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Bipolar Fuzzy Almost Quasi-Ideals in Semigroups

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Abstract. The aim of paper, we give the concept bipolar fuzzy, almost quasi-ideal in semigroups. We present the properties of bipolar fuzzy, almost quasi-ideals in semigroups. Moreover, we prove the relationship between almost quasi-ideals and bipolar fuzzy quasi-ideals in semigroups.

1. Introduction

The theory fuzzy sets are a kind of proper mathematical structure to represent a collection of objects whose boundary is vague, which was studied by Zaden in 1965 [10]. In 1994 Zhang [11] extended the concept of the fuzzy set to bipolar fuzzy sets, which is an extension of fuzzy sets whose membership degree range is $[-1, 0] \cup [0, 1]$. A bipolar fuzzy set is the membership degree of an element means that the element is irrelevant to the related property, the membership degree of an element indicates that the element somewhat satisfies the property, and the membership degree of an element indicates that the element somewhat satisfies the implicit counter-property. The ideals, introduced by her are still central concepts in ring theory, and the notion of a one-sided ideal of any algebraic structure is a generalization of the notion of an ideal. The almost ideal theory in semigroups was studied by Grosek and Satko in 1980 [2]. In 1981, Bogdanvic, [3] established definitions of almost bi-ideals in semigroups and studied properties of almost bi-ideals in semigroups. Later, Chinram gives definition definitions of the types of almost ideals in semigroups such that almost quasi-ideal [9], almost i-ideal, (m, n)-almost ideal. In 2019, Murugadas et al. [7] studied fuzzy almost quasi-ideals in semigroup. They proved the basic properties of almost quasi-ideals

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and fuzzy almost quasi-ideals in semigroups. In 2020, Chinram et al. [4] introduced almost interior and weakly almost interior ideals in semigroups and studied the properties of its. Recently N. Sarasit et al. [8] studied almost ternary subsemirings.

In this paper, we define bipolar fuzzy almost quasi-ideals in semigroup, which it is developed to study on bipolar fuzzy sets. We investigate the fundamental properties of bipolar fuzzy almost quasi-ideals in semigroups.

2. Preliminaries

In this section we give the concepts and results, which will be helpful in later sections. A *subsemigroup* of a semigroup E is a non-empty set K of E such that $KK \subseteq K$. A *left (right) ideal* of a semigroup E is a non-empty set K of E such that $EK \subseteq K$ ($KE \subseteq K$). By an *ideal* of a semigroup E, we mean a non-empty set of E which is both a left and a right ideal of E. A subsemigroup K of a semigroup E is called a *bi-ideal* of S if $KEK \subseteq K$. A subsemigroup K of a semigroup E is called a *quasi-ideal* if $KS \cap SK \subseteq K$. An *almost ideal* K of a semigroup E if $tK \cap K \neq \emptyset$ and $Kr \cap K \neq \emptyset$ for all $t, r \in E$. An *almost quasi-ideal* K of a semigroup E if $(Ks \cap sK) \cap K \neq \emptyset$ for all $s \in S$.

Theorem 2.1. *[7] Every quasi-ideal of a semigroup E is an almost quasi-ideal of E.*

For any $h_i \in [0,1]$, $i \in \mathcal{J}$, define

$$\bigvee_{i\in\mathcal{J}} h_i := \sup_{i\in\mathcal{J}}\{h_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{J}} h_i := \inf_{i\in\mathcal{J}}\{h_i\}.$$

We see that for any $h, r \in [0,1]$, we have

$$h \vee r = \max\{h, r\} \quad \text{and} \quad h \wedge r = \min\{h, r\}.$$

A **fuzzy set (fuzzy subset)** of a non-empty set E is a function
$\vartheta : E \to [0,1]$.

For any two fuzzy sets ϑ and ξ of a non-empty set E, define the symbol as follows:

(1) $\vartheta \geq \xi \Leftrightarrow \vartheta(h) \geq \xi(h)$ for all $h \in E$,
(2) $\vartheta = \xi \Leftrightarrow \vartheta \geq \xi$ and $\xi \geq \vartheta$,
(3) $(\vartheta \wedge \xi)(h) = \min\{\vartheta(h), \xi(h)\} = \vartheta(h) \wedge \xi(h)$ for all $h \in E$,
(4) $(\vartheta \vee \xi)(h) = \max\{\vartheta(h), \xi(h)\} = \vartheta(h) \vee \xi(h)$ for all $h \in E$.
(5) $\vartheta \subseteq \xi$ if $\vartheta(h) \leq \xi(h)$,
(6) $(\vartheta \cup \xi)(h) = \max\{\vartheta(h), \xi(h)\}$ and $(\vartheta \cap \xi)(h) = \min\{\vartheta(h), \xi(h)\}$ for all $h \in E$.
(7) the *support* of ϑ instead of $\mathrm{supp}(\vartheta) = \{h \in E \mid \vartheta(h) \neq 0\}$ For the symbol $\vartheta \leq \xi$, we mean $\xi \geq \vartheta$.

Definition 2.1. *[6] A **bipolar fuzzy set** (BF set) ϑ on a non-empty set E is an object having the form*

$$\vartheta := \{(h, \vartheta^p(h), \vartheta^n(h)) \mid h \in E\},$$

where $\vartheta^p : E \to [0,1]$ and $\vartheta^n : E \to [-1,0]$.

Remark 2.1. *For the sake of simplicity we shall use the symbol $\vartheta = (E; \vartheta^p, \vartheta^n)$ for the BF set $\vartheta = \{(h, \vartheta^p(h), \vartheta^n(h)) \mid h \in E\}$.*

The following example of a BF set.

Example 2.1. *Let $E = \{41, 42, 43...\}$. Define $\vartheta^p : E \to [0, 1]$ is a function*

$$\vartheta^p(u) = \begin{cases} 0 & \text{if } h \text{ is old number} \\ 1 & \text{if } h \text{ is even number} \end{cases}$$

and $\vartheta^n : E \to [-1, 0]$ is a function

$$\vartheta^n(u) = \begin{cases} -1 & \text{if } h \text{ is old number} \\ 0 & \text{if } h \text{ is even number.} \end{cases}$$

Then $\vartheta = (E; \vartheta^p, \vartheta^n)$ is a BF set.

For $h \in E$, define $F_h = \{(h_1, h_2) \in E \times E \mid h = h_1h_2\}$.
Define products $\vartheta^p \circ \xi^p$ and $\vartheta^n \circ \xi^n$ as follows: For $h \in E$

$$(\vartheta^p \circ \xi^p)(h) = \begin{cases} \bigvee_{(h_1,h_2)\in F_h} \{\vartheta^p(h_1) \wedge \xi^p(h_2)\} & \text{if } h = h_1h_2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\vartheta^n \circ \xi^n)(h) = \begin{cases} \bigwedge_{(h_1,h_2)\in F_h} \{\vartheta^n(h_1) \vee \xi^n(h_2)\} & \text{if } h = h_1h_2 \\ 0 & \text{if otherwise.} \end{cases}$$

Definition 2.2. *[6] A non-empty set K of a semigroup E. A positive characteristic function and a negative characteristic function are respectively defined by*

$$\geq^p_K : E \to [0, 1], h \mapsto \geq^p_K(h) := \begin{cases} 1 & h \in K, \\ 0 & h \notin K, \end{cases}$$

and

$$\geq^n_K : E \to [-1, 0], h \mapsto \geq^n_K(h) := \begin{cases} -1 & h \in K, \\ 0 & h \notin K. \end{cases}$$

Remark 2.2. *For the sake of simplicity we shall use the symbol $\geq_K = (E; \geq^p_K, \geq^n_K)$ for the BF set $\geq_K := \{(h, \geq^p_K(h), \geq^n_K(h)) \mid h \in K\}$.*

Lemma 2.1. *[5] Let K and M be non-empty subsets of a semigroup E. Then the following holds.*

(1) $\lambda^p_K \wedge \lambda^p_M = \lambda^p_{K\cap M}$.
(2) $\lambda^n_K \vee \lambda^n_M = \lambda^p_{K\cup M}$.
(3) $\lambda^p_K \circ \lambda^p_M = \lambda^n_{KM}$.
(4) $\lambda^n_K \wedge \lambda^n_M = \lambda^n_{KM}$.

For $h \in E$ and $(t,s) \in [0,1] \times [-1,0]$, a BF point $h_{(t,s)} = (E; x_t^p, x_s^n)$ of a set E is a BF set of E defined by

$$x_t^p(h) = \begin{cases} t & \text{if } h = x \\ 0 & \text{if } h \neq x \end{cases}$$

and

$$x_s^n(h) = \begin{cases} s & \text{if } h = x \\ 0 & \text{if } h \neq x. \end{cases}$$

Definition 2.3. *[5] A BF set $\vartheta = (E; \vartheta^p, \vartheta^n)$ on a semigroup E is called a BF subsemigroup on E if it satisfies the following conditions: $\vartheta^p(hr) \geq \vartheta^p(h) \wedge \vartheta^p(r)$ and $\vartheta^n(hr) \leq \vartheta^n(h) \vee \vartheta^n(r)$ for all $h, r \in E$.*

The following example of a BF subsemigroup.

Example 2.2. *Let E be a semigroup defined by the following table:*

·	a	b	c	d	e
a	a	a	a	a	a
a	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

Define a BF set $\vartheta = (E; \vartheta^p, \vartheta^n)$ on E as follows :

E	a	b	c	d	e
ϑ^p	*0.9*	*0.8*	*0.5*	*0.3*	*0.3*
ϑ^n	-0.8	-0.8	-0.6	-0.5	-0.3

Then $\vartheta = (E; \vartheta^p, \vartheta^n)$ is a BF subsemigroup.

Definition 2.4. *[5] A BF set $\vartheta = (E; \vartheta_p, \vartheta_n)$ on a semigroup E is called a BF left (right) ideal on E if it satisfies the following conditions: $\vartheta^p(hr) \geq \vartheta^p(r)$ $(\vartheta^p(hr) \geq \vartheta^p(h))$ and $\vartheta^n(hr) \leq \vartheta^n(r)$ $(\vartheta^n(hr) \leq \vartheta^n(h))$ for all $h, r \in E$.*

Definition 2.5. *[5] A BF set $\vartheta = (E; \vartheta_p, \vartheta_n)$ on a semigroup E is called a BF quasi-ideal on E if it satisfies the following conditions: $(\vartheta^p \circ \lambda_S^p) \wedge (\lambda_S^p \circ \vartheta^p) \geq \vartheta^p$ and $(\vartheta^n \circ \lambda_S^n) \wedge (\lambda_S^n \circ \vartheta^p) \leq \vartheta^p$.*

3. Main Results

In this section, we give define the bipolar fuzzy almost quasi-ideal in semigroups and we investigate properties of bipolar fuzzy almost qausi-ideal in semigroups.

Definition 3.1. *A BF set $\vartheta = (E; \vartheta^p, \vartheta^n)$ on a semigroup E is called a* BF almost quasi-ideal *of E if $(\vartheta^p \circ x_t^p \wedge x_t^p \circ \vartheta^p) \wedge \vartheta^p \neq 0$ and $(\vartheta^n \circ x_s^n \vee x_s^n \circ \vartheta^n) \vee \vartheta^n \neq 0$, for any BF point $x_t^p, x_s^n \in E$.*

Theorem 3.1. *Let $\vartheta = (E; \vartheta^p, \vartheta^n)$ and $\xi = (E; \xi^p, \xi^n)$ be a BF subset of a semigroup E such that $\vartheta \subseteq \xi$. If ϑ is a BF almost quasi-ideal of E, then ξ is also a BF almost quasi-ideal of E.*

Proof. Supppose that ϑ is a BF almost quasi-ideal of E and let x_t^p, x_s^n be BF points of E. By assumption, $(\vartheta^p \circ x_t^p \wedge x_t^p \circ \vartheta^p) \wedge \vartheta^p \neq 0$ and $(\vartheta^n \circ x_s^n \vee x_s^n \circ \vartheta^n) \vee \vartheta^n \neq 0$. Sine $\vartheta \subseteq \xi$ we have

$$(\vartheta^p \circ x_t^p \wedge x_t^p \circ \vartheta^p) \wedge \vartheta^p \subseteq (\xi^p \circ x_t^p \wedge x_t^p \circ \xi^p) \wedge \xi^p \neq 0$$

and

$$(\vartheta^n \circ x_s^n \vee x_s^n \circ \vartheta^n) \vee \vartheta^n \subseteq (\xi^n \circ x_s^n \vee x_s^n \circ \xi^n) \vee \xi^n \neq 0.$$

Thus, $(\xi^p \circ x_t^p \wedge x_t^p \circ \xi^p) \wedge \xi^p \neq 0$ and $(\xi^n \circ x_s^n \vee x_s^n \circ \xi^n) \vee \xi^n \neq 0$.
Hence, ξ is a BF almost quasi-ideal of E. □

Theorem 3.2. *Let $\vartheta = (E; \vartheta^p, \vartheta^n)$ and $\xi = (E; \xi^p, \xi^n)$ be BF almost quasi-ideals of a semigroup E. Then $\vartheta \vee \xi$ is also a BF almost quasi-ideal of E.*

Proof. Since $\vartheta \subseteq \vartheta \vee \xi$, by Theorem 3.1, $\vartheta \vee \xi$ is a BF almost quasi-ideal of E. □

Example 3.1. *Consider the semigroup $\mathbb{Z}_4$ under the addition.*
Define $\vartheta^p : \mathbb{Z} \to [0,1]$ and $\vartheta^n : \mathbb{Z} \to [-1,0]$ are defined by $\vartheta^p(0) = 0$, $\vartheta^p(1) = 0.4$, $\vartheta^p(2) = 0.2$, $\vartheta^p(3) = 0.4$ and $\vartheta^n(0) = 0$, $\vartheta^n(1) = -0.4$, $\vartheta^n(2) = -0.2$, $\vartheta^n(3) = -0.4$. Define $\xi^p : \mathbb{Z} \to [0,1]$ and $\xi^n : \mathbb{Z} \to [-1,0]$ are defined by $\xi^p(0) = 0.3$, $\xi^p(1) = 0.2$, $\xi^p(2) = 0$, $\xi^p(3) = 0.3$ and $\xi^n(0) = -0.3$, $\xi^n(1) = -0.2$, $\xi^n(2) = 0$, $\xi^n(3) = -0.3$. Then $\vartheta = (\mathbb{Z}_4; \vartheta^p, \vartheta^n)$ and $\xi = (\mathbb{Z}_4; \xi^p, \xi^n)$ are BF almost quasi-ideal of $\mathbb{Z}_4$. But $(\vartheta^p \wedge \xi^p)$ and $(\vartheta^n \vee \xi^n)$ are not a BF almost quasi-ideal of $\mathbb{Z}_4$.

Theorem 3.3. *Let K be a nonempty subset of a semigroup E. Then K is an almost quasi-ideal of E if and only if $\geq_K = (E; \geq_K^p, \geq_K^n)$ is a BF almost quasi-ideal of E.*

Proof. Suppose that K is an almost quasi-ideal of a semigroup E. Then $(Ks \cap sK) \cap K \neq \emptyset$ for all $s \in E$. Thus there exists $r \in E$ such that $c \in (Ks \cap sK) \cap K$. Let $x \in E$ and $t \in (0,1]$ and $s \in [-1,0)$. Then $(x_t^p \circ \geq_K^p \wedge \geq_K^p \circ x_t^p)(r) = 1$, $(x_s^n \circ \geq_K^n \vee \geq_K^n \circ x_s^n)(r) = -1$ and $\geq_K^p(r) = 1$, $\geq_K^n(c) = -1$. Thus, $(x_t^p \circ \geq_K^p \wedge \geq_K^p \circ x_t^p) \wedge \geq_K^p \neq 0$ and $(x_s^n \circ \geq_K^n \vee \geq_K^n \circ x_s^n) \vee \geq_K^n \neq 0$. Hence, $\geq_K = (E; \geq_K^p, \geq_K^n)$ is a BF almost quasi-ideal of E.

Conversely, suppose that $\geq_K = (E; \geq_K^p, \geq_K^n)$ is a BF almost quasi-ideal of E and let $x \in E$ and $t \in (0,1]$ and $s \in [-1,0)$. Then $(x_t^p \circ \geq_K^p \wedge \geq_K^p \circ x_t^p) \wedge \geq_K^p \neq 0$ and $(x_s^n \circ \geq_K^n \vee \geq_K^n \circ x_s^n) \vee \geq_K^n \neq 0$. Thus, there exists $r \in E$ such that $[(x_t^p \circ \geq_K^p \wedge \geq_K^p \circ x_t^p) \wedge \geq_K^p](r) \neq 0$ and $[(x_s^n \circ \geq_K^n \vee \geq_K^n \circ x_s^n) \vee \geq_K^n](r) \neq 0$. Hence $r \in (Ks \cap sK) \cap K$.
So $(Ks \cap sK) \cap K \neq \emptyset$. We conclude that K is an almost quasi-ideal of E. □

Theorem 3.4. *Let $\vartheta = (E; \vartheta^p, \vartheta^n)$ be a BF subset of a semigroup E. Then $\vartheta = (E; \vartheta^p, \vartheta^n)$ is a BF almost quasi-ideal of E if and only if* $\operatorname{supp}(\vartheta)$ *is an almost quasi-ideal of E.*

Proof. Assume that $\vartheta = (E; \vartheta^p, \vartheta^n)$ is a BF almost quasi-ideal of E and let $x \in E$ and $t \in (0,1]$ and $s \in [-1,0)$. Then $(\vartheta^p \circ x_t^p \wedge x_t^p \circ \vartheta^p) \wedge \vartheta^p \neq 0$ and $(\vartheta^n \circ x_s^n \vee x_s^n \circ \vartheta^n) \vee \vartheta^n \neq 0$. Thus there exists $z \in E$ such that $(x_t^p \circ \vartheta^p \wedge \vartheta^p \circ x_t^p)(z) \neq 0$ and $(x_s^n \circ \vartheta^n \vee \vartheta^n \circ x_s^n)(z) \neq 0$. So $\vartheta^p(z) \neq 0$ and $\vartheta^n(z) \neq 0$, there exists $w_1, w_2 \in E$ such that $z = w_1 w_2$ and $\vartheta^p(w_1) \neq 0$, $\vartheta^p(w_2) \neq 0$ and $\vartheta^n(w_1) \neq 0$, $\vartheta^p(w_2) \neq 0$. Thus, $(x_t^p \circ \geq^p_{supp(\vartheta)} \wedge \geq^p_{\mathrm{supp}(\vartheta)} \circ x_t^p)(z) \neq 0$ and $(x_s^n \circ \geq^n_{supp(\vartheta)} \vee \geq^n_{\mathrm{supp}(\vartheta)} \circ x_s^n)(z) \neq 0$. Hence, $(x_t^p \circ \geq^p_{supp(\vartheta)} \wedge \geq^p_{\mathrm{supp}(\vartheta)} \circ x_t^p) \wedge \geq^p_{supp(\vartheta)} \neq 0$ and $(x_s^n \circ \geq^n_{\mathrm{supp}(\vartheta)} \vee \geq^n_{supp(\vartheta)} \circ x_s^n) \vee \geq^n_{supp(\vartheta)} \neq 0$. Therefore, $\geq^p_{\mathrm{supp}(\vartheta)}$ is a BF almost quasi-ideal of E. By Theorem 3.3, $\mathrm{supp}(\vartheta)$ is an almost quasi-ideal of E.

Conversely, suppose that $\mathrm{supp}(\vartheta)$ is an almost quasi-ideal of E. By Theorem 3.3, $\geq^p_{\mathrm{supp}(\vartheta)}$ is a BF almost quasi-ideal of E. Then for any BF points $x_t^p, x_s^n \in E$, we have $(x_t^p \circ \geq^p_{supp(\vartheta)} \wedge \geq^p_{supp(\vartheta)} \circ x_t^p) \wedge \geq^p_{\mathrm{supp}(\vartheta)} \neq 0$ and $(x_s^n \circ \geq^n_{\mathrm{supp}(\vartheta)} \vee \geq^n_{\mathrm{supp}(\vartheta)} \circ x_s^n) \vee \geq^n_{\mathrm{supp}(\vartheta)} \neq 0$. Thus there exists $c \in E$ such that $[(x_t^p \circ \geq^p_{\mathrm{supp}(\vartheta)} \wedge \geq^p_{\mathrm{supp}(\vartheta)} \circ x_t^p) \wedge \geq^p_{\mathrm{supp}(\vartheta)}](c) \neq 0$ and $[(x_s^n \circ \geq^n_{\mathrm{supp}(\vartheta)} \vee \geq^n_{\mathrm{supp}(\vartheta)} \circ x_s^n) \vee \geq^n_{\mathrm{supp}(\vartheta)}](c) \neq 0$. Hence, $(x_t^p \circ \geq^p_{\mathrm{supp}(\vartheta)} \wedge \geq^p_{\mathrm{supp}(\vartheta)} \circ x_t^p)(c) = 0, \geq^p_{\mathrm{supp}(\vartheta)}(c) \neq 0$ and $(x_s^n \circ \geq^n_{\mathrm{supp}(\vartheta)} \vee \geq^n_{\mathrm{supp}(\vartheta)} \circ x_s^n)(c) = 0$, $\geq^n_{\mathrm{supp}(\vartheta)}(c) \neq 0$. Then there exists $w_1, w_2 \in \mathrm{supp}(\vartheta)$ such that $c = w_1 w_2$. Thus $\vartheta^p(w_1) \neq 0, \vartheta^p(w_2) \neq 0$ and $\vartheta^n(w_1) \neq 0, \vartheta^n(w_2) \neq 0$. So $(\vartheta^p \circ \geq^p_K \wedge \geq^p_K \circ \vartheta^p) \wedge \geq^p_K \neq 0$ and $(\vartheta^n \circ \geq^n_K \vee \geq^n_K \circ \vartheta^n) \vee \geq^n_K \neq 0$. Therefore, $\vartheta = (E; \vartheta^p, \vartheta^n)$ is a BF almost quasi-ideal of E. □

Next, we investigate minimal BF almost quasi-ideals in semigroups and study relationships between minimal almost quasi-ideals and minimal BF almost quasi-ideals of semigroups.

Definition 3.2. *An almost quasi-ideal K of a semigroup E is said* minimal *if for any almost quasi-ideal M of E if whenever $M \subseteq K$, then $M = K$.*

Definition 3.3. *A BF almost quasi-ideal $\vartheta = (E; \vartheta^p, \vartheta^n)$ of a semigroup E is said* minimal *if for any BF almost quasi-ideal $\xi = (E; \xi^p, \xi^n)$ of E if whenever $\xi \subseteq \vartheta$, then* $\mathrm{supp}(\xi) = \mathrm{supp}(\vartheta)$.

Theorem 3.5. *Let K be a nonempty subset of a semigroup E. Then K is a minimal almost quasi-ideal of E if and only if $\geq_K = (E; \geq^p_K, \geq^n_K)$ is a minimal BF almost quasi-ideal of E.*

Proof. Assume that K is a minimal almost quasi-ideal of E. Then K is an almost quasi-ideal of E. Thus by Theorem 3.3, $\geq_K = (E; \geq^p_K, \geq^n_K)$ is a BF almost quasi-ideal of E. Let $\xi = (E; \xi^p, \xi^n)$ be a BF almost quasi-ideal of E such that $\xi \subseteq \geq_K$. Then $\mathrm{supp}(\xi) \subseteq \mathrm{supp}(\geq_K) = K$. By Theorem 3.4, $\mathrm{supp}(\xi)$ is an almost quasi-ideal of E. Since K is minimal we have $\mathrm{supp}(\xi) = K = \mathrm{supp}(\geq_K)$. Therefore, $\geq_K = (E; \geq^p_K, \geq^n_K)$ is minimal BF almost quasi-ideal of E.

Conversely, suppose that $\geq_K = (E; \geq^p_K, \geq^n_K)$ is a minimal BF almost quasi-ideal of E. Then $\geq_K = (E; \geq^p_K, \geq^n_K)$ is a BF almost quasi-ideal of E. Thus by Theorem 3.3, K is an almost quasi-ideal of E. Let M be an almost quasi-ideal of E such that $M \subseteq K$. Then $\geq_M$ is a BF almost quasi-ideal of E such that $\geq_M \subseteq \geq_K$. Thus $\mathrm{supp}(\geq_M) \subseteq \mathrm{supp}(\geq_K)$. Since $\geq_K = (E; \geq^p_K, \geq^n_K)$ is a minimal BF almost quasi-ideal of E we have $\mathrm{supp}(\geq_M) = \mathrm{supp}(\geq_K)$. Thus $M = \mathrm{supp}(\geq_M) = \mathrm{supp}(\geq_K) = K$. Hence K is minimal almost quasi-ideal of E. □

Corollary 3.1. *Let E be a semigroup. Then E has no proper almost quasi-ideal if and only if* $\mathrm{supp}(\vartheta) = E$ *for every BF almost quasi-ideal $\vartheta = (E; \vartheta^p, \vartheta^n)$ of E.*

Proof. Suppose that E has no proper almost quasi-ideal and let $\vartheta = (E; \vartheta^p, \vartheta^n)$ be a BF almost quasi-ideal of E. Then by Theorem 3.4, $\mathrm{supp}(\vartheta)$ is an almost quasi-ideal of E. By assumption, $\mathrm{supp}(\vartheta) = E$.

Conversely, suppose that $\mathrm{supp}(\vartheta) = E$ and M is a proper almost quasi-ideal of E. Then by Theorem 3.3, $\geq_M = (E; \geq^p_M, \geq^n_K)$ is a BF almost quasi-ideal of E. Thus $\mathrm{supp}(\geq_M) = M \neq E$. It is a contradiction. Hence E has no proper almost quasi-ideal. □

We give definition of prime (resp., semiprime, strongly prime) almost quasi-ideal and prime (resp., semiprime strongly prime) BF almost quasi-ideal. We study the relationships between prime (resp., semiprime strongly prime) almost quasi-ideals and their bioplar fuzzification of semigroups

Definition 3.4. *Let K be an almost quasi-ideal of semigroup E. Then we called*

(1) *K is a **prime** if for any almost quasi-ideals M and L of E such that $ML \subseteq K$ implies that $M \subseteq K$ or $L \subseteq K$.*
(2) *K is a **semiprime** if for any almost quasi-ideal M of E such that $M^2 \subseteq K$ implies that $M \subseteq K$.*
(3) *K is a **strongly prime** if for any almost quasi-ideals M and L of E such that $ML \cap LM \subseteq K$ implies that $M \subseteq K$ or $L \subseteq K$.*

Definition 3.5. *A BF almost quasi-ideal $\vartheta = (E; \vartheta^p, \vartheta^n)$ on a semigroup E. Then we called*

(1) *ϑ is a **prime** if for any two BF almost quasi-ideals $\xi = (E; \xi^p, \xi^{n)}$ and $\nu = (E; \nu^p, \nu^n)$ of E such taht $\xi^p \circ \nu^p$*
$leq \vartheta^p$ and $\xi^n \circ \nu^n \geq \vartheta^n$ implies that $\xi^p \leq \vartheta^p$ and $\xi^n \geq \vartheta^n$ or $\nu^p \leq \vartheta^p$ and $\nu^n \geq \vartheta^n$.
(2) *ϑ is a **semiprime** if for any BF almost quasi-ideal $\xi = (E; \xi^p, \xi^n)$ of E such taht $\xi^p \circ \xi^p \leq \vartheta^p$ and $\xi^n \circ \xi^n \leq \vartheta^n$ implies that $\xi^p \leq \vartheta^p$ or $\xi^n \geq \vartheta^n$.*
(3) *ϑ is a **strongly prime** if for any two BF almost quasi-ideals $\xi = (E; \xi^p, \xi^n)$ and $\nu = (E; \nu^p, \nu^n)$ of E such taht $(\xi^p \circ \nu^p) \wedge (\nu^p \circ \xi^p) \leq \vartheta^p$ and $(\xi^n \circ \nu^n) \vee (\nu^n \circ \xi^n) \leq \vartheta^n$ implies that $\xi^p \subseteq \vartheta^p$ and $\xi^n \geq \vartheta^n$ or $\nu^p \leq \vartheta^p$ and $\nu^n \geq \vartheta^n$.*

It is clary, every BF strongly prime almost quasi-ideal of a semigroup is a BF prime almost quasi-ideal, and every BF prime almost quasi-ideal of a semigroup is a BF semiprime almost quasi-ideal.

Theorem 3.6. *Let K be a nonempty subset of a semigroup E. Then K is a prime (resp., semiprime) almost quasi-ideal of E if and only if $\lambda_K = (E; \lambda^p_K, \lambda^n_K)$ is a prime (resp., semiprime) BF almost quasi-ideal of E.*

Proof. Suppose that K is a prime almost quasi-ideal of E. Then K is an almost quasi-ideal of E. Thus by Theorem 3.3, $\lambda_K = (E; \lambda^p_K, \lambda^n_K)$ is a BF almost quasi-ideal of E. Let $\vartheta = (E; \vartheta^p_K, \vartheta^n_K)$ and $\xi = (E; \xi^p_K, \xi^n_K)$ be BF almost quasi-ideals such that $\vartheta^p \circ \xi^p \leq \lambda^p_K$ and $\vartheta^n \circ \xi^n \geq \lambda^n_K$. Assume that $\vartheta^p \nleq \lambda^p_K$ and $\vartheta^n \ngeq \lambda^n_K$ or $\xi^p \nleq \lambda^p_K$ and $\xi^n \ngeq \lambda^n_K$. Then there exist $h, r \in E$ such that $\vartheta^p(h) \neq 0$, $\vartheta^n(h) \neq 0$ and $\xi^p(r) \neq 0$, $\xi^n(h) \neq 0$. While $\lambda^p_K(h) = 0$, $\lambda^n_K(h) = 0$ and $\lambda^p_K(r) = 0$, $\lambda^n_K(r) = 0$. Thus $h \in \mathrm{supp}(\vartheta)$ and $r \in \mathrm{supp}(\xi)$, but $h, r \notin K$. So $\mathrm{supp}(\vartheta) \nsubseteq K$ and $\mathrm{supp}(\xi) \nsubseteq K$.

Since $\operatorname{supp}(\vartheta)$ and $\operatorname{supp}(\xi)$ are almost quasi-ideals of E we have $\operatorname{supp}(\vartheta)\operatorname{supp}(\xi) \nsubseteq K$. Thus there exists $m = de$ for some $d \in \operatorname{supp}(\vartheta)$ and $e \in \operatorname{supp}(\xi)$ such that $m \in K$. Hence $\lambda^p_K(m) = 0$ and $\lambda^n_K(m) = 0$ implies that $(\vartheta^p \circ \xi^p)(m) = 0$ and $(\vartheta^n \circ \xi^n)(m) = 0$. Since $\vartheta^p \circ \xi^p \leq \lambda^p_K$ and $\vartheta^n \circ \xi^n \leq \lambda^n_K$. we have $d \in \operatorname{supp}(\vartheta)$ and $e \in \operatorname{supp}(\xi)$. Thus $\vartheta^p(d) \neq 0$, $\vartheta^n(d) \neq 0$ and $\xi^p(e) \neq 0$, $\xi^n(e) \neq 0$. It implies that

$$(\vartheta^p \circ \xi^p)(m) = \bigvee_{(de)\in F_m} \{\vartheta^p(d) \wedge \xi^p(e)\} \neq 0$$

and

$$(\vartheta^n \circ \xi^n)(m) = \bigwedge_{(de)\in F_m} \{\vartheta^n(d) \vee \xi^n(e)\} \neq 0.$$

It is a contradiction so $\vartheta^p \leq \lambda^p_K$ and $\vartheta^n \geq \lambda^n_K$ or $\xi^p \leq \lambda^p_K$ and $\xi^n \geq \lambda^n_K$. Therefore $\lambda_K = (E; \lambda^p_K, \lambda^n_K)$ is a prime BF almost quasi-ideal of E.

Conversely, suppose that $\lambda_K = (E; \lambda^p_K, \lambda^n_K)$ is a prime BF almost quasi-ideal of E. Then $\lambda_K = (E; \lambda^p_K, \lambda^n_K)$ is a BF almost quasi-ideal of E. Thus by Theorem 3.3, K is an almost quasi-ideal of E. Let M and L be almost quasi-ideals of E such that $ML \subseteq K$. Then $\lambda_M = (E; \lambda^p_M, \lambda^n_M)$ and $\lambda_L = (E; \lambda^p_L, \lambda^n_L)$ are BF almost quasi-ideals of E. By Lemma 2.1 $\lambda^p_M \circ \lambda^p_L = \lambda^p_{ML} \leq \lambda^p_K$ and $\lambda^n_M \circ \lambda^n_L = \lambda^n_{ML} \geq \lambda^n_K$. By assumption, $\lambda^p_M \leq \lambda^p_K$ and $\lambda^n_M \geq \lambda^n_K$ or $\lambda^p_L \leq \lambda^p_K$ and $\lambda^n_L \geq \lambda^n_K$. Thus $M \subseteq K$ or $L \subseteq K$. We conclude that K is a prime almost quasi-ideal of E. □

Theorem 3.7. *Let K be a nonempty subset of a semigroup E. Then K is a strongly prime almost quasi-ideal of E if and only if $\lambda_K = (E; \lambda^p_K, \lambda^n_K)$ is a BF strongly prime almost quasi-ideal of E.*

Proof. Suppose that K is a strongly prime almost quasi-ideal of E. Then K is an almost quasi-ideal of E. Thus by Theorem 3.3, $\lambda_K = (E; \lambda^p_K, \lambda^n_K)$ is a BF almost quasi-ideal of E. Let $\vartheta = (E; \vartheta^p_K, \vartheta^n_K)$ and $\xi = (E; \xi^p_K, \xi^n_K)$ be BF almost quasi-ideals of E such that $(\vartheta^p \circ \xi^p) \wedge (\xi^p \circ \vartheta^p) \leq \lambda^p_K$ and $(\vartheta^n \circ \xi^n) \vee (\xi^n \circ \vartheta^n) \geq \lambda^n_K$. Assume that $\vartheta^p \nleq \lambda^p_K$ and $\vartheta^n \ngeq \lambda^n_K$ or $\xi^p \nleq \lambda^p_K$ and $\xi^n \ngeq \lambda^n_K$. Then there exist $h, r \in E$ such that $\vartheta^p(h) \neq 0$, $\vartheta^n(h) \neq 0$ and $\xi^p(r) \neq 0$, $\vartheta^n(h) \neq 0$. While $\lambda^p_K(h) = 0$, $\lambda^n_K(h) = 0$ and $\lambda^p_K(r) = 0$, $\lambda^n_K(r) = 0$. Thus $h \in \operatorname{supp}(\vartheta)$ and $r \in \operatorname{supp}(\xi)$, but $h, r \notin K$. So $\operatorname{supp}(\vartheta) \nsubseteq K$ and $\operatorname{supp}(\xi) \nsubseteq K$. Hence, there exists $m \in [\operatorname{supp}(\vartheta)\operatorname{supp}(\xi)] \cap [\operatorname{supp}(\vartheta)\operatorname{supp}(\xi)]$ such that $m \notin K$. Thus $\lambda^p_K(m) = 0$, $\lambda^n_K(m) = 0$.

Since $m \in \operatorname{supp}(\vartheta)\operatorname{supp}(\xi)$ and $m \in \operatorname{supp}(\xi)\operatorname{supp}(\vartheta)$ we have $m = d_1e_1$ and $m = e_2d_2$ for some $d_1, d_2 \in \operatorname{supp}(\vartheta)$ and for some $e_1, e_2 \in \operatorname{supp}(\xi)$. we have

$$(\vartheta^p \circ \xi^p)(m) = \bigvee_{(d_1e_1)\in F_m} \{\vartheta^p(d_1) \wedge \xi^p(e_1)\} \neq 0$$

and

$$(\vartheta^n \circ \xi^n)(m) = \bigwedge_{(d_1e_1)\in F_m} \{\vartheta^n(d_1) \vee \xi^n(e_1)\} \neq 0.$$

Similarly

$$(\xi^p \circ \vartheta^p)(m) = \bigvee_{(e_2d_2)\in F_m} \{\xi^p(e_2) \wedge \vartheta^p(d_2)\} \neq 0$$

and

$$(\vartheta^n \circ \xi^n)(m) = \bigwedge_{(e_2 d_2) in F_m} \{\xi^n(e_2) \vee \vartheta^n(d_2)\} \neq 0.$$

So $(\vartheta^p \circ \xi^p)(m) \wedge (\vartheta^p \circ \xi^p)(m) \neq 0$ and $(\vartheta^n \circ \xi^n)(m) \vee (\vartheta^n \circ \xi^n)(m) \neq 0$.
It is a contradiction so $(\vartheta^p \circ \xi^p)(m) \wedge (\xi^p \circ \vartheta^p)(m) = 0$ and $(\vartheta^n \circ \xi^n)(m) \vee (\xi^n \circ \vartheta^n)(m) = 0$. Hence, $\vartheta^p \leq \lambda_K^p$ and $\vartheta^n \geq \lambda_K^n$ or $\xi^p \leq \lambda_K^p$ and $\xi^n \geq \lambda_K^n$. Therefore $\lambda_K = (E; \lambda_K^p, \lambda_K^n)$ is a BF strongly prime almost quasi-ideal of E.

Conversely, suppose that $\lambda_K = (E; \lambda_K^p, \lambda_K^n)$ is a BF strongly prime almost quasi-ideal of E. Then $\lambda_K = (E; \lambda_K^p, \lambda_K^n)$ is a BF almost quasi-ideal of E. Thus by Theorem 3.3, K is an almost quasi-ideal of E. Let M and L be almost quasi-ideals of E such that $ML \cap LM \leq K$. Then $\lambda_M = (E; \lambda_M^p, \lambda_K^n)$ and $\lambda_L = (E; \lambda_L^p, \lambda_L^n)$ are BF almost quasi-ideals of E. By Lemma 2.1
$\lambda_{ML}^p = \lambda_M^p \circ \lambda_L^p$, $\lambda_{LM}^p = \lambda_L^p \circ \lambda_M^p$ and $\lambda_{ML}^n = \lambda_M^n \circ \lambda_L^n, \lambda_{LM}^n = \lambda_L^n \circ \lambda_M^n$. Thus $(\lambda_M^p \circ \lambda_L^p) \wedge (\lambda_L^p \circ \lambda_M^p) = \lambda_{ML}^p \wedge \lambda_{LM}^p = \lambda_{ML \cap LM}^p \leq \lambda_K^p$ and $(\lambda_M^n \circ \lambda_L^n) \vee (\lambda_L^n \circ \lambda_M^n) = \lambda_{ML}^n \vee \lambda_{LM}^n = \lambda_{ML \cup LM}^n \geq \lambda_K^n$. By assumption, $\lambda_M^p \leq \lambda_K^p$ and $\lambda_M^n \geq \lambda_K^n$ or $\lambda_L^p \leq \lambda_K^p$ and $\lambda_L^n \geq \lambda_K^n$. Thus $M \subseteq K$ or $L \subseteq K$. We conclude that K is a strongly prime almost quasi-ideal of E. □

4. Conclusion

In this paper, we give the concept of BF almost quasi-ideals in semigroup and we study properties of BF almost quasi-deal in semigroups. Moreover, we prove relationship between BF almost quasi-ideals and almost quasi-ideals. In the future we extend study other kinds of almost quasi-ideals and interval valued fuzzy set or class of kinds fuzzy sets.

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References

[1] S. Bogdanovic, Semigroups in Which Some Bi-Ideal Is a Group, Rev. Res. Fac. Sci.-Univ. Novi Sad, 11 (1981), 261–266.
[2] O. Grosek, L. Satko, A New Notion in the Theory of Semigroup, Semigroup Forum. 20 (1980), 233–240. https://doi.org/10.1007/bf02572683.
[3] L. Satko, O. Grosek, On Minimal A-Ideals of Semigroups, Semigroup Forum. 23 (1981), 283–295. https://doi.org/10.1007/bf02676653.
[4] N. Kaopusek, T. Kaewnoi, R. Chinram, On Almost Interior Ideals and Weakly Almost Interior Ideals of Semigroups, J. Discr. Math. Sci. Cryptography. 23 (2020), 773–778. https://doi.org/10.1080/09720529.2019.1696917.
[5] C.S. Kim, J.G. Kang, J.M. Kang, Ideal Theory of Semigroups Based on the Bipolar Valued Fuzzy Set Theory, Ann. Fuzzy Math. Inform. 2 (2012), 193–206.

[6] K. Lee, Bipolar-Valued Fuzzy Sets and Their Operations, In: Proceeding International Conference on Intelligent Technologies Bangkok, Thailand (2000), 307–312. https://cir.nii.ac.jp/crid/1573387450677828352.

[7] P. Murugadas, K. Kalpana, V. Vetrivel, Fuzzy Almost Quasi-Ideals in Semigroups, Malaya J. Mat. S (2019), 310–313. https://doi.org/10.26637/mjm0s01/0057.

[8] N. Sarasit, R. Chinram, A. Rattana, Applications of Fuzzy Sets for Almostity of Ternary Subsemirings, Int. J. Appl. Math. 36 (2023), 497–507. https://doi.org/10.12732/ijam.v36i4.5.

[9] S. Suebsung, K. Wattanatripop, R. Chinram, On Almost (m,n)- Ideals and Fuzzy Almost (m,n)-Ideals in Semigroups, J. Taibah Univ. Med. Sci. 13 (2019), 897–902.

[10] L.A. Zadeh, Fuzzy Sets, Inform. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.

[11] W.R. Zhang, Bipolar Fuzzy Sets and Relations: A Computational Framework for Cognitive Modeling and Multiagent Decision Analysis, In: Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference, IEEE, San Antonio, TX, USA, 1994: pp. 305–309. https://doi.org/10.1109/IJCF.1994.375115.