Since C_6 is strong,

$$\mu(v_1, v_2) = \mu(v_2, v_3) = \mu(v_3, v_4) = \mu(v_4, v_5) = \mu(v_6, v_1) = 0.1, \quad \mu(v_5, v_6) = 0.45.$$

Consider the vertex v_1.

The neighbours of v_1 are v_2 and v_6.

v_1 is not a boundary vertex of $v_i, 2 \leq i \leq 5, i \neq 1, 6$, since

$$d_s(v_i, v_1) < d_s(v_i, v_6), \text{ for the neighbour } v_6 \text{ of } v_1.$$

v_1 is not a boundary vertex of v_6, since

$$d_s(v_6, v_1) < d_s(v_6, v_2), \text{ for the neighbour } v_2 \text{ of } v_1.$$

Also, for the vertex v_5, there does not exist a vertex $v_j, j \neq 1, 5$ such that

$$d_s(v_5, v_j) = d_s(v_5, v_1) + d_s(v_1, v_j).$$

Thus v_1 is not an interior vertex and hence v_1 is a null vertex.

Consider the vertex v_4.

The neighbours of v_4 are v_3 and v_5.

v_4 is not a boundary vertex of $v_i, 1 \leq i \leq 6, i \neq 4, 5$ since

$$d_s(v_i, v_4) < d_s(v_i, v_5), \text{ for the neighbour } v_5 \text{ of } v_4.$$

v_4 is not a boundary vertex of v_5, since

$$d_s(v_5, v_4) < d_s(v_5, v_3), \text{ for the neighbour } v_3 \text{ of } v_4.$$

Also, for the vertex v_6, there does not exist a vertex v_j, $j \neq 4, 6$ such that

$$d_s(v_6, v_j) = d_s(v_6, v_4) + d_s(v_4, v_j).$$

Thus v_4 is not an interior vertex and hence v_4 is a null vertex.

Corollary 3.1. The null vertex in a FG G, if it exists, need not be unique.

Theorem 3.5. In the fuzzy wheel graph W_n, there exists a null vertex v_n, the apex vertex connecting the other vertices $v_i, 1 \leq i \leq (n-1)$, when

$$\mu(v_i, v_j) = \begin{cases} a, & j = n, \quad 1 \leq i \leq n-1 \\ b, & \text{otherwise} \end{cases}$$

$$\frac{2a}{n-1} < b < \frac{4a}{n-1}, \text{ when } n \text{ is odd,}$$
$$\frac{2a}{n-2} < b < \frac{4a}{n-2}, \text{ when } n \text{ is even.}$$

Proof. The wheel graph W_n is the join $K_1 + C_{n-1}$. Consider the fuzzy wheel graph $W_n, n \geq 5$ with vertices $v_i, 1 \leq i \leq n-1$, taken in order and v_n be the apex vertex connecting all the other vertices $v_i, 1 \leq i \leq n-1$.

Case (1) When n is odd, $n \geq 5$.

$$\mu(v_i, v_j) = \begin{cases} a, & j = n, \quad 1 \leq i \leq n-1 \\ b, & \text{otherwise} \end{cases}$$

$$\frac{2a}{n-1} < b < \frac{4a}{n-1}$$

Consider the apex vertex v_n. The neighbours of v_n are $v_i, 1 \leq i \leq n-1$.

$$d_s(v_i, v_n) = a$$

$$d_s\left(v_i, v_{i+\left(\frac{n-1}{2}\right)}\right) = \left(\frac{n-1}{2}\right)b, \quad i \leq \frac{n-1}{2}$$

$$d_s\left(v_i, v_{i-\left(\frac{n-1}{2}\right)}\right) = \left(\frac{n-1}{2}\right)b, \quad i > \frac{n-1}{2}$$

Since, $\frac{2a}{n-1} < b$, i.e., $a < \left(\frac{n-1}{2}\right)b$,

$$d_s(v_i, v_n) < \begin{cases} d_s\left(v_i, v_{i+\left(\frac{n-1}{2}\right)}\right), & i \leq \frac{n-1}{2} \\ d_s\left(v_i, v_{i-\left(\frac{n-1}{2}\right)}\right), & i > \frac{n-1}{2}, \end{cases}$$

for the neighbours $v_{i+\frac{n-1}{2}}, v_{i-\left(\frac{n-1}{2}\right)}$ of v_n.

So, v_n is not a boundary vertex of $v_i, 1 \leq i \leq n-1$.

We have, for $1 \leq i, j \leq (n-1), i \neq j$,

$$d_s(v_i, v_n) = d_s(v_n, v_j) = a$$

$$d_s(v_i, v_n) + d_s(v_n, v_j) = a + a = 2a$$

$$\text{Since, } b < \frac{4a}{n-1}, \quad i.e., \left(\frac{n-1}{2}\right) b < 2a$$

$$d_s(v_i, v_j) \leq \left(\frac{n-1}{2}\right) b < 2a$$

$$\text{i.e.,} \quad d_s(v_i, v_j) < 2a.$$

$$\text{So, } d_s(v_i, v_j) \neq d_s(v_i, v_n) + d_s(v_n, v_j).$$

Thus, v_n is not an interior vertex.
Hence, v_n is a null vertex.

Case (2) When n is even, $n \geq 6$.

$$\mu(v_i, v_j) = \begin{cases} a, & j = n, \quad 1 \leq i \leq n-1 \\ b, & \text{otherwise} \end{cases}$$

$$\frac{2a}{n-2} < b < \frac{4a}{n-2}$$

Consider the apex vertex v_n. The neighbours of v_n are $v_i, 1 \leq i \leq n-1$.

$$d_s(v_i, v_n) = a$$

$$d_s\left(v_i, v_{i+\left(\frac{n-2}{2}\right)}\right) = d_s\left(v_i, v_{i+\frac{n}{2}}\right) = \left(\frac{n-2}{2}\right) b, \quad i < \frac{n}{2}$$

$$d_s\left(v_i, v_{i-\left(\frac{n-2}{2}\right)}\right) = d_s\left(v_i, v_{i+\left(\frac{n-2}{2}\right)}\right) = \left(\frac{n-2}{2}\right) b, \quad i = \frac{n}{2}$$

$$d_s\left(v_i, v_{i-\left(\frac{n-2}{2}\right)}\right) = d_s\left(v_i, v_{i-\frac{n}{2}}\right) = \left(\frac{n-2}{2}\right) b, \quad i > \frac{n}{2}$$

Since, $\frac{2a}{n-2} < b$, i.e., $a < \left(\frac{n-2}{2}\right) b$,

$$d_s(v_i, v_n) < \begin{cases} d_s\left(v_i, v_{i+\left(\frac{n-2}{2}\right)}\right) \\ d_s\left(v_i, v_{i+\frac{n}{2}}\right), \end{cases} \quad i < \frac{n}{2},$$

for the neighbours $v_{i+\left(\frac{n-2}{2}\right)}, v_{i+\frac{n}{2}}$ of v_n.

$$d_s(v_i, v_n) < \begin{cases} d_s\left(v_i, v_{i-\left(\frac{n-2}{2}\right)}\right) \\ d_s\left(v_i, v_{i+\left(\frac{n-2}{2}\right)}\right), \end{cases} \quad i = \frac{n}{2},$$

for the neighbours $v_{i-\left(\frac{n-2}{2}\right)}, v_{i+\left(\frac{n-2}{2}\right)}$ of v_n.

$$d_s(v_i, v_n) < \begin{cases} d_s\left(v_i, v_{i-\left(\frac{n-2}{2}\right)}\right) & i > \dfrac{n}{2}, \\ d_s\left(v_i, v_{i-\frac{n}{2}}\right), \end{cases}$$

for the neighbours $v_{i-\left(\frac{n-2}{2}\right)}, v_{i-\frac{n}{2}}$ of v_n.
So, v_n is not a boundary vertex of $v_i, 1 \leq i \leq n-1$.

Also, we have for $1 \leq i, j \leq (n-1), i \neq j$,

$$d_s(v_i, v_n) = d_s(v_n, v_j) = a$$

$$d_s(v_i, v_n) + d_s(v_n, v_j) = a + a = 2a$$

$$\text{Since, } b < \tfrac{4a}{n-2}, \quad \text{i.e.,} \left(\tfrac{n-2}{2}\right) b < 2a$$

$$d_s(v_i, v_j) \leq \left(\tfrac{n-2}{2}\right) b < 2a$$

$$\text{i.e.,} \quad d_s(v_i, v_j) < 2a.$$

$$\text{So, } d_s(v_i, v_j) \neq d_s(v_i, v_n) + d_s(v_n, v_j).$$

i.e., v_n is not an interior vertex.
Hence, v_n is a null vertex. □

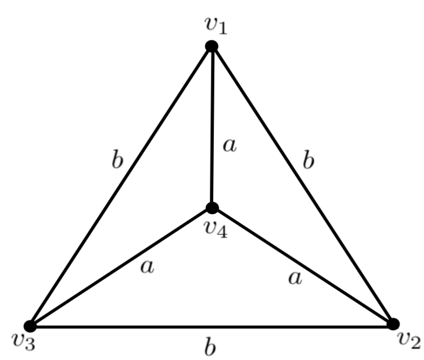

FIGURE 3. Fuzzy Wheel graph W_4

Corollary 3.2. Consider the fuzzy wheel graph $W_n, n = 4$ with

$$\mu(v_i, v_j) = \begin{cases} a, & j = 4, \quad 1 \leq i \leq 3 \\ b, & \text{otherwise} \end{cases}$$

$$a < b < 2a$$

Consider the apex vertex v_4.
The neighbours of v_4 are $v_i, 1 \leq i \leq 3$.
We have $d_s(v_i, v_4) = a, \quad d_s(v_i, v_j) = b, \quad i \neq j, \quad 1 \leq i, j \leq 3$
Since $a < b, \quad d_s(v_i, v_4) < d_s(v_i, v_j)$, for the neighbours $v_j, j = 2, 3$ of v_4.
So, v_4 is not a boundary vertex of $v_i, 1 \leq i \leq 3$.

Also, for $1 \leq i, j \leq 3, \quad i \neq j, d_s(v_i, v_j) = b$,
$d_s(v_i, v_4) + d_s(v_4, v_j) = a + a = 2a$
Since $b < 2a$,
$d_s(v_i, v_j) \neq d_s(v_i, v_4) + d_s(v_4, v_j)$.
So, v_4 is not an interior vertex.
Thus, v_4 is a null vertex.

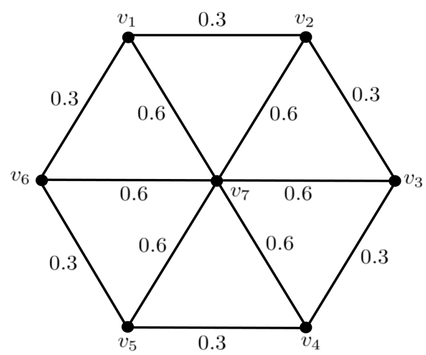

FIGURE 4. Fuzzy Wheel graph W_7

Example 3.3. Consider the fuzzy wheel graph W_n, n is odd. Consider W_7 in figure 4 having vertices $v_i, 1 \leq i \leq 6$ and v_7 as the apex vertex.

$$\mu(v_i, v_j) = \begin{cases} 0.6, & j = 7, \quad 1 \leq i \leq 6 \\ 0.3, & \text{otherwise} \end{cases}$$

Consider the apex vertex v_7.
The neighbours of v_7 are $v_i, 1 \leq i \leq 6$.
Here, $d_s(v_i, v_7) = 0.6$ for $i = 1, 2, ...6$
$d_s(v_1, v_4) = d_s(v_2, v_5) = d_s(v_3, v_6) = d_s(v_4, v_1) = d_s(v_5, v_2) = d_s(v_6, v_3) = 0.9$.
v_7 is not a boundary vertex of v_1 since $d_s(v_1, v_7) < d_s(v_1, v_4)$, for the neighbour v_4 of v_7.
Similarly v_7 is not a boundary vertex of the other vertices $v_i, 2 \leq i \leq 6$.

Also, for $1 \leq i, j \leq 6, \quad i \neq j, \quad d_s(v_i, v_j) \leq 0.9$,
$d_s(v_i, v_7) + d_s(v_7, v_j) = 1.2$
So, $d_s(v_i, v_j) \neq d_s(v_i, v_7) + d_s(v_7, v_j)$ and hence, v_7 is not an interior vertex.
Thus, v_7 is a null vertex.

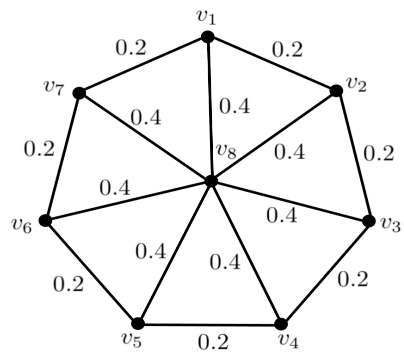

Figure 5. Fuzzy wheel graph W_8

Example 3.4. Consider the fuzzy wheel graph W_n, n is even. Consider W_8 in figure 5 having vertices $v_i, 1 \leq i \leq 7$ with v_8 as the apex vertex.

$$\mu(v_i, v_j) = \begin{cases} 0.4, & j = 8, \quad 1 \leq i \leq 7 \\ 0.2, & \text{otherwise} \end{cases}$$

Consider the apex vertex v_8.
The neighbours of v_8 are $v_i, 1 \leq i \leq 7$.
Here, $d_s(v_i, v_8) = 0.4$ for $i = 1, 2, ...7$
$d_s(v_1, v_4) = d_s(v_1, v_5) = d_s(v_2, v_5) = d_s(v_2, v_6) = d_s(v_3, v_6) = d_s(v_3, v_7) = d_s(v_4, v_1) = d_s(v_4, v_7) = d_s(v_5, v_1) = d_s(v_5, v_2) = d_s(v_6, v_2) = d_s(v_6, v_3) = 0.6$.
v_8 is not a boundary vertex of v_1 since,

$$d_s(v_1, v_8) < \begin{cases} d_s(v_1, v_4) \\ d_s(v_1, v_5), \text{ for the neighbours } v_4 \text{ and } v_5 \text{ of } v_8 \end{cases}$$

Similarly v_8 is not a boundary vertex of the other vertices $v_i, 2 \leq i \leq 7$.

For $1 \leq i, j \leq 7, \quad i \neq j, \quad d_s(v_i, v_j) \leq 0.6$
$d_s(v_i, v_8) + d_s(v_8, v_j) = 0.4 + 0.4 = 0.8$
So, $d_s(v_i, v_j) \neq d_s(v_i, v_8) + d_s(v_8, v_j)$ and hence, v_8 is not an interior vertex.
Thus, v_8 is a null vertex.

Theorem 3.6. In the fuzzy helm graph $H_n, n \geq 4$, there exists a null vertex v_{n+1}, the apex vertex connecting all the other vertices $u_i, v_i, 1 \leq i \leq n$, when

$$\mu(v_i, v_j) = \begin{cases} a, & j = n+1, \quad 1 \leq i \leq n \\ b, & \text{otherwise} \end{cases}$$

$$\mu(u_i, v_i) = b, \quad \forall i, \text{ where,}$$

$$\frac{2a}{n-1} < b < \frac{4a}{n-1}, \text{ when } n \text{ is odd,}$$

$$\frac{2a}{n} < b < \frac{4a}{n}, \text{ when } n \text{ is even.}$$

Proof. Consider the fuzzy helm graph $H_n, n \geq 4$ with vertices $u_i, v_i, 1 \leq i \leq n$ and v_{n+1} is the apex vertex connecting the other vertices.

Case (1) When n is odd, $n \geq 5$.

$$\mu(v_i, v_j) = \begin{cases} a, & j = n+1, \quad 1 \leq i \leq n \\ b, & \text{otherwise} \end{cases}$$

$$\mu(u_i, v_i) = b, \quad \forall i$$

$$\tfrac{2a}{n-1} < b < \tfrac{4a}{n-1}$$

Consider the apex vertex v_{n+1}. The neighbours of v_{n+1} are $v_i, 1 \leq i \leq n$.

$$d_s(v_i, v_{n+1}) = a$$

$$d_s\left(v_i, v_{i+\left(\frac{n-1}{2}\right)}\right) = d_s\left(v_i, v_{i+\left(\frac{n+1}{2}\right)}\right) = \left(\tfrac{n-1}{2}\right)b, \quad i < \tfrac{n+1}{2}$$

$$d_s\left(v_i, v_{i+\left(\frac{n-1}{2}\right)}\right) = d_s\left(v_i, v_{i-\left(\frac{n-1}{2}\right)}\right) = \left(\tfrac{n-1}{2}\right)b, \quad i = \tfrac{n+1}{2}$$

$$d_s\left(v_i, v_{i-\left(\frac{n-1}{2}\right)}\right) = d_s\left(v_i, v_{i-\left(\frac{n+1}{2}\right)}\right) = \left(\tfrac{n-1}{2}\right)b, \quad i > \tfrac{n+1}{2}$$

Since, $\frac{2a}{n-1} < b$, i.e., $a < \left(\frac{n-1}{2}\right)b$,

$$d_s(v_i, v_{n+1}) < \begin{cases} d_s\left(v_i, v_{i+\left(\frac{n-1}{2}\right)}\right) \\ d_s\left(v_i, v_{i+\left(\frac{n+1}{2}\right)}\right), \end{cases} \quad i < \frac{n+1}{2},$$

for the neighbours $v_{i+\left(\frac{n-1}{2}\right)}, v_{i+\left(\frac{n+1}{2}\right)}$ of v_{n+1}.

$$d_s(v_i, v_{n+1}) < \begin{cases} d_s\left(v_i, v_{i+\left(\frac{n-1}{2}\right)}\right) \\ d_s\left(v_i, v_{i-\left(\frac{n-1}{2}\right)}\right), \end{cases} \quad i = \frac{n+1}{2},$$

for the neighbours $v_{i+\left(\frac{n-1}{2}\right)}, v_{i-\left(\frac{n-1}{2}\right)}$ of v_{n+1}.

$$d_s(v_i, v_{n+1}) < \begin{cases} d_s\left(v_i, v_{i-\left(\frac{n-1}{2}\right)}\right) \\ d_s\left(v_i, v_{i-\left(\frac{n+1}{2}\right)}\right), \end{cases} \quad i > \frac{n+1}{2},$$

for the neighbours $v_{i-\left(\frac{n-1}{2}\right)}, v_{i-\left(\frac{n+1}{2}\right)}$ of v_{n+1}.

So, v_{n+1} is not a boundary vertex of $v_i, 1 \leq i \leq n$.

$$d_s(u_i, v_{n+1}) = a + b$$

$$d_s\left(u_i, v_{i+\left(\frac{n-1}{2}\right)}\right) = d_s\left(u_i, v_{i+\left(\frac{n+1}{2}\right)}\right) = \left(\tfrac{n+1}{2}\right)b, \quad i < \tfrac{n+1}{2}$$

$$d_s\left(u_i, v_{i+\left(\frac{n-1}{2}\right)}\right) = d_s\left(u_i, v_{i-\left(\frac{n-1}{2}\right)}\right) = \left(\tfrac{n+1}{2}\right)b, \quad i = \tfrac{n+1}{2}$$

$$d_s\left(u_i, v_{i-\left(\frac{n-1}{2}\right)}\right) = d_s\left(u_i, v_{i-\left(\frac{n+1}{2}\right)}\right) = \left(\frac{n+1}{2}\right)b, \quad i > \frac{n+1}{2}$$

Since $a < \left(\frac{n-1}{2}\right)b$,

$$a + b < \left(\frac{n-1}{2}\right)b + b.$$

i.e., $a + b < (\frac{n+1}{2})b$. Therefore,

$$d_s(u_i, v_{n+1}) < \begin{cases} d_s\left(u_i, v_{i+\left(\frac{n-1}{2}\right)}\right) \\ d_s\left(u_i, v_{i+\left(\frac{n+1}{2}\right)}\right), \end{cases} \quad i < \frac{n+1}{2},$$

for the neighbours $v_{i+\left(\frac{n-1}{2}\right)}, v_{i+\left(\frac{n+1}{2}\right)}$ of v_{n+1}.

$$d_s(u_i, v_{n+1}) < \begin{cases} d_s\left(u_i, v_{i+\left(\frac{n-1}{2}\right)}\right) \\ d_s\left(u_i, v_{i-\left(\frac{n-1}{2}\right)}\right), \end{cases} \quad i = \frac{n+1}{2},$$

for the neighbours $v_{i+\left(\frac{n-1}{2}\right)}, v_{i-\left(\frac{n-1}{2}\right)}$ of v_{n+1}.

$$d_s(u_i, v_{n+1}) < \begin{cases} d_s\left(u_i, v_{i-\left(\frac{n-1}{2}\right)}\right) \\ d_s\left(u_i, v_{i-\left(\frac{n+1}{2}\right)}\right), \end{cases} \quad i > \frac{n+1}{2},$$

for the neighbours $v_{i-\left(\frac{n-1}{2}\right)}, v_{i-\left(\frac{n+1}{2}\right)}$ of v_{n+1}.

So, v_{n+1} is not a boundary vertex of $u_i, 1 \leq i \leq n$.

Thus, v_{n+1} is not a boundary vertex of $H_n, n \geq 5$.

Also, for $1 \leq i, j \leq n, i \neq j$

$$d_s(v_i, v_{n+1}) = d_s(v_{n+1}, v_j) = a$$

$$d_s(v_i, v_{n+1}) + d_s(v_{n+1}, v_j) = a + a = 2a$$

Since $\quad b < \frac{4a}{n-1}, \quad i.e., (\frac{n-1}{2})b < 2a.$

$$d_s(v_i, v_j) \leq \left(\frac{n-1}{2}\right)b < 2a.$$

i.e., $\quad d_s(v_i, v_j) < 2a.$

So, $d_s(v_i, v_j) \neq d_s(v_i, v_{n+1}) + d_s(v_{n+1}, v_j)$.

$$d_s(u_i, v_{n+1}) = d_s(v_{n+1}, u_j) = a + b$$

$$d_s(u_i, v_{n+1}) + d_s(v_{n+1}, u_j) = 2(a + b)$$

Since, $\quad d_s(u_i, u_j) < 2a + 2b$

i.e., $d_s(u_i, u_j) < 2(a + b)$.

$$d_s(u_i, u_j) \neq d_s(u_i, v_{n+1}) + d_s(v_{n+1}, u_j)$$

$$\text{Also, } d_s(u_i, v_{n+1}) = a + b, \quad d_s(v_{n+1}, v_j) = a$$

$$d_s(u_i, v_{n+1}) + d_s(v_{n+1}, v_j) = 2a + b$$

$$\text{But, } d_s(u_i, v_j) < 2a + b$$

$$\text{So, } \quad d_s(u_i, v_j) \neq d_s(u_i, v_{n+1}) + d_s(v_{n+1}, v_j)$$

Thus, v_{n+1} is not an interior vertex.

Hence, v_{n+1} is a null vertex.

Case (2) When n is even, $n \geq 4$.

$$\mu(v_i, v_j) = \begin{cases} a, & j = n+1, \quad 1 \leq i \leq n \\ b, & \text{otherwise} \end{cases}$$

$$\mu(u_i, v_i) = b, \quad \forall i$$

$$\frac{2a}{n} < b < \frac{4a}{n}$$

Consider the apex vertex v_{n+1}. The neighbours of v_{n+1} are $v_i, 1 \leq i \leq n$.

$$d_s(v_i, v_{n+1}) = a$$

$$d_s(v_i, v_{i+\frac{n}{2}}) = (\tfrac{n}{2})b, \quad i \leq \tfrac{n}{2}$$

$$d_s(v_i, v_{i-\frac{n}{2}}) = (\tfrac{n}{2})b, \quad i > \tfrac{n}{2}$$

Since, $\frac{2a}{n} < b$, i.e., $a < \frac{n}{2}b$.

$$d_s(v_i, v_{n+1}) < \begin{cases} d_s(v_i, v_{i+\frac{n}{2}}), i \leq \frac{n}{2} \\ d_s(v_i, v_{i-\frac{n}{2}}), i > \frac{n}{2}, \end{cases}$$

for the neighbours $v_{i+\frac{n}{2}}, v_{i-\frac{n}{2}}$ of v_{n+1}.

So, v_{n+1} is not a boundary vertex of $v_i, 1 \leq i \leq n$.

$$d_s(u_i, v_{n+1}) = a + b$$

$$d_s(u_i, v_{i+\frac{n}{2}}) = (\tfrac{n}{2})b + b = (\tfrac{n}{2} + 1)b, \quad i \leq \tfrac{n}{2}$$

$$d_s(u_i, v_{i-\frac{n}{2}}) = (\tfrac{n}{2} + 1)b, \quad i > \tfrac{n}{2}$$

$$\text{Since } a < \tfrac{n}{2}b, \quad a + b < \tfrac{n}{2}b + b,$$

$$\text{i.e., } a + b < (\tfrac{n}{2} + 1)b,$$

$$d_s(u_i, v_{n+1}) < \begin{cases} d_s(u_i, v_{i+\frac{n}{2}}), & i \leq \frac{n}{2} \\ d_s(u_i, v_{i-\frac{n}{2}}), & i > \frac{n}{2} \end{cases}$$

So, v_{n+1} is not a boundary vertex of $u_i, 1 \leq i \leq n$.

Thus, v_{n+1} is not a boundary vertex of $H_n, n \geq 3$.

Also we have, for $1 \leq i, j \leq n, i \neq j$

$$d_s(v_i, v_{n+1}) = d_s(v_{n+1}, v_j) = a$$

$$d_s(v_i, v_{n+1}) + d_s(v_{n+1}, v_j) = a + a = 2a$$

$$\text{Since,} \quad b < \frac{4a}{n}, \quad i.e., (\tfrac{n}{2})b < 2a,$$

$$d_s(v_i, v_j) \leq (\tfrac{n}{2})b < 2a.$$

$$i.e., \quad d_s(v_i, v_j) < 2a.$$

$$\text{So,} \quad d_s(v_i, v_j) \neq d_s(v_i, v_{n+1}) + d_s(v_{n+1}, v_j).$$

$$d_s(u_i, v_{n+1}) = d_s(v_{n+1}, u_j) = a + b$$

$$d_s(u_i, v_{n+1}) + d_s(v_{n+1}, u_j) = 2(a + b)$$

$$\text{But, } d_s(u_i, u_j) < 2a + 2b$$

$$\text{i.e., } d_s(u_i, u_j) < 2(a + b)$$

$$\text{So,} \quad d_s(u_i, u_j) \neq d_s(u_i, v_{n+1}) + d_s(v_{n+1}, u_j)$$

$$d_s(u_i, v_{n+1}) = a + b, \quad d_s(v_{n+1}, v_j) = a$$

$$d_s(u_i, v_{n+1}) + d_s(v_{n+1}, v_j) = 2a + b$$

$$\text{But, } d_s(u_i, v_j) < 2a + b$$

$$\text{So,} \quad d_s(u_i, v_j) \neq d_s(u_i, v_{n+1}) + d_s(v_{n+1}, v_j)$$

Thus, v_{n+1} is not an interior vertex.

Hence, v_{n+1} is a null vertex.

□

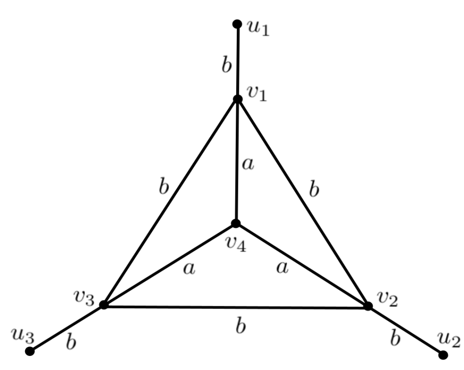

FIGURE 6. Fuzzy Helm graph H_3

Corollary 3.3. Consider the fuzzy helm graph $H_n, n = 3$ with

$$\mu(v_i, v_j) = \begin{cases} a, & j = 4, \quad 1 \le i \le 3 \\ b, & \text{otherwise} \end{cases}$$

$$\mu(u_i, v_i) = b, \quad \forall i$$

$$a < b < 2a$$

Consider the apex vertex v_4.
The neighbours of v_4 are $v_i, 1 \le i \le 3$.
We have $d_s(v_i, v_4) = a, \quad d_s(v_i, v_j) = b, 1 \le i, j \le 3, i \ne j$.
Since $a < b, \quad d_s(v_i, v_4) < d_s(v_i, v_j)$, for the neighbours $v_j, j = 2, 3$ of v_4.
i.e., v_4 is not a boundary vertex of $v_i, 1 \le i \le 3$,

We have $d_s(u_i, v_4) = a + b, \quad d_s(u_i, v_j) = 2b$.
Since $a < b, \quad a + b < 2b$
$d_s(u_i, v_4) < d_s(u_i, v_j)$, for the neighbours $v_j, j = 2, 3$ of v_4.
i.e., v_4 is not a boundary vertex of $u_i, 1 \le i \le 3$.

Also, for $1 \le i, j \le 3, \quad i \ne j$
$d_s(v_i, v_j) = b$.
$d_s(v_i, v_4) + d_s(v_4, v_j) = a + a = 2a$.
Since $b < 2a$,
$d_s(v_i, v_j) \ne d_s(v_i, v_4) + d_s(v_4, v_j)$.

$d_s(u_i, v_j) = b + b = 2b$.
$d_s(u_i, v_4) + d_s(v_4, v_j) = a + b + a = 2a + b$.
Since $b < 2a, \quad i.e., 2b < 2a + b$,
$d_s(u_i, v_j) \ne d_s(u_i, v_4) + d_s(v_4, v_j)$.

$d_s(u_i, u_j) = 3b$,
$d_s(u_i, v_4) + d_s(v_4, u_j) = 2(a + b)$,
Since $b < 2a, \quad i.e., 3b < 2(a + b)$,
$d_s(u_i, u_j) \ne d_s(u_i, v_4) + d_s(v_4, u_j)$.

So, v_4 is not an interior vertex.
Thus, v_4 is a null vertex.

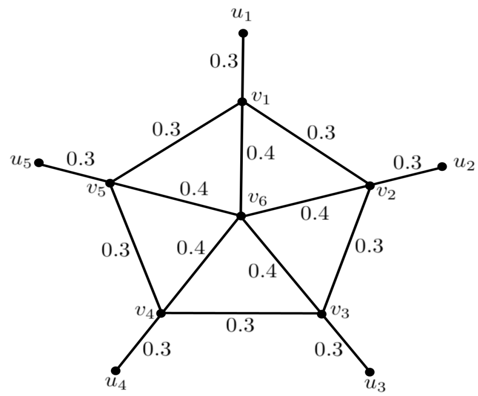

FIGURE 7. Fuzzy Helm graph H_5

Example 3.5. Consider the fuzzy helm graph H_n, n is odd.
Consider H_5 in figure 7 having vertices $u_i, v_i, i = 1, 2, ...5$ and v_6 as the apex vertex.

$$\mu(v_i, v_j) = \begin{cases} 0.4, & j = 6, \quad 1 \leq i \leq 5 \\ 0.3, & \text{otherwise} \end{cases}$$

$$\mu(u_i, v_i) = 0.3, \quad \forall i$$

Consider the apex vertex v_6.
The neighbours of v_6 are $v_i, 1 \leq i \leq 5$
$d_s(v_i, v_6) = 0.4$ for $i = 1, 2, ...5$
$d_s(v_1, v_3) = d_s(v_1, v_4) = 0.6.$

$$d_s(v_1, v_6) < \begin{cases} d_s(v_1, v_3) \\ d_s(v_1, v_4), & \text{for the neighbours } v_3 \text{ and } v_4 \text{ of } v_6 \text{ .} \end{cases}$$

So, v_6 is not a boundary vertex of v_1.
Similarly v_6 is not a boundary vertex of the vertices $v_i, 2 \leq i \leq 5$.

$d_s(u_i, v_6) = 0.7$ for $i = 1, 2, ...5$
$d_s(u_1, v_3) = d_s(u_1, v_4) = 0.9$

$$d_s(u_1, v_6) < \begin{cases} d_s(u_1, v_3) \\ d_s(u_1, v_4), & \text{for the neighbours } v_3 \text{ and } v_4 \text{ of } v_6, \end{cases}$$

So, v_6 is not a boundary vertex of u_1.
Similarly v_6 is not a boundary vertex of the vertices $u_i, 2 \leq i \leq 5$.

For $i \neq j, 1 \leq i, j \leq 5$,
Since, $d_s(v_i, v_j) \leq 0.6, \quad d_s(v_i, v_6) + d_s(v_6, v_j) = 0.8,$

$d_s(v_i, v_j) \neq d_s(v_i, v_6) + d_s(v_6, v_j)$.

Since, $d_s(u_i, v_j) \leq 0.9, \quad d_s(u_i, v_6) + d_s(v_6, v_j) = 1.1$,

$d_s(u_i, v_j) \neq d_s(u_i, v_6) + d_s(v_6, v_j)$.

Since, $d_s(u_i, u_j) \leq 1.2, \quad d_s(u_i, v_6) + d_s(v_6, u_j) = 1.4$,

$d_s(u_i, u_j) \neq d_s(u_i, v_6) + d_s(v_6, u_j)$.

Thus, v_6 is not an interior vertex.

Hence, v_6 is a null vertex.

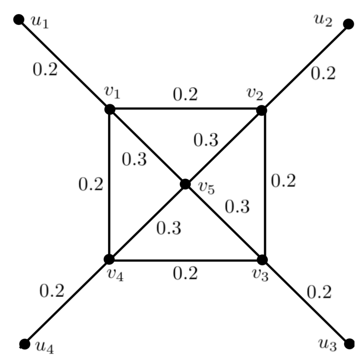

FIGURE 8. Fuzzy Helm graph H_4

Example 3.6. Consider the fuzzy helm graph H_n, n is even.

Consider H_4 in figure 8 having vertices $u_i, v_i, i = 1, 2, ...4$ and v_5 as the apex vertex.

$$\mu(v_i, v_j) = \begin{cases} 0.3, & j = 5, \quad 1 \leq i \leq 4 \\ 0.2, & \text{otherwise} \end{cases}$$

$$\mu(u_i, v_i) = 0.2, \quad \forall i$$

Consider the apex vertex v_5.

The neighbours of v_5 are $v_i, 1 \leq i \leq 4$

Here, $d_s(v_i, v_5) = 0.3$ for $i = 1, 2, ...4$

$d_s(v_1, v_3) = 0.4$

v_5 is not a boundary vertex of v_1 since, $d_s(v_1, v_5) < d_s(v_1, v_3)$, for the neighbour v_3 of v_5.

Similarly v_5 is not a boundary vertex of the vertices $v_i, 2 \leq i \leq 4$.

$d_s(u_i, v_5) = 0.5$ for $i = 1, 2, ...4$

$d_s(u_1, v_3) = 0.6$

v_5 is not a boundary vertex of u_1 since, $d_s(u_1, v_5) < d_s(u_1, v_3)$, for the neighbour v_3 of v_5

Similarly v_5 is not a boundary vertex of the vertices $u_i, 2 \leq i \leq 4$.

Since, $d_s(v_i, v_j) \leq 0.4, \quad d_s(v_i, v_5) + d_s(v_5, v_j) = 0.6$,

$d_s(v_i, v_j) \neq d_s(v_i, v_5) + d_s(v_5, v_j)$.

$d_s(u_i, v_j) \leq 0.6, \quad d_s(u_i, v_5) + d_s(v_5, v_j) = 0.5 + 0.3 = 0.8$,

$d_s(u_i, v_j) \neq d_s(u_i, v_5) + d_s(v_5, v_j)$.

$d_s(u_i, u_j) \leq 0.8, \quad d_s(u_i, v_5) + d_s(v_5, u_j) = 0.5 + 0.5 = 1,$

$d_s(u_i, u_j) \neq d_s(u_i, v_5) + d_s(v_5, u_j)$.

Thus, v_5 is not an interior vertex.

Hence, v_5 is a null vertex.

4. Conclusions

We presented the novel idea of null vertex in FGs, which is a vertex distinct from boundary vertex and interior vertex. Null vertex is a vertex which is neither a boundary vertex nor an interior vertex. We investigated that null vertex in a fuzzy graph if it exists need not be unique. We established the presence of null vertex in some graphs such as complete FGs, fuzzy cycles, fuzzy wheel graphs and fuzzy helm graphs.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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