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Optimal Impulse Control for Systems Deriven by Stochastic Delayed Differential Equations

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Abstract. In this paper we study the problem of optimal impulse control for stochastic systems with delay in the case when the value function of the impulse problem depends only on the initial data of the given process through its initial value (value at zero) and some weighted averages. A verification theorem for such impulse control problem is given. As an example the optimal stream of dividends with transaction costs is solved.

1. Introduction

A stochastic impulse control policy can be characterized by the following factors: The first factor is known to be the random dates at which the considered policies are exercised while the second one is the size of the applied policies. Such characterization indicates that times and size are factors that can be studied separately by depending on the nature of the considered applications such as forest economic and cash flow management. In such applications both the timing and size of an admissible impulse policy have to be simultaneously determined. The mathematical analysis of stochastic impulse control in most cases is based on a combination of dynamic programming techniques and quasi-variational inequalities. The approach is general, its typically results into functional inequalities which, depending on the nature of the considered problem.

2. Problem Formulation

Suppose that- if there are no interventions- the state $X(t)$ we consider is described by a 1-dimension stochastic delayed differential equations of the form:

$$dX(t) = b(X(t), Y(t), Z(t))dt + \sigma(X(t), Y(t), Z(t))d\beta(t), t \geq 0, \tag{2.1}$$

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where $b : \mathbb{R}^3 \to \mathbb{R}$ and $\sigma : \mathbb{R}^3 \to \mathbb{R}$are given functions and

$$Y(t) = \int_{-\delta}^{0} \exp(\lambda s) X(t+s) ds;$$

$$Z(t) = X(t-\delta),$$

δ-constant delay, $\lambda \in \mathbb{R}$ is also a constant, $(\Omega, F, F_t, \beta(t) = \beta(t,\omega), t \geq 0, \omega \in \Omega)$ is a 1-dimensional Brownian motion. For $-\delta \leq s \leq 0$, we set the initial condition is to be

$$X(s) = \xi(s) \in C[-\delta, 0]. \tag{2.2}$$

The solution of (2.1) given the initial path ξ is denoted by $X^{\xi}(t)$. For existence and uniqueness of solution of such systems we refer to [3,5].

In what follows, we refer to the law of the solution $X^{\xi}(t)$ of (2.1) by $\rho^{\xi}(t)$ and the corresponding expectation by E^{ξ}.

Suppose that at any time t and any state X we are free to intervene and give the system an impulse $\eta \in H \subset C(\mathbb{R} \times C[-\delta, 0])$ the set of admissible impulse values. We assume that $\eta \in H$ has the form

$$\eta(X, \xi) = \eta(x, y(\xi))$$

where

$$y = y(\xi) := \int_{-\delta}^{0} \exp(\lambda s) \xi(s) ds. \tag{2.3}$$

An impulse control for this system is a double (possibly finite) sequence

$$\nu = (\tau_1, \tau_2, \ldots, \tau_k, \ldots; \eta_1, \eta_2, \ldots, \eta_j, \ldots)_{k \leq N}, N \leq \infty,$$

where $0 \leq \tau_1 \leq \tau_2 \leq \ldots$ are F_t-stopping times and $\eta_1, \eta_2, \ldots$ are the corresponding impulses at these times assumed to be F_{τ_j}-measurable for all j.
If the impulse control v is applied to system (2.1)-(2.2), the process $X^{(\xi,v)}(t)$ is defined by

$$dX^{(\xi,v)}(t) = b(X(t), Y(t), Z(t)) + \sigma(X(t), Y(t), Z(t)) d\beta(t), \tau_k \leq t \leq \tau_{k+1} \leq T^*. \tag{2.4}$$

$$(X(\tau_{k+1}), Y(\tau_{k+1})) = \Gamma(X(\tau_{k+1}), Y(\tau_{k+1}), \eta_{k+1}); k = 0, 1, \ldots; \tau_{k+1} \leq T^* \tag{2.5}$$

where $T^* = T^*(\omega)$ is the explosion time of $X^{(\xi,v)}$ defined by

$$T^*(\omega) = \lim_{R \to \infty} (\inf\{t \geq 0; |X^{(\xi,v)}(t)| \geq \mathbb{R}\}).$$

$\Gamma : \mathbb{R} \times C[-\delta, 0] \times H \to \mathbb{R}$ is a given function. Let $S \in \mathbb{R}^3$ be a given Borel set (solvency set) with the property that

$$\overline{S} = (\overline{S^0})$$

where S^0 and S^- denotes for the interior and closure of S respectively. Define

$$T = \inf\{t \in (0, T^*(\omega)); (s+t, X(t), Y(t)) \ \ not \ \ belongs \ \ to \ \ S\}$$

where

$$Y(t) := \int_{-\delta}^{0} \exp(\lambda s) X(t+s) ds.$$

Suppose that the profit rate is a function $u : \mathbb{R}^3 \to \mathbb{R}$ which is continuous, increasing and concave. Let $g : \partial S \to \mathbb{R}$ be a given bequest function, where ∂S denotes for the boundary of S. Moreover, suppose that the profit performing an intervention K where $K : \mathbb{R}^3 \times H \to \mathbb{R}; K(t, x, y, \eta)$ is a given function.

Let Δ be the set of admissible controls includes the set of impulse controls $v = (\tau_1, \tau_2, \ldots; \eta_1, \eta_2, \ldots)$ such that $X^{(\xi,v)}(t) \in S$ for all $t \leq T, T^* = \infty$ and $\lim \tau_n = T a.s. p^{s,\xi,v}$ for all s, ξ, v where $p^{s,\xi,v}$ is the law of the time space harvested process

$$W(t) = W^{\xi,v} = (s + t, X^{\xi,v}(t))$$

(if $N < \infty$ we assume that $\tau_N = T$ a.s.).

Assume the following conditions to be hold:

$$E^{s,\xi,v}[\int_0^T [|u(s+t, X(t), Y(t))|dt] \leq \infty \forall s, \xi, v, v \in \Delta,$$

$$E^{s,\xi,v}[|g(s+t, X(t), Y(t))|\chi_{\{T<\infty\}}] < \infty \forall s, \xi, v, v \in \Delta,$$

$$E^{s,\xi,v}[\sum_{\tau_k<T} |K(s+t, X(t_k^-) Y(t_k), \eta_k)] < \infty \forall s, \xi, v, v \in \Delta.$$

Then the total expected profit $J^{(s,\xi)}$ when $v \in \Delta$ is applied to the system (2.1)-(2.2) is defined by:

$$\begin{aligned} J^{(s,\xi)} =& E^{(s,\xi)}[\int_0^T [u(s+t, X(t), Y(t))dt] \\ &+ g(s+t, X(t), Y(t))\chi_{\{T<\infty\}}] \\ &+ [\sum_{\tau_k<T} K(s+t, X(t_k^-) Y(t_k), \eta_k)]. \end{aligned} \tag{2.6}$$

Now, our optimal impulse control problem for systems (2.1)-(2.2) is to find the value function $\Phi(s, \xi)$ and the optimal impulse control $v^* \in \Delta$ such that:

$$\Phi(s, \xi) = \sup_{v \in \Delta} J^v(s, \xi) = J^{v^*}(s, \xi). \tag{2.7}$$

A problem of this type for systems without delay had been studied in [2,4].

Problem (2.7) in general is infinite dimensional. The purpose of this paper is to reduce problem (2.7) for system (2.1)-(2.2) to finite dimensional one when we restrict its value function Φ to depends only on the initial path ξ through the three linear functionals namely:

$$X = X(\xi) := \xi(0). \tag{2.8}$$

$$Y = Y(\xi) := \int_{-\delta}^{0} \exp(\lambda s) \xi(s) ds \tag{2.9}$$

and

$$Z = Z(\xi) := \xi(-\delta). \tag{2.10}$$

In this case Φ can be written as:

$$\Phi(s, \xi) = \Psi(s, x, y, z)$$

where $\Psi : \mathbb{R}^4 \to \mathbb{R}$.

3. A Quasi-variational Inequality Formulation

In this section we prove a verification theorem for problem (2.7) in view of (2.8)-(2.10). To start with, first let $X_t(S) = X(t+s)$ for $t \geq 0, -\delta \leq S \leq 0$ to be the segment of the path of X from $t-\delta$ to t.

Lemma 3.1. *(Itô Formula)*
Suppose that the function $F : \mathbb{R} \times \mathbb{R} \times C[-\delta, 0] \to \mathbb{R}$ has the form:

$$F(t, x, y) = f(t, x, y(\eta)); (t, x, \eta) \in \mathbb{R} \times \mathbb{R} \times C[-\delta, 0]$$

where f is some function in $C^{1,2,1}(\mathbb{R}^3)$ and

$$y(\eta) = \int_{-\delta}^{0} \exp(\lambda s)\eta(s)ds, \ \ \lambda \ \ is \ \ a \ \ constant.$$

Define

$$G(t) = F(S + t, X^{\xi}(t), X_t^{\xi}(.)).$$

Then

$$G(t) = Lfdt + \frac{\partial f}{\partial x}\sigma(x, y, y)dB(t) + \frac{\partial f}{\partial y}[x - e^{(-\lambda sz)} - \lambda y]dt \tag{3.1}$$

where the differential operator L acting on f as

$$Lf = Lf(u, x, y, z) = \frac{\partial f}{\partial u} + b(x, y, z)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x, y, z)\frac{\partial^2 f}{\partial x^2}$$

where

$$u = s + t, Z = x(X_t^{\xi}(.)),$$

$$Y = y(X_t^{\xi}(.)) = \int_{-\delta}^{0} e^{(\lambda s)} X^{\xi}(t+s)ds,$$

$$Z = z(X_t^{\xi}(.)) = X^{\xi}(t - \delta).$$

Proof. See [1]. □

Lemma 3.2. *(Dynkin Formula). Suppose that $\ell \in C_0^{1,2}(\mathbb{R}^3), t \geq 0$. Then*

$$E^{s,\xi}[\ell(t + s, X^{\xi}(t), Y(X_t^{\xi}(.))] = \ell(s, \xi(s), y(\xi)) + E^{s,\xi}[\int_0^t \{Lf + \frac{\partial f}{\partial y}[x - e^{-\lambda\delta z}] - \lambda y]\}dr \tag{3.2}$$

where $L\ell(u, x, y, z)$ and all functions are evaluated at $u = s + r, x = X^{\xi}(r), Y = y(X_r^{\xi}(.)), z = X^{\xi}(r - \delta)$. The proof of this fact follows from Itô formula (Lemma 3.1).

The intervention operator. If we denote by G for the space of all measurable functions $\ell : S \to \mathbb{R}$, then the intervention operator denoted by M where $M : G \to G$ is denoted to be:

$$M\ell(x, y) = \sup\{\ell(\Gamma(x, y, \eta) + K(x, y, \eta); \eta \in H, \Gamma(x, y, \eta) \in S\} \tag{3.3}$$

where $\ell \in G$ and $(x, y) \in S$.

Lemma 3.3. *(Approximation):*
Let $D \subseteq S$ be open, ∂S is Lipschitz and $\varphi : \overline{S} \to \mathbb{R}$ satisfy the following
(i) $\varphi \in C^{1,2,1}(S^0) \bigcap C(\overline{S})$.
(ii) $\varphi \in C^{1,2,1}(S^0 \setminus \partial D)$.
(iii) Second order derivatives of φ with respect to x are locally bounded near ∂D. Then there exists a sequence of functions φ_j, $j = 1, 2, \dots$ such that
(a) $\varphi_j \to \varphi$ uniformly on compact subsets $\overline{S}$ as $j \to \infty$.
(b) $\ell\varphi_j \to \ell\varphi$ uniformly on compact subsets of $S^0 \setminus \partial D$ as $j \to \infty$, where ℓ is:

$$\ell\varphi(s, x, y, z) = \frac{\partial\varphi}{\partial s} + b(x, y, z)\frac{\partial\varphi}{\partial x} + \frac{1}{2}\sigma^2(x, y, z)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial\varphi}{\partial y}[x - e^{-(\lambda S)z} - \lambda y]. \tag{3.4}$$

(c) $\{\ell\varphi_j\}_{j=1}^{\infty}$ is locally bounded on S^0.

Proof. See [2]. □

Theorem 3.1. *(Verification theorem): Let $S = \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$. Suppose that a nonnegative $\varphi : \overline{S} \to \mathbb{R}$ exists such that:*
(i) $\varphi \in C^{1,2,1}(S^0) \cap C(\overline{S})$.
(ii) $\varphi(s, x, y) \geq M\varphi(s, x, y) \forall (x, y) \in S$.
Let the continuation region D be defined by:

$$D = \{(s, x, y) \in S : \varphi(s, x, y) > M\varphi(s, x, y)\}.$$

Suppose the following hold:
(iii)

$$E^{\xi}[\int_0^T \chi_{\partial D}(s, x^{\xi}(t), y(x_t^{\xi}(.))dt] = 0 \forall \xi \geq 0.$$

Suppose that the continuation region D has the form:
(iv) $D := \{(s, x, y) \in S : \omega(x, y) \subseteq \omega^*\}$ for some function $\omega : \mathbb{R}^2 \to \mathbb{R}$ and some constant ω^* and ∂D is Lipschitz surface.
(v) $\varphi \in C^{1,2,1}(S^0 \setminus \partial D)$.
(vi) $\ell\varphi + \eta \leq 0$ on $(S^0 \setminus \partial D)$
(vii) The family $\{\phi^-(s, X^{\xi}(\tau), y(X_{\tau}^{\xi}(.))); \tau \leq T\}$ is uniformly integrable with respect to $\rho^{s,\xi}$ for all $\xi \geq 0$ and for all $v \in \nu$.
(viii) $\varphi(s, X^{\xi}(\tau), y(X_{\tau}^{\xi}(.)) \to g(s, X^{\xi}(T), y(X_T^{\xi}))\chi_{\{T<\infty\}}$ as $t \to T$ a.s. $\rho^{s,\xi}$ for all $v \in \nu$.
Then $\varphi(s, \xi(0), y(\xi)) \geq \Phi(s, \xi) \forall\ \xi \geq 0$.
Suppose that in addition to (i)-(viii) the following to be hold:

(ix) $\ell\varphi + u = 0$ *on* $D \cap S^0$ *for all* z.
(x) η^* *exists for all* $\xi \geq 0$ *where*

$$\eta^* \in \nu^* := \nu^* := (\tau_1^*, \tau_2^*, \dots, \eta_1^*, \eta_2^*, \dots),$$

where

$$\tau_{k+1}^* = \inf\{t > \tau_k^*; (s, X^\xi(t), y(X_t^\xi).)) \ \ not \ \ in \ \ D\} \wedge T.$$

(xi) $\varphi(s, X^{\xi,v^*}(\tau), y(X_\tau^{\xi,v^*}(.))_{t\leq T}$ *is* $\rho^{s,\xi}$ *uniformly integrable for all* $\xi \geq 0$.
Then with $y(\xi) = \int_{-\delta}^0 e^{(\lambda s)}\xi(s)ds$,
$\varphi(s, \xi(0), y(\xi)) = \Phi(s, \xi) \ \ for \ \ all \ \ \xi \geq 0$ *and* v^* *is an optimal impulse control.*

Proof. As in [4] for systems without delay we give the following details of the proof for systems with constant delay as follows:
First when (i)-(iv) is satisfied and by Lemma 3.2, we can find such sequence of functions φ_j, $j = 1, 2\dots$ in $C^{1,2,1}(\overline{S^0}) \cap C(\overline{S})$, such that
(a) $\varphi_j \to \varphi$ uniformly on compact subsets of $\overline{S}$, $j \to \infty$.
(b) $\ell\varphi_j \to \ell\varphi$ uniformly on compact subsets of $\overline{S^0} \setminus \partial D$ as $j \to \infty$.
(c) $\{\ell\varphi_j\}_{j=1}^\infty$ is locally bounded on $\overline{S^0}$.
let $v = (\tau_1, \tau_2, \dots; \eta_1, \eta_2, \dots) \in \nu$ and for $R \geq 0$ let

$$T_R = R \wedge \inf\{t \geq 0; (X^\xi(t), y(X_t^\xi(.)) \geq R)\}$$

and $\theta_{j+1} = \theta_{j+1}^R = \max(\tau(\tau_{j+1} \wedge \tau_R), j = 1, 2\dots)$. Then by applying Lemma 3.2 we have

$$\begin{aligned} E^{s,\xi}[\varphi_j(s + \tau_i, X(\tau_i), Y(\tau_i))] - E^{s,\xi}[\varphi_j(s, X, \theta_{j+1}^-), Y_\theta(\theta_{j+1})] \\ = -E^{s,\xi}[\int_{\tau_i}^{\theta_{i+1}} \ell\varphi_j dt], i, j = 1, 2, \dots. \end{aligned} \tag{3.5}$$

By (a),(b) and (c) as $j \to \infty$ taking into account (iii), (v) and (vi) we deduce that:

$$\begin{aligned} E^{s,\xi}[\varphi(s + \tau_i), X(\tau_i), Y(\tau_i)] - E^{s,\xi}[\varphi(s, X(\theta_{i+1}^-), Y(\theta_{i+1}))] \\ = -E^{s,\xi}[\int_{\tau_i}^{\theta_{i+1}} \ell\varphi dt] \\ \geq E^{s,\xi}[\int_{\tau_i}^{\theta_{i+1}} u(s + t, X(t), Y(t))dt]. \end{aligned} \tag{3.6}$$

From Fatou's lemma we have

$$\begin{aligned} E^{s,\xi}[\varphi(s + \tau_i), X(\tau_i), Y(\tau_i)] - E^{s,\xi}[\varphi(s, X(\tau_{i+1}^-), Y(\tau_{i+1}))] \\ \geq E^{s,\xi}[\int_{\tau_i}^{\theta_{i+1}} u(s + t, X(t), Y(t))dt]. \end{aligned} \tag{3.7}$$

By taking summation from $i = 0$ to $i = m$, we get

$$\begin{aligned}\varphi(s,\xi(0),y(\xi))+&\sum_{i=0}^{m} E^{s,\xi}[\varphi(s+\tau_i,X(\tau_i),Y(\tau_i))]\\ &-\varphi(s,X(\tau_i^-),Y(\tau_i)-\varphi(s,X(\tau_{m+1}^-),Y(\tau_{m+1}))\\ &\geq E^{s,\xi}[\int_0^{\tau_{m+1}} u(s+t,X(t),Y(t))dt].\end{aligned} \tag{3.8}$$

Now

$$\begin{aligned}\varphi(s+\tau_i,X(\tau_i),Y(\tau_i)) =&\varphi(\Gamma(s+\tau_i,X(\tau_i^-),Y(\tau_i),\eta_i))\\ &\leq M\varphi(s+\tau_i,X(\tau_i^-),Y(\tau_i)\\ &-K(\Gamma(s+\tau_i,X(\tau_i^-),Y(\tau_i),\eta_i)),\tau_i<T\end{aligned} \tag{3.9}$$

and

$$\varphi(s+\tau_i,X(\tau_i),Y(\tau_i)) = \varphi(s+\tau_i,X(\tau_i),Y(\tau_i)) \ \ if \ \ s+\tau_i = T \tag{3.10}$$

and therefore

$$\begin{aligned}\varphi(s,\xi(0),y(\xi))+&\sum_{i=0}^{m} E^{s,\xi}[M\varphi(s+\tau_i,X(\tau_i^-),Y(\tau_i))-\varphi(s+\tau_i,X(\tau_i^-),Y(\tau_i))_{\chi_{\tau_i}<T}]\\ &\geq E^{s,\xi}[\int_0^{\tau_{m+1}} u(Ss+t,X(t),Y(t))dt]\\ &+\varphi(s+\tau_{m+1},X(\tau_{m+1}^-),Y(\tau_{m+1}))\\ &+\sum_{i=0}^{m} K((s+\tau_i,X(\tau_i^-),Y(\tau_i),\eta_i))\end{aligned} \tag{3.11}$$

Hence by (ii) we get

$$M\varphi(s+\tau_i,X(\tau_i^-),Y(\tau_i)-\varphi(s+\tau_i,X(\tau_i^-),Y(\tau_i)\leq 0.$$

By (3.11) we deduce that

$$\begin{aligned}\varphi(s,\xi(0),y(\xi) \geq& E^{s,\xi}[\int_0^{T_{m+1}} u(s+t,X(t),Y(t))dt]\\ &+g(s+t,X(t),Y(t))_{\chi_T<\infty}\\ &+\sum_{i=0}^{N} K((s+t,X(\tau_i),Y(\tau_i),\eta_i))\end{aligned}$$

i.e.,$\varphi(s,\xi(0),y(\xi)) \geq J^v(s,\xi)$.
Since v is taken arbitrary, then (a) is satisfied. Next, we assume that conditions (ix)-(xi) are satisfied. Then by (3.7)-(3.8) applied to $v = v^*$ we have (2.7) hold by (ix) since $(X(t), Y(t)) \in D$ for $t \in (\tau_i, \theta_{i+1})$. Equality in (3.8)-(3.9) is obtained by (x). Equality in (2.7) is also obtained by our choice of η^*. Since $(X(t_i^-), Y(t_i) \in \partial D$ and $\varphi = M\varphi$ outside D we conclude that equality in (3.11)-(3.12) is obtained.
Hence

$$\varphi(s,\xi(0),y(\xi)) \geq J^v(s,\xi) = \Phi(s,\xi)$$

and v^* is optional. □

4. Application

This application is an extension to the no-delay case of a problem of optional stream of dividends with transaction costs. Suppose that if we make no interventions the amount $X(t) = X^{\xi}(t)$ available (cash flow) is given by

$$dX(t) = [\theta X(t) + \alpha Y(t) + \beta Z(t)]dt + \sigma[X(t) + \beta E^{\lambda\delta} Y(t)]d\beta(t), t \geq 0 \tag{4.1}$$

$$X(s) = \xi(s), -\delta \leq s \leq 0 \tag{4.2}$$

where

$$Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s)ds, Z(t) = X(t-\delta).$$

Suppose that at any time t we are free to take out dividend η from $X(t)$ by applying the transaction cost $K(\eta) = c + \gamma\eta$, where $c > 0$ and $\gamma > 0$ are constants. The constant c is called the fixed part and the quantity $\gamma\eta$ is called the proportional part, respectively, of the transaction cost. The resulting cash flow $X^{(\xi,v)}$ is given by (4.1)-(4.2) and for $\tau_i \leq t \leq \tau_{i+1}$

$$X(\tau_{i+1}) = X(\tau_{i+1}^-) - c - (1+\gamma)\eta_i, i = 0, 1, 2, \ldots \tag{4.3}$$

Let

$$S = \{(s, x, y) : x + \beta e^{\lambda s} y \geq 0\}. \tag{4.4}$$

So that

$$T^*(\omega) = \inf\{t \geq 0 : X(t) + \beta e^{\lambda s} Y(t) \leq 0\}.$$

Define

$$J^{s,\xi} = E^{s,\xi}[\int_0^T e^{-\rho(s+t)}(X(t) + \beta e^{\lambda s} Y(t))^k dt + \sum_{\tau_i < T} e^{-\rho(s+t+\tau_i)} \eta_i], \tag{4.5}$$

ρ is a constant (discounted exponent). The functional $J^{s,\xi}$ represents the total expected discounted dividend up to time $T, k \in (0,1)$ is a constant. The problem is to find the optional impulse $v^* \in \nu$ and the value function Φ such that

$$\Phi(s,\xi) = \sup_{v \in \nu} J^v(s,\xi) = J^{v^*}(s,\xi)$$

we try to find a function $\varphi(s, \xi(0), y(\xi))$ of the form

$$\varphi(s,x,y) = \exp(-\rho s)\psi(x,y)$$

satisfying the conditions of Theorem 3.1. Assume that the continuation region D has the form:

$$D = \{(s,x,y) : 0 \leq x + \beta e^{\lambda\delta} y \leq \omega^*\} \tag{4.6}$$

where $\psi(x,y), \omega^*$ are to be determined.
According to this choice of ψ we have

$$\begin{aligned} e^{\rho s}\ell\psi(s,x,y,z) =& -\rho\psi + (\theta x + \alpha y)\frac{\partial\psi}{\partial x} + \frac{1}{2}\sigma^2[x+\beta e^{\lambda s}y]^2\frac{\partial^2\psi}{\partial x^2} \\ &+ (x-\lambda y)\frac{\partial\psi}{\partial y} + z[\beta\frac{\partial\psi}{\partial x} - e^{-\lambda\delta}\frac{\partial\psi}{\partial y} + (x+\beta e^{\lambda\delta}y)^k = 0 \end{aligned} \tag{4.7}$$

for all $z \geq 0$ and all $(x,y) \in D$. This is only possible if

$$\beta\frac{\partial\psi}{\partial x} - e^{-\lambda s}\frac{\partial\psi}{\partial Y} = 0. \tag{4.8}$$

The general solution of (4.8) is $\psi(x,y) = h(\omega)$ for some $h : \mathbb{R} \to \mathbb{R}$, where

$$\omega(x,y) = x + \beta e^{\lambda\delta}y. \tag{4.9}$$

By substituting for this ψ into (4.7) we obtain:

$$\begin{aligned} &-\rho h(\omega) + (\theta+\beta e^{\lambda\delta})[x + (\theta+\beta e^{-\lambda\delta})(\alpha - \lambda\beta e^{\lambda\delta})y]h' \\ &+ \frac{1}{2}\sigma^2(x+\beta e^{\lambda\delta}y)^2h''(\omega) + (x+\beta e^{\lambda\delta}y)^k = 0. \end{aligned} \tag{4.10}$$

Equation (4.10) has a solution depending on ω if and only if:

$$x + (\theta + \beta e^{-\lambda\delta}(\alpha - \lambda\beta e^{\lambda\delta}))y = \omega(x,y)$$

i.e.,

$$\alpha = \beta e^{\lambda\delta}(\lambda + \theta + \beta e^{\lambda\delta}). \tag{4.11}$$

If we assume (4.11) holds, then (4.10) takes the form:

$$-\rho h(\omega) + (\theta+\beta e^{\lambda\delta})\omega h'(\omega) + \frac{1}{2}\sigma^2\omega^2h''(\omega) + \omega^k = 0 \tag{4.12}$$

which has a general solution

$$h(\omega) = C\omega^{r_1} + D\omega^{r_2} + K\omega^k \tag{4.13}$$

for some arbitrary constants C, D, where

$$r_1 = \sigma^{-2}[\frac{1}{2}\sigma^2 - p \pm \sqrt{(\gamma - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2}; i = 1,2 \tag{4.14}$$

where $p := (\theta + \beta e^{\lambda\delta})$ are the solutions of the equation:

$$\frac{1}{2}\sigma^2r^2 + (p - \frac{1}{2}\sigma^2)r - \rho = 0; r_1 \leq 0 \leq r_2$$

and

$$K = -(\frac{1}{2}\sigma^2k^2 + (p - \frac{1}{2}\sigma^2)r - \rho).$$

If we assume that

$$\rho \geq p := (\theta + \beta e^{\lambda\delta}). \tag{4.15}$$

Then

$$r_2 > 1 \tag{4.16}$$

which implies that $K > 0$ since $0 < k < 1$. Choose $h(\omega)$ of the form:

$$h(\omega) = C\omega^r + K\omega^k \tag{4.17}$$

where r is given by (4.14). If this is the case, then our solution $\psi(s,x,y)$ gets the form

$$\psi(s,x,y) = e^{-\rho s}\phi(x,y); \omega \leq \omega_1,$$

$$\psi(s,x,y) = M\psi(s,x,y); \omega \geq \omega_1,$$

where $M\psi$ is:

$$M\phi(x,y) = \sup\{\phi(x-c-(1+\gamma)\eta, y) + \eta\}. \tag{4.18}$$

The supremum of

$$g(\eta) := \phi(x-c-(1+\gamma)\eta, y) + \eta \tag{4.19}$$

is $\hat{\eta} = \hat{\eta}(x,y)$ such that

$$g^{'}(\hat{\eta}) = \phi_{D_1}(x-c-(1+\gamma)\hat{\eta}, y) = \frac{1}{1+\gamma} \tag{4.20}$$

or

$$\phi_{D_1} = h^{'}(\omega_0) = \frac{1}{1+\gamma} \tag{4.21}$$

where D_1 is the derivative of ϕ with respect to the first variable and ω_0, x_0 are as follows:

$$\omega_0 = \omega(x-c-(1+\gamma)\hat{\eta}, y) \tag{4.22}$$

$$x_0 = x-c-(1+\gamma)\hat{\eta}. \tag{4.23}$$

By equation (4.19) we have

$$\phi(x,y) = h(\omega) = \phi(x_0,y) + \hat{\eta} = h(\omega_0) + \hat{\eta}, \omega \geq \omega_1.$$

In particular,

$$\phi_{D_1}(x_1,y) = h^{'}(\omega_1) = \frac{1}{1+\gamma} \tag{4.24}$$

where $\omega_1 = \omega(x_1, y)$ and

$$\phi(x_1,y) = \phi(x_0,y) + \frac{(x-x_0)-c}{(1+\gamma)}. \tag{4.25}$$

To summarize we put

$$\phi(x,y) = h(\omega) = c\omega^r + K\omega^k; \omega \leq \omega_1 \tag{4.26}$$

$$\phi(x,y) = h(\omega_0) + \frac{1}{1+\gamma}(\omega-\omega_0) - \frac{c}{1+\gamma}; \omega \geq \omega_1 \tag{4.27}$$

where ω_0, ω_1 are determined by (4.21), (4.23), (4.25) and (4.26), i.e.,

$$\gamma c\omega_0^{\gamma-1} = -kK\omega_0^{k-1} + \frac{1}{1+\gamma} \tag{4.28}$$

$$\gamma c\omega_1^{\gamma-1} = -kK\omega_1^{k-1} + \frac{1}{1+\gamma} \tag{4.29}$$

$$c(\omega_1^\gamma - \omega_0^r) = \frac{1}{1+r\gamma}(x_1 - x_0) + K(\omega_0^k - \omega_1^k) - \frac{c}{1+\gamma}. \tag{4.30}$$

To study the solutions of (4.27)-(4.30), we first consider the function

$$F(\omega) = \gamma c \omega_0^{\gamma-1} + kK\omega_0^{k-1} - \frac{1}{1+\gamma}$$

and

$$F^{'}(\omega) = \gamma(\gamma-1)c\omega^{\gamma-2} + k(k-1)K\omega^{k-2}$$

so that $F^{''}(\omega) = 0$ if and only if

$$\gamma(\gamma-1)c\omega^{\gamma-2} = k(k-1)K\omega^{k-2}$$

which has a unique solution

$$\omega = \overline{\omega} = [\frac{k(1-k)K}{c\gamma(\gamma-1)}]^{\frac{1}{\gamma-k}} > 0. \tag{4.31}$$

Since

$$F^{'''}(\omega) = \gamma(\gamma-1)(\gamma-2)c\omega^{\gamma-3} + k(k-1)(k-2)K\omega^{k-3} < 0$$

for $\omega < \overline{\omega}$ and $F^{'}(\omega) > 0$ for $\omega < \overline{\omega}$ we see that $\omega = \overline{\omega}$ is a maximum point for $f(\omega)$. From this we conclude that (4.27) and (4.30) have exactly two solutions $\omega_0 = \omega(x_0, y)$ as given in (4.32) and ω_1 such that $0 < \omega_0 < \overline{\omega} < \omega_1$. We know choose $\omega_1 = \overline{\omega}$ and $x_0 = x - c - (1+r)\hat{\eta}, c$ is chosen such that (4.26)-(4.27) defines a continuous function at $\omega = \omega_1$, i.e.,

$$\phi(x_1, y) = \phi(x_0, y) + \frac{x_1 - x_0 - c}{1+r}$$

i.e.,

$$h(\omega_1) = \frac{\omega_1 - \omega_0 - c}{1+\gamma}$$

or

$$h(\omega_1) - h(\omega_0)\frac{\omega_1 - \omega_0 - c}{1+\gamma}.$$

Hence

$$c = (\omega_1^\gamma - \omega_0^\gamma)[\frac{\omega_1 - \omega_0 - c}{1+\gamma} - K(\omega_1^k - \omega_0^k)] \tag{4.32}$$

and

$$\hat{\eta} = \frac{x - x_0 - c}{1+r} = \frac{\omega_1 - \omega_0 - c}{1+\gamma}. \tag{4.33}$$

Now, we verify condition (ii)-Theorem 1; i.e., $\phi(x, y) = M\phi(x, y)$ on S.
First we assume that $\omega > \omega_1$. Then if:

$$\omega_0 = \omega(x - c - (1+\gamma)\eta, y) \geq \omega_1$$

we have by equations (4.26)-(4.27).

$$\phi(x - c - (1+r), \eta, y) + \eta = \phi(x, y) - \frac{c}{1+\gamma}$$

On the other hand, by by (4.34)

$$\begin{aligned}&\sup \phi(x-c-(1+\gamma)\eta, y)+\eta ; \omega(x-c-(1+\gamma)\eta, y)<\omega_1 \\ &=\phi(x_0, y)+\frac{x-x_0-c}{1+\gamma}=\phi(x, y).\end{aligned}$$

This proves (ii) for $\omega>\omega_1$. To verify (ii) for $\omega<\omega_1$, note that $\phi_{D_1}(x, y)>\frac{1}{1+\gamma}$ for $0, \omega<\omega_1$ and therefore

$$M \phi(x, y)=\phi(x_0, y)+\frac{x-x_0-c}{1+\gamma}, \omega_0<\omega<\omega_1.$$

Since

$$\frac{\partial}{\partial \omega}[c \omega^\gamma+K \omega^k-h(\omega_0)-\frac{\omega-\omega_0-c}{1+\gamma}]<0, \omega_0<\omega<\omega_1.$$

We see that $\phi 9 x, y) . M \phi(x, y)$ for $\omega_0<\omega<\omega_1$, since ϕ is increasing, we have

$$\phi(x, y) \geq M \phi(x, y) \quad for \quad \omega<\omega_0.$$

Verification of condition (iv)-Theorem 3.1:

$$\rho h(\omega)+p \omega \frac{\partial h}{\partial \omega}+\frac{1}{2} \sigma^2 \omega^2 \frac{\partial^2 h}{\partial \omega^2}+\omega^k \leq 0, \omega<\omega_1.$$

In our case, this reduces to check that

$$\rho(h(\omega_0))+\frac{\omega-\omega_0-c}{1+r}+\frac{p}{1+r} \omega+\omega^k \leq 0, \omega>\omega_1.$$

Since $\ell \phi(x, y)=0$ for $\omega=\omega_1$ and $0<k<1$ we see that $\ell \phi(x, y) \leq 0$ for $\omega>\omega_1$ if and only if

$$\rho>\theta=\beta e^{\lambda \delta}:=p$$

which is (4.17). We conclude that the value function of problem (2.6) is given by

$$\Phi(S, \xi)=e^{-\rho S} \phi(x, y)$$

where ϕ is given by (4.26)-(4.27). The corresponding optimal impulse control v^* is the following: Define

$$D=\{(x, y)=0 \leq x+\beta e^{\lambda \delta} y \leq \omega_1\}. \tag{4.34}$$

In case $X(0^-)<x_1(=\omega_1-\beta e^{\lambda \delta} y)$, then wait until the first time $t=\tau_1^*$ when $x(t)+\beta e^{\lambda \delta} y(t)=\omega_1=\omega(x_1, y)$, then pay out the dividend

$$\hat{\eta_1}=\frac{\omega_1-\omega-c}{1+\gamma}=\frac{x_1-x_0-c}{1+\gamma}.$$

This bring $x(t)+\beta e^{\lambda \delta} y(t)$ down the level

$$\omega_0=\omega(x_0, y)=x-c-(1+r) \hat{\eta}+\beta e^{\lambda \delta} y,$$

i.e., $x_0=\omega_0-\beta e^{\lambda \delta} y$. Again do nothing until the next time $t=\tau_2^*$ when $x(t)=x_1$ and again we harvest the amount $\hat{\eta_2}=\hat{\eta_2}$, etc. If initially x has a value $x(0^-) \geq x_1$ then $\tau_1^*=0$ and we immediately pay out

$$\hat{\eta_1}=\frac{x(0^-)-x_0-c}{1+\gamma}.$$

In other words, if $\omega > \omega_1$, then

$$\omega - \omega_1 = x + \beta e^{\lambda\delta} - \omega_1 = x - x_1,$$

i.e., we harvest exactly enough to bring x-level down to the value x_1. We summarize what we have proved in the following theorem:

Theorem 4.1. *Suppose that $\theta, \alpha, \beta, \delta, \lambda$ and ρ satisfy the following conditions:*

$$\rho \geq p \;\; where \;\; p := \theta + \beta e 6\lambda\delta \tag{4.35}$$

and

$$\alpha = \beta e^{\lambda\delta}(\lambda + \theta + \beta e^{\lambda\delta}). \tag{4.36}$$

Then, with $\omega + \omega(x, y) = x + \beta e^{\lambda\delta} y$,

$$\Phi(s, \xi) = \varphi(s, x, y) = e^{-\rho s}\phi(x, y); 0 < \omega \leq \omega_1,$$

$$\Phi(s, \xi) = \varphi(s, x, y) = e^{-\rho s}[\phi(x_0, y) + \frac{x - x_0 - c}{1 + r}]; \omega \geq \omega_1$$

where $\phi, \omega_0, \omega_1, c$ are given by (4.26)-(4.27), (4.22), (4.32) and (4.33) respectively, $v^* = (\tau_1^*, \tau_2^*, \ldots; \eta_1^*, \eta_2^*, \ldots)$ is the optimal impulse control.

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