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A Fractional Elliptic System With Strongly Coupled Critical Terms and Concave-Convex Nonlinearities

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Abstract. By the Nehari method and variational method, two positive solutions are obtained for a fractional elliptic system with strongly coupled critical terms and concave-convex nonlinearities. Recent results from the literature are extended to the fractional case.

1. Introduction

In this paper, we are concerned with the study of the existence of positive solutions for the following elliptic system involving the fractional Laplacian for a given $0 < s < 1$:

$$\begin{cases} (-\Delta)^s u = \frac{\eta_1\alpha_1}{2_s^*}|u|^{\alpha_1-2}|v|^{\beta_1}u + \frac{\eta_2\alpha_2}{2_s^*}|u|^{\alpha_2-2}|v|^{\beta_2}u + \lambda\frac{|u|^{q-2}u}{|x|^\gamma}, & x \in \Omega, \\ (-\Delta)^s v = \frac{\eta_1\beta_1}{2_s^*}|u|^{\alpha_1}|v|^{\beta_1-2}v + \frac{\eta_2\beta_2}{2_s^*}|u|^{\alpha_2}|v|^{\beta_2-2}v + \mu\frac{|v|^{q-2}v}{|x|^\gamma}, & x \in \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\eta_1, \eta_2, \lambda, \mu$, are positive, $2_s^* := \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent, $N > 2s$, $\alpha_1 + \beta_1 = 2_s^*$, $\alpha_2 + \beta_2 = 2_s^*$, and $(-\Delta)^s$ is the spectral fractional Laplacian, in Ω, defined in section 2.

$|u|^{\alpha_i-2}u|v|^{\beta_i}$ and $|u|^{\alpha_i}|v|^{\beta_i-2}v, i = 1, 2$ are called strongly coupled terms. When $\eta_1 = \eta_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $\gamma = 0$, problem (1.1) becomes the following elliptic system:

$$\begin{cases} (-\Delta)^s u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta + \lambda|u|^{q-2}u & \text{in } \Omega, \\ (-\Delta)^s v = \frac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v + \mu|v|^{q-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

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Systems similar to (1.2), but involving Laplacian or p-Laplacian operators, have been studied extensively in recent years, for example, the authors in [13] proved that the system admits at least two positive solutions when (λ,μ) belongs to certain subset of $\mathbb{R}^2$. Later, Hsu [12] obtained the same results for the p-Laplacian elliptic system. There are other multiplicity results for critical elliptic equations involving concave–convex nonlinearities, see for example [2,4].

Caffarelli and Silvestre [8] gave a new formulation of the fractional Laplacians through Dirichlet-Neumann maps. This is commonly used in the recent literature since it allows us to write nonlocal problems in a local way and this permits to us use the variational methods for those kinds. To our best knowledge, there are just a few results in the literature on the fractional system (1.1) with both concave-convex nonlinearities and the strongly-coupled critical terms. Motivated by the results mentioned above, in this paper we extend the work [18], where the fibering and Nehari manifold methods are applicable to obtain two positive solutions for

$$\begin{cases} -\Delta u = \dfrac{\eta_1\alpha_1}{2^*}|u|^{\alpha_1-2}|v|^{\beta_1}u + \dfrac{\eta_2\alpha_2}{2^*}|u|^{\alpha_2-2}|v|^{\beta_2}u + \lambda\dfrac{|u|^{q-2}u}{|x|^{\gamma}}, & x\in\Omega,\\ -\Delta v = \dfrac{\eta_1\beta_1}{2^*}|u|^{\alpha_1}|v|^{\beta_1-2}v + \dfrac{\eta_2\beta_2}{2^*}|u|^{\alpha_2}|v|^{\beta_2-2}v + \mu\dfrac{|v|^{q-2}v}{|x|^{\gamma}}, & x\in\Omega,\\ u=v=0, & x\in\partial\Omega. \end{cases} \tag{1.3}$$

We point out that we adopt in the paper the spectral (or regional) definition of the fractional laplacian in a bounded domain and not the integral definition. We shall refer the interested reader to [14] for a careful comparison between these two different notions. To formulate the main result, we consider

$$C_\theta = \{(\lambda,\mu)\in\mathbb{R}^2_+ : 0<\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}<\theta\},$$

$$\Lambda_1 = \left(\frac{2^*_s-q}{2^*_s-2}\Theta\right)^{-\frac{2}{2-q}}\left[\frac{2-q}{(2^*_s-q)}\left(k_s\mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{2^*_s}{2}}\right]^{\frac{2}{2^*_s-2}} \tag{1.4}$$

where k_s is a normalization constant and Θ, $\mathcal{S}_{\eta,\alpha,\beta}$ are constants that will be introduced later. We assume that

$$(\mathcal{F}):\quad 1<q<2 \ \text{ and } \ 0\le\gamma<N+sq-\frac{qN}{2}.$$

Our main results dealing with the problem (1.1) are the following.

Theorem 1.1. *Assume that $(\mathcal{F})$ holds, Then*
(i) system (1.1) has at least one positive solution for all $(\lambda,\mu)\in C_{\Lambda_1}$.
(ii) there is $\Lambda_2<\Lambda_1$ such that (1.1) has at least two positive solutions for $(\lambda,\mu)\in C_{\Lambda_2}$.

The paper is organized as follows. In a preliminary Section 2 we describe the appropriate functional setting for the study of problem (1.1), including the definition of an equivalent problem. In Section 3 we show that the Palais-Smale condition holds for the energy functional associated with (1.1) at energy levels in a suitable range related to the best Sobolev constants. In Section 4 we investigate the existence of Palais-Smale sequences and we obtain solutions to some related local minimization problems. Finally, the proof of Theorem 1.1 is given in Section 5.

2. Functional setting and definitions

In this section, we collect some preliminary facts in order to establish the functional setting. We denote the upper half-space in $\mathbb{R}^{N+1}_+$ by

$$\mathbb{R}^{N+1}_+ := \left\{ z = (x,y) = (x_1, \cdots, x_N, y) \in \mathbb{R}^{N+1} : y > 0 \right\},$$

the half cylinder standing on a bounded smooth domain $\Omega \subset \mathbb{R}^N$ by $\mathcal{C}_\Omega := \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$, the cylinder with base Ω and its lateral boundary by $\partial_L \mathcal{C}_\Omega := \partial\Omega \times (0,\infty)$. Let e_k be an orthonnormal basis of $L^2(\Omega)$ with $\|e_k\|_{L^2(\Omega)} = 1$ forming a spectral decomposition of $-\Delta$ in Ω with zero Dirichlet boundary conditions and λ_k be the corresponding eigenvalues. Let

$$H_0^s(\Omega) = \left\{ u = \sum_{j=1}^{\infty} a_j e_j \in L^2(\Omega) : \|u\|_{H_0^s(\Omega)} = \left(\sum_{j=1}^{\infty} a_j^2 \lambda_j^s \right)^{1/2} < \infty \right\}.$$

For any $u \in H_0^s(\Omega)$, the spectral fractional Laplacian $(-\Delta)^s$, is defined by

$$(-\Delta)^s u = \sum_{i \in \mathbb{N}} a_i \lambda_i^s e_i.$$

Note that $\|u\|_{H_0^s(\Omega)} = \left\| (-\Delta)^{s/2} u \right\|_{L^2(\Omega)}$. The dual space $H^{-s}(\Omega)$ is defined in the standard way, as well as the inverse operator $(-\Delta)^{-s}$.

Definition 2.1. *We say that $(u,v) \in H_0^s(\Omega) \times H_0^s(\Omega)$ is a weak solution of (1.1) such that for all $(\varphi_1, \varphi_2) \in H_0^s(\Omega) \times H_0^s(\Omega)$, it holds*

$$\begin{aligned} &\int_\Omega \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi_1 + (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} \varphi_2 \right) dx \\ &= \int_\Omega \left(\frac{\eta_1 \alpha_1}{2_s^*} |u|^{\alpha_1 - 2} |v|^{\beta_1} u \varphi_1 + \frac{\eta_2 \alpha_2}{2_s^*} |u|^{\alpha_2 - 2} |v|^{\beta_2} u \varphi_1 \right) dx \\ &+ \int_\Omega \left(\frac{\eta_1 \beta_1}{2_s^*} |u|^{\alpha_1} |v|^{\beta_1 - 2} v \varphi_2 + \frac{\eta_2 \beta_2}{2_s^*} |u|^{\alpha_2} |v|^{\beta_2 - 2} v \varphi_2 \right) dx \\ &+ \int_\Omega \left(\lambda \frac{|u|^{q-2} u}{|x|^\gamma} \varphi_1 + \mu \frac{|v|^{q-2} v}{|x|^\gamma} \varphi_2 \right) dx. \end{aligned} \tag{2.1}$$

Associated with problem (1.1), we consider the energy functional

$$\mathcal{J}_{\lambda,\mu}(u,v) := \frac{1}{2} \int_\Omega \left(|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 \right) dx - \frac{1}{2_s^*} Q(u,v) - \frac{1}{q} K(u,v), \tag{2.2}$$

where

$$Q(u,v) := \int_\Omega \left(\eta_1 |u|^{\alpha_1} |v|^{\beta_1} + \eta_2 |u|^{\alpha_2} |v|^{\beta_2} \right) dx$$

and

$$K(u,v) := \int_\Omega \left(\lambda \frac{|u|^q}{|x|^\gamma} + \mu \frac{|v|^q}{|x|^\gamma} \right) dx.$$

The functional is well defined in $H_0^s(\Omega) \times H_0^s(\Omega)$, and moreover, the critical points of the functional $\mathcal{J}_{\lambda,\mu}$ correspond to weak solutions of (1.1).

We now conclude the main ingredients of a recently developed technique which can deal with fractional power of the Laplacian. To treat the nonlocal problem (1.1), we will study a corresponding extension problem, so that we can investigate problem (1.1) by studying a local problem via classical variational methods.

We first define the extension operator and fractional Laplacian for functions in $H_0^s(\Omega)$. We refer the reader to [3,5,9] and to the referencess therein.

Definition 2.2. *For a function $u \in H_0^s(\Omega)$, we denote its s-harmonic extension $w = E_s(u)$ to the cylinder C_Ω as the solution of the problem*

$$\begin{cases} -\operatorname{div}\left(y^{1-2s}\nabla w\right) = 0 & \text{in } C_\Omega \\ w = 0 & \text{on } \partial_L C_\Omega \\ w = u. & \text{on } \Omega \times \{0\}, \end{cases}$$

and

$$(-\Delta)^s u(x) = -k_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}(x,y),$$

where k_s is a normalization constant. Define $H_{0,L}^s(C_\Omega)$ as the closure of $C_0^\infty(C_\Omega)$ under the norm

$$\|w\|_{H_{0,L}^s(C_\Omega)} := \left(k_s \int_{C_\Omega} y^{1-2s}|\nabla w|^2 dxdy\right)^{1/2}.$$

We will use the following notations:

$$\mathcal{H} := H_0^s(\Omega) \times H_0^s(\Omega), \quad \tilde{\mathcal{H}} := H_{0,L}^s(C_\Omega) \times H_{0,L}^s(C_\Omega),$$

$$L_s w = -\operatorname{div}\left(y^{1-2s}\nabla w\right), \qquad \frac{\partial w}{\partial \nu^s} := -k_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y},$$

$$\text{and } \|(w_1,w_2)\|^2_{\tilde{\mathcal{H}}} = \|w_1\|^2_{H_{0,L}^s(C_\Omega)} + \|w_2\|^2_{H_{0,L}^s(C_\Omega)}.$$

By the arguments above, we can reformulate our problem (1.1) in terms of the extension problem as follows

$$\begin{cases} L_s w_1 = 0, \quad L_s w_1 = 0 & \text{in } C_\Omega \\ w_1 = w_2 = 0 & \text{on } \partial_L C_\Omega \\ w_1 = u, \quad w_2 = v & \text{on } \Omega\times\{0\} \\ \frac{\partial w_1}{\partial \nu^s} = \frac{\eta_1\alpha_1}{2_s^*}|w_1|^{\alpha_1-2}|w_2|^{\beta_1}w_1 + \frac{\eta_2\alpha_2}{2_s^*}|w_1|^{\alpha_2-2}|w_2|^{\beta_2}w_1 + \lambda\frac{|w_1|^{q-2}w_1}{|x|^\gamma} & \text{on } \Omega\times\{0\} \\ \frac{\partial w_2}{\partial \nu^s} = \frac{\eta_1\beta_1}{2_s^*}|w_1|^{\alpha_1}|w_2|^{\beta_1-2}w_2 + \frac{\eta_2\beta_2}{2_s^*}|w_1|^{\alpha_2}|w_2|^{\beta_2-2}w_2 + \mu\frac{|w_2|^{q-2}w_2}{|x|^\gamma} & \text{on } \Omega\times\{0\}, \end{cases} \tag{2.3}$$

where $(w_1,w_2) = (E_s(u), E_s(v))$.

An energy solution to this problem is a function $(w_1, w_2) \in \tilde{\mathcal{H}}$ such that

$$\begin{aligned} &k_s \int_{C_\Omega} y^{1-2s} \nabla w_1 \cdot \nabla \varphi_1 dxdy + k_s \int_{C_\Omega} y^{1-2s} \nabla w_2 \cdot \nabla \varphi_2 dxdy \\ &= \int_{\Omega\times\{0\}} \left(\frac{\eta_1 \alpha_1}{2_s^*} |w_1|^{\alpha_1 - 2} |w_2|^{\beta_1} w_1 \varphi_1 + \frac{\eta_2 \alpha_2}{2_s^*} |w_1|^{\alpha_2 - 2} |w_2|^{\beta_2} w_1 \varphi_1 \right) dx \\ &+ \int_{\Omega\times\{0\}} \left(\frac{\eta_1 \beta_1}{2_s^*} |w_1|^{\alpha_1} |w_2|^{\beta_1 - 2} w_2 \varphi_2 + \frac{\eta_2 \beta_2}{2_s^*} |w_1|^{\alpha_2} |w_2|^{\beta_2 - 2} w_2 \varphi_2 \right) dx \\ &+ \int_{\Omega\times\{0\}} \left(\lambda \frac{|w_1|^{q-2} w_1}{|x|^\gamma} \varphi_1 + \mu \frac{|w_2|^{q-2} w_2}{|x|^\gamma} \varphi_2 \right) dx, \end{aligned} \tag{2.4}$$

for all $(\varphi_1, \varphi_2) \in \tilde{\mathcal{H}}$.

If (w_1, w_2) satisfies (2.3), then the trace $(u, v) = (w_1(\cdot, 0), w_2(\cdot, 0)) \in \mathcal{H}$ is an energy solution to problem (1.1). The converse is also true. By the equivalence of these two formulations, we will use both formulations in the sequel to their best advantage. Finally, the energy functional associated with problem (2.3) is the following,

$$\begin{aligned} \mathcal{I}_{\lambda,\mu}(w) := \mathcal{I}_{\lambda,\mu}(w_1, w_2) = \frac{1}{2} \|(w_1, w_2)\|_{\tilde{\mathcal{H}}}^2 - \frac{1}{q} \int_{\Omega\times\{0\}} \left(\lambda \frac{|w_1|^q}{|x|^\gamma} + \mu \frac{|w_2|^q}{|x|^\gamma} \right) dx \\ - \frac{1}{2_s^*} \int_{\Omega\times\{0\}} \left(\eta_1 |w_1|^{\alpha_1} |w_2|^{\beta_1} + \eta_2 |w_1|^{\alpha_2} |w_2|^{\beta_2} \right) dx. \end{aligned}$$

Clearly, the critical points of $\mathcal{I}_{\lambda,\mu}$ in $\tilde{\mathcal{H}}$ correspond to critical points of $\mathcal{J}_{\lambda,\mu} : \mathcal{H} \to \mathbb{R}$.

The following lemma is due to [5], which reflect the relationship between the spaces $H_0^s(\Omega)$ and $H_{0,L}^s(C_\Omega)$.

Lemma 2.1. *i)* $\|u\|_{H_0^s(\Omega)} = \left\|E_s(u)\right\|_{H_{0,L}^s(C_\Omega)}$, *for all* $u \in H_0^s(\Omega)$.
ii)For any $1 \leq r \leq 2_s^*$ *and any* $w \in H_{0,L}^s(C_\Omega)$, *it holds*

$$\left(\int_\Omega |u(x)|^r dx \right)^{\frac{2}{r}} \leq C \int_{C_\Omega} y^{1-2s} |\nabla w(x, y)|^2 dxdy, \quad u := \operatorname{Tr}(w), \tag{2.5}$$

for some positive constant $C = C(r, s, N, \Omega)$. *Furthermore, the space* $H_{0,L}^s(C_\Omega)$ *is compactly embedded into* $L^r(\Omega)$, *for every* $r < 2_s^*$.

Remark. When $r = 2_s^*$, the best constant in (2.5) is denoted by $\mathcal{S}(s, N)$, that is

$$\mathcal{S}(s, N) := \inf_{w \in H_{0,L}^s(C_\Omega) \setminus \{0\}} \frac{\int_{C_\Omega} y^{1-2s} |\nabla w(x, y)|^2 dxdy}{\left(\int_\Omega |w(x, 0)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}}. \tag{2.6}$$

It is not achieved in any bounded domain and, for all $w \in H^s\left(\mathbb{R}_+^{N+1}\right)$,

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w(x, y)|^2 dxdy \geq \mathcal{S}(s, N) \left(\int_{\mathbb{R}^N} |w(x, 0)|^{\frac{2N}{N-2s}} dx \right)^{\frac{N-2s}{N}}. \tag{2.7}$$

However, $\mathcal{S}(s,N)$ is indeed achieved for the case $\Omega = \mathbb{R}^N$ when $w_\varepsilon(x,y) = E(u_\varepsilon)$, where u_ε takes the form

$$u_\varepsilon(x) := \frac{\varepsilon^{(N-2s)/2}}{(\varepsilon^2 + |x|^2)^{(N-2s)/2}}, \quad \varepsilon > 0, x \in \mathbb{R}^N. \tag{2.8}$$

Let $U(x) = \left(1 + |x|^2\right)^{\frac{2s-N}{2}}$ and let $\mathcal{W}$ be the extension of U [3,13]. Then

$$\mathcal{W}(x,y) = E_s(U) = C_{N,s} y^{2s} \int_{\mathbb{R}^N} \frac{U(z)dz}{(|x-z|^2 + y^2)^{\frac{N+2s}{2}}},$$

is the extreme function for the fractional Sobolev inequality (2.7). The constant $\mathcal{S}(s,N)$ given in (2.6) takes the exact value

$$\mathcal{S}(s,N) = \frac{2\pi^s \Gamma\left(\frac{N+2s}{2}\right)\Gamma(1-s)\left(\Gamma\left(\frac{N}{2}\right)\right)^{\frac{2s}{N}}}{\Gamma(s)\Gamma\left(\frac{N-2s}{2}\right)(\Gamma(N))^s}.$$

Now, we consider the following minimization problem

$$\mathcal{S}_{\eta,\alpha,\beta} := \inf_{(w_1,w_2)\in\tilde{\mathcal{H}}\setminus\{0\}} \frac{\int_{C_\Omega} y^{1-2s}\left(|\nabla w_1|^2 + |\nabla w_2|^2\right) dxdy}{\left(\int_{\Omega\times\{0\}} \left(\eta_1 |w_1|^{\alpha_1}|w_2|^{\beta_1} + \eta_2 |w_1|^{\alpha_2}|w_2|^{\beta_2}\right) dx\right)^{2/2_s^*}}. \tag{2.9}$$

We define

$$f(\tau) := \frac{1+\tau^2}{(\eta_1\tau^{\beta_1} + \eta_2\tau^{\beta_2})^{\frac{2}{2_s^*}}}, \quad \tau > 0. \tag{2.10}$$

Since f is continuous on $(0,\infty)$ such that $\lim_{\tau\to 0^+} f(\tau) = \lim_{\tau\to+\infty} f(\tau) = +\infty$, then there exists $\tau_0 > 0$ such that

$$f(\tau_0) := \min_{\tau>0} f(\tau) > 0. \tag{2.11}$$

Using ideas from [1], we establish a relationship between $\mathcal{S}(s,N)$ and $\mathcal{S}_{\eta,\alpha,\beta}$.

Lemma 2.2. *For the constants $\mathcal{S}(s,N)$ and $\mathcal{S}_{\eta,\alpha,\beta}$ introduced in (2.6) and (2.9), it holds*

$$\mathcal{S}_{\eta,\alpha,\beta} = f(\tau_0)\,\mathcal{S}(s,N). \tag{2.12}$$

In particular, the constant $\mathcal{S}_{\eta,\alpha,\beta}$ is achieved for $\Omega = \mathbb{R}^N$.

Proof. Let $\{z_n\} \subset H^s_{0,L}(C_\Omega)$ be a minimization sequence for $\mathcal{S}(s,N)$. Consider the sequences $w_{1,n} := z_n$ and $w_{2,n} := \tau_0 z_n$ in $H^s_{0,L}(C_\Omega)$. By (2.9), we have

$$\frac{1+\tau_0^2}{\left(\eta_1\tau_0^{\beta_1} + \eta_2\tau_0^{\beta_2}\right)^{\frac{2}{2_s^*}}} \frac{\int_{C_\Omega} y^{1-2s}|\nabla z_n(x,y)|^2 dxdy}{\left(\int_\Omega |z_n(x,0)|^{2_s^*} dx\right)^{\frac{2}{2_s^*}}} \geq S_{\eta,\alpha,\beta}. \tag{2.13}$$

Letting $n \to +\infty$ yields

$$f(\tau_0)\,\mathcal{S}(s,N) \geq S_{\eta,\alpha,\beta}. \tag{2.14}$$

On the other hand, let $\{(w_{1,n}, w_{2,n})\} \subset \tilde{\mathcal{H}}\backslash\{0\}$ be a minimizing sequence for $\mathcal{S}_{\eta,\alpha,\beta}$. Set $h_n := s_n w_{2,n}$ for $s_n > 0$ with $\int_{\Omega\times\{0\}} |w_{1,n}|^{2_s^*}dx = \int_{\Omega\times\{0\}} |h_n|^{2_s^*}dx$. Then Young's inequality yields

$$\begin{aligned}\int_{\Omega\times\{0\}} |w_{1,n}|^{\alpha}|h_n|^{\beta}dx &\leq \frac{\alpha}{2_s^*}\int_{\Omega\times\{0\}} |w_{1,n}|^{2_s^*}dx + \frac{\beta}{2_s^*}\int_{\Omega\times\{0\}} |h_n|^{2_s^*}dx \\ &= \int_{\Omega\times\{0\}} |h_n|^{2_s^*}dx = \int_{\Omega\times\{0\}} |w_{1,n}|^{2_s^*}dx.\end{aligned}$$

In turn, we can estimate

$$\begin{aligned}&\frac{\int_{C_\Omega} y^{1-2s}\left(|\nabla w_{1,n}(x,y)|^2 + |\nabla w_{2,n}(x,y)|^2\right)dxdy}{\left(\int_{\Omega\times\{0\}} \left(\eta_1|w_{1,n}|^{\alpha_1}|w_{2,n}|^{\beta_1} + \eta_2|w_{1,n}|^{\alpha_2}|w_{2,n}|^{\beta_2}\right)dx\right)^{2/2_s^*}} \\ &\geq \frac{\int_{C_\Omega} y^{1-2s}|\nabla w_{1,n}(x,y)|^2dxdy}{\left(\left(\eta_1 s_n^{-\beta_1} + \eta_2 s_n^{-\beta_2}\right)\int_{\Omega\times\{0\}} |w_{1,n}|^{2_s^*}\right)^{\frac{2}{2_s^*}}} \\ &+ \frac{s_n^{-2}\int_{C_\Omega} y^{1-2s}|\nabla h_n(x,y)|^2dxdy}{\left(\left(\eta_1 s_n^{-\beta_1} + \eta_2 s_n^{-\beta_2}\right)\int_{\Omega\times\{0\}} |h_n|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}} \\ &\geq f\left(s_n^{-1}\right)\mathcal{S}(s,N) \\ &\geq f\left(\tau_0\right)\mathcal{S}(s,N).\end{aligned}$$

Passing to the limit in the last inequality we obtain

$$f\left(\tau_0\right)\mathcal{S}(s,N) \leq \mathcal{S}_{\eta,\alpha,\beta}.$$

Which together with (2.14) implies that

$$S_{\eta,\alpha,\beta} = f\left(\tau_0\right)\mathcal{S}(s,N).$$

□

Let $R_0 > 0$ be a constant such that $\Omega \subset B(0,R_0)$, where $B(0,R_0) = \left\{x \in \mathbb{R}^N : |x| < R_0\right\}$. By Hölder's inequality and (2.6), for all $(w_1, w_1) \in \tilde{H}$, $1 < q < 2$ and $0 \leq \gamma < N + sq - \frac{qN}{2}$, we get

$$\begin{aligned}\int_{\Omega\times\{0\}} \frac{w_1^q}{|x|^{\gamma}}dx &\leq \left(\int_{\Omega\times\{0\}} |w_1|^{q\cdot\frac{2_s^*}{q}}dx\right)^{\frac{q}{2_s^*}}\left(\int_{\Omega}\left(\frac{1}{|x|^{\gamma}}\right)^{\frac{2_s^*}{2_s^*-q}}dx\right)^{\frac{2_s^*-q}{2_s^*}} \\ &\leq \left(k_s\mathcal{S}(s,N)\right)^{-\frac{q}{2}}\|w_1\|^q_{H^s_{0,L}(C_\Omega)}\left(\int_{B(0,R_0)}\left(\frac{1}{|x|^{\gamma}}\right)^{\frac{2_s^*}{2_s^*-q}}dx\right)^{\frac{2_s^*-q}{2_s^*}} \\ &\leq \left(k_s\mathcal{S}(s,N)\right)^{-\frac{q}{2}}\|w_1\|^q_{H^s_{0,L}(C_\Omega)}\left(\int_0^{R_0}\frac{r^{N-1}}{|r|^{\frac{2_s^*\gamma}{2_s^*-q}}}dr\right)^{\frac{2_s^*-q}{2_s^*}} \\ &= \Theta\|w_1\|^q_{H^s_{0,L}(C_\Omega)}\end{aligned} \tag{2.15}$$

and

$$\int_{\Omega\times\{0\}} \frac{w_2^q}{|x|^\gamma} dx \leq \Theta \|w_2\|^q_{H^s_{0,L}(\mathcal{C}_\Omega)}, \tag{2.16}$$

where

$$\Theta := \left(\frac{2N - qN + 2sq}{2N\left(N - \gamma - \frac{qN}{2} + sq\right)} \right)^{\frac{2^*_s - q}{2^*_s}} R_0^{N-\gamma-\frac{qN}{2}+sq} \left(k_s \mathcal{S}(s,N)\right)^{-\frac{q}{2}}. \tag{2.17}$$

Now we are looking for the solutions of problem (1.1). Equivalently, we consider the solutions of problem (2.3). Since the energy functional $\mathcal{I}_{\lambda,\mu}$ is not bounded on $\tilde{\mathcal{H}}$, it is useful to consider the functional on the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} := \left\{ z \in \tilde{\mathcal{H}}\backslash\{0\} : \left\langle \mathcal{I}'_{\lambda,\mu}(z), z \right\rangle = 0 \right\}. \tag{2.18}$$

Thus, $z = (w_1, w_2) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\left\langle \mathcal{I}'_{\lambda,\mu}(z), z \right\rangle = \|z\|^2_{\tilde{\mathcal{H}}} - K(z) - Q(z) = 0, \tag{2.19}$$

where

$$Q(z) := \int_{\Omega\times\{0\}} \left(\eta_1 |w_1|^{\alpha_1} |w_2|^{\beta_1} + \eta_2 |w_1|^{\alpha_2} |w_2|^{\beta_2} \right) dx$$

and

$$K(z) := \int_{\Omega\times\{0\}} \left(\lambda \frac{|w_1|^q}{|x|^\gamma} + \mu \frac{|w_2|^q}{|x|^\gamma} \right) dx.$$

Define $\Phi(z) = \left\langle \mathcal{I}'_{\lambda,\mu}(z), z \right\rangle$, then for all $z = (w_1, w_2) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\begin{aligned} \langle \Phi'(z), z \rangle &= 2\|z\|^2_{\tilde{\mathcal{H}}} - 2^*_s Q(z) - qK(z) \\ &= (2-q)\|z\|^2_{\tilde{\mathcal{H}}} - (2^*_s - q)\, Q(z) \\ &= (2 - 2^*_s)\, \|z\|^2_{\tilde{\mathcal{H}}} + (2^*_s - q)\, K(z). \end{aligned} \tag{2.20}$$

Thus, it is natural to split $\mathcal{N}_{\lambda,\mu}$ into three parts corresponding to local minima, local maxima and points of inflection, i.e.

$$\begin{aligned} \mathcal{N}^+_{\lambda,\mu} &= \left\{ z \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'(z), z \rangle > 0 \right\}, \\ \mathcal{N}^-_{\lambda,\mu} &= \left\{ z \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'(z), z \rangle < 0 \right\}, \\ \mathcal{N}^0_{\lambda,\mu} &= \left\{ z \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'(z), z \rangle = 0 \right\}. \end{aligned} \tag{2.21}$$

It is clear that all critical points of $\mathcal{I}_{\lambda,\mu}$ must, lie on $\mathcal{N}_{\lambda,\mu}$ and, as we will see below, local minimizers on $\mathcal{N}_{\lambda,\mu}$ are actually critical points of $\mathcal{I}_{\lambda,\mu}$. We have the following results.

Lemma 2.3. *The energy functional $\mathcal{I}_{\lambda,\mu}$ is bounded below and coercive on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Let $z = (w_1, w_2) \in \mathcal{N}_{\lambda,\mu}$. From (2.15), (2.16) and (2.19) by the Hölder inequality, we get

$$\begin{aligned} \mathcal{I}_{\lambda,\mu}(z) &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right) \|z\|^2_{\tilde{\mathcal{H}}} - \left(\frac{1}{q} - \frac{1}{2_s^*}\right) K(z) \\ &\geq \frac{s}{N} \|z\|^2_{\tilde{\mathcal{H}}} - \left(\frac{1}{q} - \frac{1}{2_s^*}\right) \left(\lambda \|w_1\|^q_{H^s_{0,L}(C_\Omega)} + \mu \|w_2\|^q_{H^s_{0,L}(C_\Omega)}\right) \Theta \qquad (2.22) \\ &\geq \frac{s}{N} \|z\|^2_{\tilde{\mathcal{H}}} - \left(\frac{1}{q} - \frac{1}{2_s^*}\right) \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z\|^q_{\tilde{\mathcal{H}}} \Theta, \end{aligned}$$

where Θ is given by (2.17). Since $1 < q < 2$, the functional $\mathcal{I}_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$. □

Lemma 2.4. *(Natural Constraint). Suppose that z_0 is a local minimizer of $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ and that $z_0 \notin \mathcal{N}^0_{\lambda,\mu}$, then $\mathcal{I}'_{\lambda,\mu}(z_0) = 0$ in $\tilde{\mathcal{H}}^{-1}$.*

Proof. Suppose that $z_0 = (w_{0,1}, w_{0,2})$ is a local minimizer of $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, then $\mathcal{I}_{\lambda,\mu}(z_0) = \min_{z \in \mathcal{N}_{\lambda,\mu}} \mathcal{I}_{\lambda,\mu}(z)$ and (2.20) holds. Furthermore, by the theory of Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that $\mathcal{I}'_{\lambda,\mu}(z_0) = \theta \Phi'(z_0)$. As $z_0 \in \mathcal{N}_{\lambda,\mu}$, we get

$$0 = \left\langle \mathcal{I}'_{\lambda,\mu}(z_0), z_0 \right\rangle = \theta \left\langle \Phi'(z_0), z_0 \right\rangle.$$

Since $z_0 \notin \mathcal{N}^0_{\lambda,\mu}$, $\langle \Phi'(z_0), z_0 \rangle \neq 0$. Consequently, $\theta = 0$ and $\mathcal{I}'_{\lambda,\mu}(z_0) = 0$ in $\tilde{\mathcal{H}}^{-1}$. □

Let Λ_1 be the positive number defined in (1.4). Then we have the following result.

Lemma 2.5. *Assume that $(\lambda, \mu) \in C_{\Lambda_1}$. Then $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.*

Proof. Assume by contradiction that there exist $\lambda > 0$ and $\mu > 0$ with $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_1$ and such that $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$. Let $z \in \mathcal{N}^0_{\lambda,\mu}$. Then, by virtue of (2.20), we get

$$\|z\|^2_{\tilde{\mathcal{H}}} = \frac{2_s^* - q}{2 - q} Q(z), \quad \|z\|^2_{\tilde{\mathcal{H}}} = \frac{2_s^* - q}{2_s^* - 2} K(z).$$

By Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned} \|z\|_{\tilde{\mathcal{H}}} &\geq \left[\frac{2-q}{(2_s^* - q)} \left(k_s \mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{2_s^*}{2}}\right]^{\frac{1}{2_s^*-2}}, \\ \|z\|_{\tilde{\mathcal{H}}} &\leq \left(\frac{2_s^* - q}{2_s^* - 2} \Theta\right)^{\frac{1}{2-q}} \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{1}{2}}, \end{aligned}$$

which leads to the inequality

$$\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \geq \left(\frac{2_s^* - q}{2_s^* - 2} \Theta\right)^{-\frac{2}{2-q}} \left[\frac{2-q}{(2_s^* - q)} \left(k_s \mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{2_s^*}{2}}\right]^{\frac{2}{2_s^*-2}} = \Lambda_1,$$

contradicting the assumption. □

From Lemma 2.5, if $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_1$, we can write $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$ and define

$$\alpha_{\lambda,\mu} := \inf_{z\in\mathcal{N}_{\lambda,\mu}} \mathcal{I}_{\lambda,\mu}(z), \quad \alpha^+_{\lambda,\mu} := \inf_{z\in\mathcal{N}^+_{\lambda,\mu}} \mathcal{I}_{\lambda,\mu}(z), \quad \alpha^-_{\lambda,\mu} := \inf_{z\in\mathcal{N}^-_{\lambda,\mu}} \mathcal{I}_{\lambda,\mu}(z).$$

Moreover, we have the following properties about the Nehari manifold $\mathcal{N}_{\lambda,\mu}$.

Theorem 2.1. *The following facts holds*
(i) If $(\lambda,\mu) \in \mathcal{C}_{\Lambda_1}$, then we have $\alpha_{\lambda,\mu} \leq \alpha^+_{\lambda,\mu} < 0$;
(ii) If $(\lambda,\mu) \in \mathcal{C}_{(q/2)^{2/(2-q)}\Lambda_1}$, then we have $\alpha^-_{\lambda,\mu} > c_0$ for some positive constant c_0 depending on λ,μ,N,q,s and Θ.

Proof. (i) Let $z = (w_1, w_2) \in \mathcal{N}^+_{\lambda,\mu}$. By (2.18), (2.20) and (2.21), it follows that

$$\frac{2-q}{2^*_s - q}\|z\|^2_{\tilde{\mathcal{H}}} > Q(z). \tag{2.23}$$

According to (2.18) and (2.23), we have that

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z) &= \left(\frac{1}{2} - \frac{1}{q}\right)\|z\|^2_{\tilde{\mathcal{H}}} + \left(\frac{1}{q} - \frac{1}{2^*_s}\right)Q(z)\\ &< \left[\left(\frac{1}{2} - \frac{1}{q}\right) + \left(\frac{1}{q} - \frac{1}{2^*_s}\right)\frac{2-q}{2^*_s - q}\right]\|z\|^2_{\tilde{\mathcal{H}}}\\ &= -\frac{(2-q)s}{qN}\|z\|^2_{\tilde{\mathcal{H}}} < 0.\end{aligned}$$

Therefore, by the definition of $\alpha_{\lambda,\mu}$, $\alpha^+_{\lambda,\mu}$, we can deduce that $\alpha_{\lambda,\mu} \leq \alpha^+_{\lambda,\mu} < 0$.

(ii) Suppose that $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in \left(0, \left(\frac{q}{2}\right)^{\frac{2}{2-q}}\Lambda_1\right)$ and $z = (w_1, w_2) \in \mathcal{N}^-_{\lambda,\mu}$. By (2.9), (2.20) and (2.21), one has

$$\frac{2-q}{2^*_s - q}\|z\|^2_{\tilde{\mathcal{H}}} < Q(z) \leq \left(k_s\mathcal{S}_{\eta,\alpha,\beta}\right)^{-\frac{2^*_s}{2}}\|z\|^{2^*_s}_{\tilde{\mathcal{H}}},$$

which implies that

$$\|z\|_{\tilde{\mathcal{H}}} > \left(\frac{2-q}{2^*_s - q}\right)^{\frac{1}{2^*_s-2}}\left(k_s\mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{N}{4s}}. \tag{2.24}$$

From the last inequality we infer that

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z) &= \frac{2^*_s - 2}{22^*_s}\|z\|^2_{\tilde{\mathcal{H}}} - \frac{2^*_s - q}{q2^*_s}K(z)\\ &\geq \|z\|^q_{\tilde{\mathcal{H}}}\left[\frac{s}{N}\|z\|^{2-q}_{\tilde{\mathcal{H}}} - \left(\frac{2^*_s - q}{2^*_s q}\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\Theta\right]\\ &> \left(\frac{2-q}{2\left(2^*_s - q\right)}\right)^{\frac{q}{2^*_s-2}}\left(k_s\mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{qN}{4s}}\left[\frac{2^*_s - 2}{22^*_s}\left(k_s\mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{(2-q)N}{4s}}\left(\frac{2-q}{2\left(2^*_s - q\right)}\right)^{\frac{2-q}{2^*_s-2}}\right.\\ &\left.-\frac{2^*_s - q}{q2^*_s}\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\Theta\right].\end{aligned}$$

Thus, if $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < (q/2)^{\frac{2}{2-q}}\Lambda_1$, then

$$\mathcal{I}_{\lambda,\mu}(z) > c_0, \quad \text{for all } z \in \mathcal{N}^-_{\lambda,\mu},$$

for some positive constant $c_0 = c_0(\lambda, \mu, q, N, s, \Theta)$. □

Theorem 2.2. *Let $(\lambda, \mu) \in \mathcal{C}_{\Lambda_1}$. Then, for every $z = (w_1, w_2) \in \tilde{\mathcal{H}}$ with $Q(z) > 0$ then there exist (unique) $t^- = t^-(z) > 0$ and $t^+ = t^+(z) > 0$ such that*

$$t^+ z \in \mathcal{N}^+_{\lambda,\mu}, \quad t^- z \in \mathcal{N}^-_{\lambda,\mu}.$$

In particular, we have

$$t^+ < t_{\max} < t^-, \quad t_{\max} := \left(\frac{(2-q)\|z\|^2_{\tilde{\mathcal{H}}}}{(2^*_s - q)\, Q(z)} \right)^{\frac{1}{2^*_s - 2}}$$

as well as

$$\mathcal{I}_{\lambda,\mu}\left(t^+ z\right) = \min_{0<t<t_{\max}} \mathcal{I}_{\lambda,\mu}(tz), \quad \mathcal{I}_{\lambda,\mu}\left(t^- z\right) = \max_{t>0} \mathcal{I}_{\lambda,\mu}(tz).$$

Proof. For each $z \in \tilde{\mathcal{H}}$ such that $Q(z) > 0$, and for all $t \geq 0$, we have

$$\left\langle \mathcal{I}'_{\lambda,\mu}(tz), tz \right\rangle = t^2\|z\|^2_{\tilde{\mathcal{H}}} - t^{2^*_s} Q(z) - t^q K(z).$$

We define $g, h : \mathbb{R}^+ \to \mathbb{R}$ by

$$g(t) := t^{2-q}\|z\|^2_{\tilde{\mathcal{H}}} - t^{2^*_s - q} Q(z) - K(z),$$
$$h(t) := t^{2-q}\|z\|^2_{\tilde{\mathcal{H}}} - t^{2^*_s - q} Q(z).$$

Clearly, we obtain $h(0) = 0$, and $h(t) \to -\infty$ as $t \to \infty$. Because

$$h'(t) = t^{1-q}\left[(2-q)\|z\|^2_{\tilde{\mathcal{H}}} - (2^*_s - q)\, t^{2^*_s - 2} Q(z)\right], \quad \text{for all } t > 0,$$

solving $h'(t) = 0$, we obtain

$$\bar{t}_{\max} = \left[\frac{(2-q)\|z\|^2_{\tilde{\mathcal{H}}}}{(2^*_s - q)\, Q(z)} \right]^{\frac{1}{2^*_s - 2}} > 0.$$

Easy computations show that $h'(t) > 0$ for all $0 < t < \bar{t}_{\max}$ and $h'(t) < 0$ for all $t > \bar{t}_{\max}$. Thus $h(t)$ attains its maximum at $\bar{t}_{\max}$, that is,

$$h(\bar{t}_{\max}) = \left[\frac{(2-q)\|z\|^2_{\tilde{\mathcal{H}}}}{(2^*_s - q)\, Q(z)} \right]^{\frac{2-q}{2^*_s - 2}} \frac{2^*_s - 2}{2^*_s - q}\|z\|^2_{\tilde{\mathcal{H}}}.$$

Then from (2.17), (2.15) and (2.16), by the Holder inequality, one gets

$$\begin{aligned} g(\bar{t}_{\max}) &= h(\bar{t}_{\max}) - K(w_1, w_2) \\ &= \left[\frac{(2-q)\|z\|^2_{\tilde{\mathcal{H}}}}{(2^*_s - q)\, Q(z)} \right]^{\frac{2-q}{2^*_s - 2}} \frac{2^*_s - 2}{2^*_s - q}\|z\|^2_{\tilde{\mathcal{H}}} - \int_{\Omega\times\{0\}} \left(\lambda \frac{w_1^q}{|x|^\gamma} + \mu \frac{w_2^q}{|x|^\gamma} \right) dx \end{aligned} \tag{2.25}$$

$$
\begin{aligned}
&\geq \left[\frac{(2-q)\|z\|_{\tilde{\mathcal{H}}}^{2}}{(2_s^*-q)\,\|z\|_{\tilde{\mathcal{H}}}^{2_s^*}\left(k_s S_{\eta,\alpha,\beta}\right)^{-\frac{2_s^*}{2}}} \right]^{\frac{2-q}{2_s^*-2}} \frac{2_s^*-2}{2_s^*-q}\|z\|_{\tilde{\mathcal{H}}}^{2} \\
&-\left(\lambda\|w_1\|_{H_{0,L}^s(C_\Omega)}^{q} + \mu\|w_2\|_{H_{0,L}^s(C_\Omega)}^{q}\right)\Theta \\
&\geq \left(\frac{2-q}{2_s^*-q}\right)^{\frac{2-q}{2_s^*-2}} \left(k_s S_{\eta,\alpha,\beta}\right)^{\frac{2_s^*(2-q)}{2(2_s^*-2)}} \frac{2_s^*-2}{2_s^*-q}\|z\|_{\tilde{\mathcal{H}}}^{q} - \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z\|_{\tilde{\mathcal{H}}}^{q}\Theta \\
&> 0,
\end{aligned}
$$

where Θ is as in (2.17) and the last inequality holds for every $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Lambda_1)$. It follows that there exist $t^+$and t^-such that

$$g\left(t^+\right) = g\left(t^-\right) \text{ and } \quad g'\left(t^+\right) > 0 > g'\left(t^-\right),$$

for $0 < t^+ < \bar{t}_{\max} < t^-$. We have $t^+z \in \mathcal{N}_{\lambda,\mu}^{+}$, $t^-z \in \mathcal{N}_{\lambda,\mu}^{-}$and

$$\mathcal{I}_{\lambda,\mu}\left(t^-z\right) \geq \mathcal{I}_{\lambda,\mu}(tz) \geq \mathcal{I}_{\lambda,\mu}\left(t^+z\right),$$

for each $t \in [t^+, t^-]$, and $\mathcal{I}_{\lambda,\mu}\left(t^+z\right) \leq \mathcal{I}_{\lambda,\mu}(tz)$ for each $t \in [0, t^+]$. Thus

$$\mathcal{I}_{\lambda,\mu}\left(t^+z\right) = \min_{0\leq t\leq \bar{t}_{\max}} \mathcal{I}_{\lambda,\mu}(tz), \quad \mathcal{I}_{\lambda,\mu}\left(t^-z\right) = \max_{t\geq \bar{t}_{\max}} \mathcal{I}_{\lambda,\mu}(tz).$$

□

3. The Palais-Smale condition

In this section, we will find the range of c where the $(PS)_c$ condition holds for the functional $\mathcal{I}_{\lambda,\mu}$.

Definition 3.1. *Let $c \in \mathbb{R}$ and $\mathcal{I}_{\lambda,\mu} \in C^1(\tilde{\mathcal{H}}, \mathbb{R})$.*
(i) $\{z_n\}$ is a $(PS)_c$-sequence in $\tilde{\mathcal{H}}$ for $\mathcal{I}_{\lambda,\mu}$ if $\mathcal{I}_{\lambda,\mu}(z_n) = c + o(1)$ and $\mathcal{I}'_{\lambda,\mu}(z_n) = o(1)$ strongly in $\tilde{\mathcal{H}}^{-1}$ as $n \to \infty$.
(ii) We say that $\mathcal{I}_{\lambda,\mu}$ satisfies the $(PS)_c$ condition if any $(PS)_c$-sequence $\{z_n\}$ for $\mathcal{I}_{\lambda,\mu}$ has a convergent subsequence in $\tilde{\mathcal{H}}$.

We shall need the following preliminary result.

Lemma 3.1. *(Uniform Lower Bound). Let $\{z_n\} \subset \tilde{\mathcal{H}}$ is a $(PS)_c$-sequence for $\mathcal{I}_{\lambda,\mu}$ with $z_n \rightharpoonup z$ in $\tilde{\mathcal{H}}$ and $\mathcal{I}'_{\lambda,\mu}(z) = 0$ and there exists a positive constant C_0 such that*

$$\mathcal{I}_{\lambda,\mu}(z) \geq -C_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right), \tag{3.1}$$

where

$$C_0 = \frac{2-q}{2}\left[\left(\frac{2N-qN+2sq}{4s}\right)\Theta\right]^{\frac{2}{2-q}}.$$

Proof. Consider $z_n = (w_{1,n}, w_{2,n}) \subset \tilde{\mathcal{H}}$ and $z = (w_1, w_2) \in \tilde{\mathcal{H}}$. If $\{z_n\}$ is a $(PS)_c$-sequence for $\mathcal{I}_{\lambda,\mu}$ with $z_n \rightharpoonup z$ in $\tilde{\mathcal{H}}$, then $w_{1,n} \rightharpoonup w_1$ and $w_{2,n} \rightharpoonup w_2$ in $H^s_{0,L}(\mathcal{C}_\Omega)$, as $n \to \infty$. Then, by virtue of Sobolev embedding theorem (Lemma 2.1), we also have $w_{1,n}(\cdot,0) \to w_1(\cdot,0)$ and $w_{2,n}(\cdot,0) \to w_2(\cdot,0)$ strongly in $L^q(\Omega)$, as $n \to \infty$. Of course, up to a further subsequence, $w_{1,n}(\cdot,0) \to w_1(\cdot,0)$ and $w_{2,n}(\cdot,0) \to w_2(\cdot,0)$ a.e. in Ω. It is standard to check that $\mathcal{I}'_{\lambda,\mu}(z) = 0$. This implies that $\left\langle \mathcal{I}'_{\lambda,\mu}(z), z \right\rangle = 0$, namely

$$k_s \int_{\mathcal{C}_\Omega} y^{1-2s} \left(|\nabla w_1|^2 + |\nabla w_2|^2\right) dxdy = K(w_1,w_2) + Q(w_1,w_2). \tag{3.2}$$

Consequently, we get

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z) = \left(\frac{1}{2} - \frac{1}{2^*_s}\right) k_s \int_{\mathcal{C}_\Omega} y^{1-2s} \left(|\nabla w_1|^2 + |\nabla w_2|^2\right) dxdy \\ - \left(\frac{1}{q} - \frac{1}{2^*_s}\right) \int_{\Omega\times\{0\}} \left(\lambda \frac{|w_1|^q}{|x|^\gamma} + \mu \frac{|w_2|^q}{|x|^\gamma}\right) dx.\end{aligned}$$

Combining (2.15), (2.16) and the Young inequality, we have

$$\begin{aligned}K(z) &\le \left(\lambda \|w_1\|^q_{H^s_{0,L}(\mathcal{C}_\Omega)} + \mu \|w_2\|^q_{H^s_{0,L}(\mathcal{C}_\Omega)}\right)\Theta \\ &= \left(\left[\frac{2}{q}\frac{s}{N}\left(\frac{1}{q} - \frac{1}{2^*_s}\right)^{-1}\right]^{\frac{q}{2}} \|w_1\|^q_{H^s_{0,L}(\mathcal{C}_\Omega)}\right)\left(\left[\frac{2}{q}\frac{s}{N}\left(\frac{1}{q} - \frac{1}{2^*_s}\right)^{-1}\right]^{-\frac{q}{2}} \lambda\Theta\right) \\ &\quad + \left(\left[\frac{2}{q}\frac{s}{N}\left(\frac{1}{q} - \frac{1}{2^*_s}\right)^{-1}\right]^{\frac{q}{2}} \|w_2\|^q_{H^s_{0,L}(\mathcal{C}_\Omega)}\right)\left(\left[\frac{2}{q}\frac{s}{N}\left(\frac{1}{q} - \frac{1}{2^*_s}\right)^{-1}\right]^{-\frac{q}{2}} \mu\Theta\right) \\ &\le \frac{s}{N}\left(\frac{1}{q} - \frac{1}{2^*_s}\right)^{-1} \left(\|w_1\|^2_{H^s_{0,L}(\mathcal{C}_\Omega)} + \|w_2\|^2_{H^s_{0,L}(\mathcal{C}_\Omega)}\right) + \widehat{C}\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right) \\ &= \frac{s}{N}\left(\frac{1}{q} - \frac{1}{2^*_s}\right)^{-1} \|z\|^2_{\tilde{\mathcal{H}}} + \widehat{C}\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right),\end{aligned} \tag{3.3}$$

with

$$\widehat{C} = \frac{2-q}{2}\left(\left[\frac{2}{q}\frac{s}{N}\left(\frac{1}{q} - \frac{1}{2^*_s}\right)^{-1}\right]^{-\frac{q}{2}} \Theta\right)^{\frac{2}{2-q}} = \frac{2-q}{2}\left[\left(\frac{2N - qN + 2sq}{4s}\right)^{\frac{q}{2}} \Theta\right]^{\frac{2}{2-q}}.$$

We obtain

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z) &= \left(\frac{1}{2} - \frac{1}{2^*_s}\right)\|z\|^2_{\tilde{\mathcal{H}}} - \left(\frac{1}{q} - \frac{1}{2^*_s}\right) K(z) \\ &\ge \frac{s}{N}\|z\|^2_{\tilde{\mathcal{H}}} - \frac{s}{N}\|z\|^2_{\tilde{\mathcal{H}}} - C_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right) \\ &= -C_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right).\end{aligned}$$

Then (3.1) follows from (3.3) with $C_0 = \left(\frac{1}{q} - \frac{1}{2^*_s}\right)\widehat{C}$. □

.

Lemma 3.2. *If $\{z_n\} \subset \tilde{\mathcal{H}}$ is a $(PS)_c$-sequence for $\mathcal{I}_{\lambda,\mu}$, then $\{z_n\}$ is bounded in $\tilde{\mathcal{H}}$.*

Proof. Let $z_n = (w_{1,n}, w_{2,n}) \subset \tilde{\mathcal{H}}$ be a $(PS)_c$-sequence for $\mathcal{I}_{\lambda,\mu}$ and suppose, by contradiction, that $\|z\|_{\tilde{\mathcal{H}}} \to \infty$, as $n \to \infty$. Put

$$\widetilde{z}_n = (\widetilde{w}_{1,n}, \widetilde{w}_{2,n}) := \frac{z_n}{\|z\|_{\tilde{\mathcal{H}}}} = \left(\frac{w_{1,n}}{\|z\|_{\tilde{\mathcal{H}}}}, \frac{w_{2,n}}{\|z\|_{\tilde{\mathcal{H}}}}\right).$$

We may assume that $\widetilde{z}_n \rightharpoonup \widetilde{z} = (\widetilde{w}_1, \widetilde{w}_2)$ in $\tilde{\mathcal{H}}$. This implies that $\widetilde{w}_{1,n}(\cdot,0) \to \widetilde{w}_1(\cdot,0)$ and $\widetilde{w}_{2,n}(\cdot,0) \to \widetilde{w}_2(\cdot,0)$ strongly in $L^r(\Omega)$ for all $1 \le r < 2_s^*$ and, thus,

$$\int_{\Omega\times\{0\}} \frac{\lambda|\widetilde{w}_{1,n}|^q}{|x|^\gamma} + \frac{\mu|\widetilde{w}_{2,n}|^q)}{|x|^\gamma} dx = \int_{\Omega\times\{0\}} \frac{\lambda|\widetilde{w}_1|^q}{|x|^\gamma} + \frac{\mu|\widetilde{w}_2|^q)}{|x|^\gamma} dx + o_n(1).$$

Since $\{z_n\}$ is a $(PS)_c$ sequence for $\mathcal{I}_{\lambda,\mu}$ and $\|z\|_{\tilde{\mathcal{H}}} \to \infty$, there following hold:

$$\begin{aligned} &\frac{k_s}{2}\int_{C_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy - \frac{\|z\|_{\tilde{\mathcal{H}}}^{q-2}}{q}\int_{\Omega\times\{0\}} \frac{\lambda|\widetilde{w}_{1,n}|^q}{|x|^\gamma} + \frac{\mu|\widetilde{w}_{2,n}|^q)}{|x|^\gamma} dx \\ &- \frac{\|z\|_{\tilde{\mathcal{H}}}^{2_s^*-2}}{2_s^*}\int_{\Omega\times\{0\}} \eta_1|\widetilde{w}_{1,n}|^{\alpha_1}|\widetilde{w}_{2,n}|^{\beta_1} + \eta_2|\widetilde{w}_{1,n}|^{\alpha_2}|\widetilde{w}_{2,n}|^{\beta_2}dx = o_n(1), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &k_s\int_{C_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy - \|z\|_{\tilde{\mathcal{H}}}^{q-2}\int_{\Omega\times\{0\}} \frac{\lambda|\widetilde{w}_{1,n}|^q}{|x|^\gamma} + \frac{\mu|\widetilde{w}_{2,n}|^q)}{|x|^\gamma} dx \\ &- \|z\|_{\tilde{\mathcal{H}}}^{2_s^*-2}\int_{\Omega\times\{0\}} \left(\eta_1|\widetilde{w}_{1,n}|^{\alpha_1}|\widetilde{w}_{2,n}|^{\beta_1} + \eta_2|\widetilde{w}_{1,n}|^{\alpha_2}|\widetilde{w}_{2,n}|^{\beta_2}\right)dx = o_n(1). \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), as $n \to \infty$, we obtain

$$\begin{aligned} &k_s\int_{C_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy \\ &= \frac{2\,(2_s^*-q)}{q\,(2_s^*-2)}\|z\|_{\tilde{\mathcal{H}}}^{q-2}\int_{\Omega\times\{0\}} \frac{\lambda|\widetilde{w}_{1,n}|^q}{|x|^\gamma} + \frac{\mu|\widetilde{w}_{2,n}|^q)}{|x|^\gamma} dx + o_n(1). \end{aligned} \tag{3.6}$$

In view of $1 < q < 2$ and $\|z\|_{\tilde{\mathcal{H}}} \to \infty$, (3.6) implies that

$$k_s\int_{C_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy \to 0,$$

as $n \to \infty$, which contradicts to the fact that $\left\|\widetilde{z}_n\right\| = 1$ for any $n \ge 1$. □

Lemma 3.3. *Suppose that $(\mathcal{F})$ holds and $0 \le \gamma < N + sq - \frac{qN}{2}$, for all $-\infty < c < c_\infty = \frac{s}{N}\left(S_{\eta,\alpha,\beta}\right)^{\frac{N}{2s}} - C_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)$, then $\mathcal{I}_{\lambda,\mu}$ satisfies the $(PS)_c$ condition in $\tilde{\mathcal{H}}$, where C_0 is given by Lemma 3.1.*

Proof. Let $\{z_n\} \subset \tilde{\mathcal{H}}$ be a $(PS)_c$-sequence satisfying $\mathcal{I}_{\lambda,\mu}(z_n) = c + o(1)$ and $\mathcal{I}'_{\lambda,\mu}(z_n) = o(1)$, where $z_n = (w_{1,n}, w_{2,n})$. By Lemma 3.2 , we see that $\{z_n\}$ is bounded in $\tilde{\mathcal{H}}$. Passing to a subsequence (still denoted by $\{z_n\}$), there exists $z = (w_1, w_2) \in \tilde{\mathcal{H}}$ such that

$$\begin{cases} w_{1,n} \rightharpoonup w_1, \quad w_{2,n} \rightharpoonup w_2, & \text{weakly in } H^s_{0,L}\left(\mathcal{C}_\Omega\right), \\ w_{1,n}(\cdot,0) \to w_1(\cdot,0), \quad w_{2,n}(\cdot,0) \to w_2(\cdot,0), & \text{strongly in } L^r(\Omega)\ (1 \le r < 2^*_s), \\ w_{1,n}(\cdot,0) \to w_1(\cdot,0), \quad w_{2,n}(\cdot,0) \to w_2(\cdot,0), & \text{a.e. in } \Omega. \end{cases} \tag{3.7}$$

Hence, we have

$$\int_{\Omega\times\{0\}} \frac{\lambda|w_{1,n}|^q}{|x|^\gamma} + \frac{\mu|w_{2,n}|^q)}{|x|^\gamma} dx = \int_{\Omega\times\{0\}} \frac{\lambda|w_1|^q}{|x|^\gamma} + \frac{\mu|w_2|^q)}{|x|^\gamma} dx + o_n(1).$$

Set $\widehat{w}_{1,n} := w_{1,n} - w_1$, $\widehat{w}_{2,n} := w_{2,n} - w_2$ and $\widehat{z}_n := (\widehat{w}_{1,n}, \widehat{w}_{2,n})$. From Brézis-Lieb's lemma [6], it follows that

$$\left\|\widehat{z}_n\right\|^2_{\tilde{\mathcal{H}}} = \|z\|^2_{\tilde{\mathcal{H}}} - \|z\|^2_{\tilde{\mathcal{H}}} + o_n(1), \tag{3.8}$$

and by Lemma (2.1) in [11] one has

$$\int_{\Omega\times\{0\}} |\widehat{w}_{1,n}|^{\alpha_i} |\widehat{w}_{2,n}|^{\beta_i} dx = \int_{\Omega\times\{0\}} |w_{1,n}|^{\alpha_i} |w_{2,n}|^{\beta_i} dx - \int_{\Omega\times\{0\}} |w_1|^{\alpha_i} |w_2|^{\beta_i} dx + o(1), \quad i = 1, 2. \tag{3.9}$$

Since $\mathcal{I}_{\lambda,\mu}(z_n) = c + o(1)$ and $\mathcal{I}'_{\lambda,\mu}(z_n) = o(1)$, and by (3.7) to (3.9), we can deduce that

$$\frac{1}{2}\left\|\widehat{z}_n\right\|^2_{\tilde{\mathcal{H}}} - \frac{1}{2^*_s} \int_{\Omega\times\{0\}} \left(\eta_1 |\widehat{w}_{1,n}|^{\alpha_1} |\widehat{w}_{2,n}|^{\beta_1} + \eta_2 |\widehat{w}_{1,n}|^{\alpha_2} |\widehat{w}_{2,n}|^{\beta_2}\right) dx = c - \mathcal{I}_{\lambda,\mu}(z) + o_n(1), \tag{3.10}$$

and

$$\begin{aligned} \left\|\widehat{z}_n\right\|^2_{\tilde{\mathcal{H}}} &- \int_{\Omega\times\{0\}} \left(\eta_1 |\widehat{w}_{1,n}|^{\alpha_1} |\widehat{w}_{2,n}|^{\beta_1} + \eta_2 |\widehat{w}_{1,n}|^{\alpha_2} |\widehat{w}_{2,n}|^{\beta_2}\right) dx \\ &= \left\langle \mathcal{I}'_{\lambda,\mu}(z_n), z_n \right\rangle - \left\langle \mathcal{I}'_{\lambda,\mu}(z), z \right\rangle + o_n(1) = o_n(1). \end{aligned} \tag{3.11}$$

Now, we can assume that

$$\lim_{n\to\infty} \left\|\widehat{z}_n\right\|^2 = \lim_{n\to\infty} Q\left(\widehat{z}_n\right) = l. \tag{3.12}$$

If $l = 0$, the proof is complete. Assume $l > 0$, then it follows from (3.12) and the definition of $S_{\eta,\alpha,\beta}$ that

$$\left\|\widehat{z}_n\right\|^2_{\tilde{\mathcal{H}}} \ge S_{\eta,\alpha,\beta} Q^{\frac{2}{2^*_s}}\left(\widehat{z}_n\right),$$

$$k_s \mathcal{S}_{\eta,\alpha,\beta} \ell^{\frac{2}{2^*_s}} = k_s \mathcal{S}_{\eta,\alpha,\beta} \lim_{n\to\infty} Q^{\frac{2}{2^*_s}}\left(\widehat{z}_n\right) \le \lim_{n\to\infty} \left\|\widehat{z}_n\right\|^2_{\tilde{\mathcal{H}}} = \ell,$$

which implies that

$$\ell \ge \left(k_s \mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{N}{2s}}. \tag{3.13}$$

In addition, from (3.12) to (3.13) and Lemma 3.1, we have

$$c = \left(\frac{1}{2} - \frac{1}{2^*_s}\right)\ell + \mathcal{I}_{\lambda,\mu}(z) \ge \frac{s}{N}\left(k_s \mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{N}{2s}} - C_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right) = c_\infty,$$

which contradicts the definition of c. Therefore, $l = 0$ and $(w_{1,n}, w_{2,n}) \to (w_1, w_2)$ strongly in $\tilde{\mathcal{H}}$. The proof is complete. □

4. Existence of Palais-Smale sequences

Lemma 4.1. *Let $(\lambda,\mu)\in \mathcal{C}_{\Lambda_1}$. Then, for any $z\in \mathcal{N}_{\lambda,\mu}$, there exists $r>0$ and a differentiable map $\xi: B(0;r)\subset \tilde{\mathcal{H}}\to \mathbb{R}^+$such that $\xi(0)=1$ and $\xi(h)(z-h)\in \mathcal{N}_{\lambda,\mu}$ for every $h\in B(0;r)$. Let us set*

$$
\begin{aligned}
\mathcal{D}_1 &:= 2k_s \int_{\Omega} y^{1-2s}\left(\nabla w_1\cdot\nabla h_1+\nabla w_2\cdot\nabla h_2\right)dxdy\\
\mathcal{D}_2 &:= q\int_{\Omega\times\{0\}}\left(\frac{\lambda|w_1|^{q-2}w_1h_1}{|x|^{\gamma}}+\frac{\mu|w_2|^{q-2}w_2h_2}{|x|^{\gamma}}\right)dx\\
\mathcal{D}_3 &:= \int_{\Omega\times\{0\}}\left(\alpha_1\eta_1|w_1|^{\alpha_1-2}w_1h_1|w_2|^{\beta_1}+\beta_1\eta_1|w_1|^{\alpha_1}|w_2|^{\beta_1-2}w_2h_2\right)dx\\
&+\int_{\Omega\times\{0\}}\left(\alpha_2\eta_2|w_1|^{\alpha_2-2}w_1h_1|w_2|^{\beta_2}+\beta_2\eta_2|w_1|^{\alpha_2}|w_2|^{\beta_2-2}w_2h_2\right)dx,
\end{aligned}
$$

for all $(h_1,h_2)\in\tilde{\mathcal{H}}$ and $(w_1,w_2)\in\tilde{\mathcal{H}}$. Then

$$
\langle \xi'(0),h\rangle=\frac{\mathcal{D}_3+\mathcal{D}_2-\mathcal{D}_1}{(2-q)\|z\|^2_{\tilde{\mathcal{H}}}-(2_s^*-q)\,Q(w_1,w_2)} \tag{4.1}
$$

for all $(h_1,h_2)\in\tilde{\mathcal{H}}$.

Proof. The proof is almost the same as in [17]. For $z=(w_1,w_2)\in\mathcal{N}_{\lambda,\mu}$, define a function $F_z:\mathbb{R}\times\tilde{\mathcal{H}}\to\mathbb{R}$ by

$$
\begin{aligned}
F_z(\xi,w)=&\left\langle \mathcal{I}'_{\lambda,\mu}(\xi(z-p)),\xi(z-p)\right\rangle\\
=&\xi^2\|z-p\|^2_{\tilde{\mathcal{H}}}-\xi^{2_s^*}Q(z-p)-\xi^qK(z-p).
\end{aligned}
$$

Then $F_z(1,0)=\left\langle \mathcal{I}'_{\lambda,\mu}(z),z\right\rangle=0$ and, by Lemma 2.5 , we have

$$
\begin{aligned}
\frac{d}{d\xi}F_z(1,0)&=\langle\Phi'(z),z\rangle\\
&=(2-2_s^*)\,\|z\|^2_{\tilde{\mathcal{H}}}+(2_s^*-q)\,Q(z)\neq 0.
\end{aligned}
$$

According to the implicit function theorem, there exist $\eta>0$ and a differentiable function $\xi: B(0;\eta)\subset\tilde{\mathcal{H}}\to\mathbb{R}$ such that $\xi(0)=1$ and formula (4.1) holds, via direct computation. Moreover,

$$
F_z(\xi(h),h)=0\quad \text{for all } h\in B(0;\eta),
$$

which is equivalent to

$$
\left\langle \mathcal{I}'_{\lambda,\mu}(\xi(h)(z-h)),\xi(h)(z-h)\right\rangle=0\quad\text{for all } h\in B(0;\eta),
$$

that is $\xi(h)(z-h)\in\mathcal{N}_{\lambda,\mu}$. □

Lemma 4.2. *Let $(\lambda,\mu)\in\mathcal{C}_{\Lambda_1}$. Then, for any $z\in\mathcal{N}^-_{\lambda,\mu}$, there exists $r>0$ and a differentiable map $\xi^-: B(0;r)\subset\tilde{\mathcal{H}}\to\mathbb{R}^+$such that $\xi^-(0)=1$ and $\xi^-(h)(z-h)\in\mathcal{N}^-_{\lambda,\mu}$ for every $h\in B(0;r)$.*

Proof. Arguing as for the proof of Lemma 4.1, there exists $r > 0$ and a differentiable function $\xi^- : B(0;r) \subset \tilde{\mathcal{H}} \to \mathbb{R}^+$such that $\xi^-(0) = 1$, $\xi^-(h)(z-h) \in \mathcal{N}_{\lambda,\mu}$ for all $h \in B(0;r)$ and formula (4.1) holds. Since

$$\langle \Phi'(z), z\rangle = (2-q)\|z\|^2_{\tilde{\mathcal{H}}} - (2^*_s - q)\, Q(z) < 0,$$

by the continuity of the functions Φ' and ξ^-, up to reducing the size of $r > 0$, we get

$$\langle \Phi'\left(\xi^-(h)(z-h)\right), \xi^-(h)(z-h)\rangle < 0.$$

This implies that the functions $\xi^-(h)(z-h)$ belong to $\mathcal{N}^-_{\lambda,\mu}$. □

Proposition 4.1. *The following facts hold.*
(i) Let $(\lambda,\mu) \in \mathcal{C}_{\Lambda_1}$. Then there is a $(PS)_{\alpha_{\lambda,\mu}}$-sequence $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$ for $\mathcal{I}_{\lambda,\mu}$.
(ii) Let $(\lambda,\mu) \in \mathcal{C}_{(q/2)^{2/(2-q)}\Lambda_1}$. Then there is a $(PS)_{\alpha^-_{\lambda,\mu}}$-sequence $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$ for $\mathcal{I}_{\lambda,\mu}$.

Proof. (i) By Lemma 2.3 and Ekeland Variational Principle [10], there exists a minimizing sequence $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$\mathcal{I}_{\lambda,\mu}(z_n) < \alpha_{\lambda,\mu} + \frac{1}{n},$$

$$\mathcal{I}_{\lambda,\mu}(z_n) < \mathcal{I}_{\lambda,\mu}(w) + \frac{1}{n}\|w - z_n\|_{\tilde{\mathcal{H}}}, \quad \text{for each } w \in \mathcal{N}_{\lambda,\mu}. \tag{4.2}$$

Taking n large and using $\alpha_{\lambda,\mu} < 0$, we have

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z_n) &= \left(\frac{1}{2} - \frac{1}{2^*_s}\right)\|z\|^2_{\tilde{\mathcal{H}}} - \left(\frac{1}{q} - \frac{1}{2^*_s}\right)\int_{\Omega\times\{0\}}\left(\lambda\frac{|w_{1,n}|^q}{|x|^\gamma} + \mu\frac{|w_{2,n}|^q}{|x|^\gamma}\right)dx \\ &< \alpha_{\lambda,\mu} + \frac{1}{n} < \frac{\alpha_{\lambda,\mu}}{2}.\end{aligned} \tag{4.3}$$

This yiclds that

$$\begin{aligned}-\frac{q2^*_s}{2\left(2^*_s - q\right)}\alpha_{\lambda,\mu} &< \int_{\Omega\times\{0\}}\left(\lambda\frac{|w_{1,n}|^q}{|x|^\gamma} + \mu\frac{|w_{2,n}|^q}{|x|^\gamma}\right)dx \\ &\le \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\|z\|^q_{\tilde{\mathcal{H}}}\Theta.\end{aligned} \tag{4.4}$$

Consequently, $z_n \neq 0$ and combining with (4.3) and (4.4) and using Hölder inequality

$$\|z\|_{\tilde{\mathcal{H}}} > \left[-\frac{q2^*_s}{2\left(2^*_s - q\right)\Theta}\alpha_{\lambda,\mu}\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{q-2}{2}}\right]^{\frac{1}{q}},$$

and

$$\|z\|_{\tilde{\mathcal{H}}} < \left[\frac{2\left(2^*_s - q\right)}{q\left(2^*_s - 2\right)}\Theta\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\right]^{\frac{1}{2-q}}. \tag{4.5}$$

Now we prove that

$$\left\|\mathcal{I}'_{\lambda,\mu}(z_n)\right\|_{\tilde{\mathcal{H}}^{-1}} \to 0, \text{ as } n \to \infty.$$

Fix $n \in \mathbb{N}$. By applying Lemma 4.2 to z_n, we obtain the function $\xi_n : B(0; r_n) \to \mathbb{R}^+$ for some $r_n > 0$, such that $\xi_n(h)(z_n - h) \in \mathcal{N}_{\lambda,\mu}$. Take $0 < \sigma < r_n$. Let $w \in \tilde{\mathcal{H}}$ with $w \neq 0$ and put $h^* = \frac{\sigma w}{\|w\|_{\tilde{\mathcal{H}}}}$. We set $h_\rho = \xi_n(h^*)(z_n - h^*)$, then $h_\sigma \in \mathcal{N}_{\lambda,\mu}$, and we have from (4.2) that

$$\mathcal{I}_{\lambda,\mu}(h_\sigma) - \mathcal{I}_{\lambda,\mu}(z_n) \geq -\frac{1}{n}\|h_\sigma - z_n\|_{\tilde{\mathcal{H}}}.$$

By the Mean Value Theorem, we get

$$\left\langle \mathcal{I}'_{\lambda,\mu}(z_n), h_\sigma - z_n \right\rangle + o\left(\|h_\sigma - z_n\|_{\tilde{\mathcal{H}}}\right) \geq -\frac{1}{n}\|h_\sigma - z_n\|_{\tilde{\mathcal{H}}}.$$

Thus, we have

$$\left\langle \mathcal{I}'_{\lambda,\mu}(z_n), -h^* \right\rangle + (\xi_n(h^*) - 1)\left\langle \mathcal{I}'_{\lambda,\mu}(z_n), z_n - h^* \right\rangle \geq -\frac{1}{n}\|h_\sigma - z_n\|_{\tilde{\mathcal{H}}} + o\left(\|h_\sigma - z_n\|_{\tilde{\mathcal{H}}}\right).$$

Whence, from $\xi_n(h^*)(z_n - h^*) \in \mathcal{N}_{\lambda,\mu}$, it follows that

$$\begin{aligned} -\sigma\left\langle \mathcal{I}'_{\lambda,\mu}(z_n), \frac{w}{\|w\|_{\tilde{\mathcal{H}}}} \right\rangle + (\xi_n(h^*) - 1)\left\langle \mathcal{I}'_{\lambda,\mu}(z_n) - \mathcal{I}'_{\lambda,\mu}(h_\sigma), z_n - h^* \right\rangle \\ \geq -\frac{1}{n}\|h_\sigma - z_n\|_{\tilde{\mathcal{H}}} + o\left(\|h_\sigma - z_n\|_{\tilde{\mathcal{H}}}\right). \end{aligned}$$

So, we get

$$\begin{aligned} \left\langle \mathcal{I}'_{\lambda,\mu}(z_n), \frac{w}{\|w\|_{\tilde{\mathcal{H}}}} \right\rangle &\leq \frac{1}{n\sigma}\|h_\sigma - z_n\|_{\tilde{\mathcal{H}}} + \frac{o\left(\|h_\sigma - z_n\|_{\tilde{\mathcal{H}}}\right)}{\sigma} \\ &\quad + \frac{(\xi_n(h^*) - 1)}{\sigma}\left\langle \mathcal{I}'_{\lambda,\mu}(z_n) - \mathcal{I}'_{\lambda,\mu}(h_\sigma), z_n - h^* \right\rangle. \end{aligned} \tag{4.6}$$

Since $\|h_\sigma - z_n\|_{\tilde{\mathcal{H}}} \leq \sigma|\xi_n(h^*)| + |\xi_n(h^*) - 1|\|z\|_{\tilde{\mathcal{H}}}$ and

$$\lim_{\sigma\to 0}\frac{|\xi_n(h^*) - 1|}{\sigma} \leq \left\|\xi'_n(0)\right\|_{\tilde{\mathcal{H}}}.$$

For fixed $n \in \mathbb{N}$, if we let $\sigma \to 0$ in (4.6), then by virtue of (4.5) we can choose a constant $C > 0$ independent of σ such that

$$\left\langle \mathcal{I}'_{\lambda,\mu}(z_n), \frac{w}{\|w\|_{\tilde{\mathcal{H}}}} \right\rangle \leq \frac{C}{n}\left(1 + \left\|\xi'_n(0)\right\|_{\tilde{\mathcal{H}}}\right).$$

Thus, we are done once we prove that $\left\|\xi'_n(0)\right\|_{\tilde{\mathcal{H}}}$ remains uniformly bounded. By (4.1), (4.5) and Hölder inequality, we have

$$|\langle \xi'_n(0), h\rangle| \leq \frac{C_1\|h\|_{\tilde{\mathcal{H}}}}{(2-q)\|z\|^2_{\tilde{\mathcal{H}}} - Q(w_{1,n}, w_{2,n})}$$

for some $C_1 > 0$. We only need to prove that

$$|(2-q)\|z\|^2_{\tilde{\mathcal{H}}} - (2^*_s - q) - Q(w_{1,n}, w_{2,n})| \geq C_2,$$

for some $C_2 > 0$ and n large enough. We argue by contradiction. Suppose that there exists a subsequence $\{z_n\}$ such that

$$(2-q)\|z\|^2_{\tilde{\mathcal{H}}} - 2(2^*_s - q)\, Q(w_{1,n}, w_{2,n}) = o_n(1). \tag{4.7}$$

By virtue of (4.7) and the fact that $z_n \in \mathcal{N}_{\lambda,\mu}$, we have

$$\|z\|^2_{\tilde{\mathcal{H}}} = \frac{(2^*_s - q)}{2-q} Q(w_{1,n}, w_{2,n}) + o_n(1), \quad \|z\|^2_{\tilde{\mathcal{H}}} = \frac{2^*_s - q}{2^*_s - 2} K(z_n) + o_n(1).$$

Taking into account that $\mathcal{I}_{\lambda,\mu}(z_n) \to \alpha_{\lambda,\mu} < 0$ as $n \to \infty$, we have $\|z\|_{\tilde{\mathcal{H}}} \nrightarrow 0$ as $n \to \infty$. Then, arguing as in the proof of Lemma 2.5 yields $(\lambda, \mu) \notin \mathcal{C}_{\Lambda_1}$, a contradiction. Then,

$$\left\langle \mathcal{I}'_{\lambda,\mu}(z_n), \frac{w}{\|w\|_{\tilde{\mathcal{H}}}} \right\rangle \le \frac{C}{n}.$$

This proves (i). By Lemma 4.2, one can prove (ii), but we shall omit the details here. □

Now, we establish the existence of a local minimizer for $\mathcal{I}_{\lambda,\mu}$ in $\mathcal{N}^+_{\lambda,\mu}$.

Proposition 4.2. *Let $(\lambda, \mu) \in \mathcal{C}_{\Lambda_1}$. Then $\mathcal{I}_{\lambda,\mu}$ has a local minimizer z^+ in $\mathcal{N}^+_{\lambda,\mu}$ satisfying the following conditions:*
(i) $\mathcal{I}_{\lambda,\mu}(z^+) = \alpha_{\lambda,\mu} = \alpha^+_{\lambda,\mu} < 0$;
(ii) z^+ is a positive solution of (2.3).

Proof. By (i) of Proposition 4.1, there exists a minimizing sequence $\{z_n\} = \{(w_{1,n}, w_{2,n})\}$ for $\mathcal{I}_{\lambda,\mu}$ in $\mathcal{N}_{\lambda,\mu}$ such that, as $n \to \infty$,

$$\mathcal{I}_{\lambda,\mu}(z_n) = \alpha_{\lambda,\mu} + o_n(1) \text{ and } \mathcal{I}'_{\lambda,\mu}(z_n) = o_n(1) \text{ in } \tilde{\mathcal{H}}^{-1}. \tag{4.8}$$

By Lemma 2.3, we see that $\mathcal{I}_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$, and $\{z_n\}$ is bounded in $\tilde{\mathcal{H}}$. Then there exists a subsequence, still denoted by $\{z_n\}$ and $z^+ = \left(w_1^+, w_2^+\right) \in \tilde{\mathcal{H}}$ such that, as $n \to \infty$,

$$\begin{cases} w_{1,n} \rightharpoonup w_1^+, w_{2,n} \rightharpoonup w_2^+, \text{weakly in } H^s_{0,L}(\mathcal{C}_\Omega), \\ w_{1,n} \to w_1^+, w_{2,n} \to w_2^+, \text{strongy in } L^r(\Omega) \text{ for all } 1 \le r < 2^*_s, \\ w_{1,n} \to w_1^+, w_{2,n} \to w_2^+, \text{a.e. in } \Omega, \end{cases}$$

up to subsequences. This implies that, as $n \to \infty$,

$$K(z_n) = K\left(z^+\right) + o_n(1). \tag{4.9}$$

We claim that $z^+$is a nontrivial solution of (2.3). From (4.8) and (4.9), it is easy to verify that $z^+$is a weak solution of (2.3). From $z_n \in \mathcal{N}_{\lambda,\mu}$ and (2.3) we deduce that

$$K(z_n) = \frac{q(2^*_s - 2)}{2(2^*_s - q)} \|z\|^2_{\tilde{\mathcal{H}}} - \frac{q 2^*_s}{2^*_s - q} \mathcal{I}_{\lambda,\mu}(z_n). \tag{4.10}$$

Let $n \to \infty$ in (4.10), by (4.8), (4.9) and $\alpha_{\lambda,\mu} < 0$, we have

$$K\left(z^+\right) \ge -\frac{q 2^*_s}{2^*_s - q} \alpha_{\lambda,\mu} > 0.$$

Therefore, $z^+ \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial solution of (2.3). Now we show that $z_n \to z^+$strongly in $\tilde{\mathcal{H}}$ and $\mathcal{I}_{\lambda,\mu}(z^+) = \alpha_{\lambda,\mu}$. Since $z^+ \in \mathcal{N}_{\lambda,\mu}$, then by (4.10), we obtain

$$\begin{aligned}\alpha_{\lambda,\mu} &\leq \mathcal{I}_{\lambda,\mu}\left(z^+\right)\\ &= \frac{s}{N}\|z^+\|^2_{\tilde{\mathcal{H}}} - \frac{2^*_s - q}{q2^*_s}K\left(z^+\right)\\ &\leq \lim_{n\to\infty}\left(\frac{s}{N}\|z\|^2_{\tilde{\mathcal{H}}} - \frac{2^*_s - q}{q2^*_s}K(z_n)\right)\\ &= \lim_{n\to\infty}\mathcal{I}_{\lambda,\mu}(z_n) = \alpha_{\lambda,\mu}.\end{aligned}$$

This implies that $\mathcal{I}_{\lambda,\mu}(z^+) = \alpha_{\lambda,\mu}$ and $\lim_{n\to\infty}\|z\|^2_{\tilde{\mathcal{H}}} = \|z^+\|^2_{\tilde{\mathcal{H}}}$. Set $\widehat{z_n} = z_n - z^+$. Then, that

$$\left\|\widehat{z_n}\right\|^2_{\tilde{\mathcal{H}}} = \|z\|^2_{\tilde{\mathcal{H}}} - \|z^+\|^2_{\tilde{\mathcal{H}}} + o_n(1).$$

Hence, $z_n \to z^+$in $\tilde{\mathcal{H}}$. We claim that $z^+ \in \mathcal{N}^+_{\lambda,\mu}$. Assume by contradiction that $z^+ \in \mathcal{N}^-_{\lambda,\mu}$. Then, by Theorem 2.2 , there exist (unique) $t_1^+$and $t_1^-$with $t_1^+ z^+ \in \mathcal{N}^+_{\lambda,\mu}$and $t_1^- z^+ \in \mathcal{N}^-_{\lambda,\mu}$ In particular, we have $t_1^+ < t_1^- = 1$. Since

$$\frac{d}{dt}\mathcal{I}_{\lambda,\mu}\left(tz^+\right)|_{t=t_1^+} = 0, \text{ and } \frac{d^2}{dt^2}\mathcal{I}_{\lambda,\mu}\left(tz^+\right)|_{t=t_1^+} > 0,$$

there exists $t_1^+ < t^* \leq t_1^-$such that $\mathcal{I}_{\lambda,\mu}\left(t_1^+ z^+\right) < \mathcal{I}_{\lambda,\mu}(t^* z^+)$. By Theorem 2.2, we have

$$\mathcal{I}_{\lambda,\mu}\left(t_1^+ z^+\right) < \mathcal{I}_{\lambda,\mu}\left(t^* z^+\right) \leq \mathcal{I}_{\lambda,\mu}\left(t_1^- z^+\right) = \mathcal{I}_{\lambda,\mu}\left(z^+\right),$$

a contradiction. Since $\mathcal{I}_{\lambda,\mu}(z^+) = \mathcal{I}_{\lambda,\mu}\left(|w_1^+|, |w_2^+|\right)$and $\left(|w_1^+|, |w_2^+|\right) \in \mathcal{N}_{\lambda,\mu}$, by Lemma 2.4 we may assume that $z^+$is a nontrivial nonnegative solution of (2.3). Then by the Strong Maximum Principle [[9], Lemma 2.4], we have $w_1^+, w_2^+ > 0$ in $\mathcal{C}_\Omega$, hence, $z^+$is a positive solution for (2.3). □

Next we will use $w_\varepsilon = E_s(u_\varepsilon)$, the family of minimizers for the trace inequality (2.7), where u_ε is given in (2.8). Without loss of generality, we may assume that $0 \in \Omega$. We then define the cut-off function $\phi \in C_0^\infty(\mathcal{C}_\Omega)$, $0 \leq \phi \leq 1$ and for small fixed $\rho > 0$,

$$\phi(x,y) = \begin{cases} 1, & (x,y) \in B_\rho, \\ 0, & (x,y) \notin \overline{B_{2\rho}}, \end{cases}$$

where $B_\rho = \left\{(x,y) : |x|^2 + y^2 < \rho^2, y > 0\right\}$. We take ρ so small that $\overline{B_{2\rho}} \subset \overline{\mathcal{C}_\Omega}$. Recall $\mathcal{W}$ is the extension of U introduced in Section 2, we have (cf. [3]) $|\nabla\mathcal{W}(x,y)| \leq Cy^{-1}\mathcal{W}(x,y)$. Let

$$U_\varepsilon(x) = \frac{1}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}}, \quad \varepsilon > 0.$$

Then the extension of $U_\varepsilon(x)$ has the form

$$\mathcal{W}_\varepsilon(x,y) = c_{N,s}y^{2s}\int_{\mathbf{R}^N}\frac{U_\varepsilon(z)dz}{(|x-z|^2 + y^2)^{\frac{N+2s}{2}}} = \varepsilon^{2s-N}\mathcal{W}\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right).$$

Notice that $\phi\mathcal{W}_\varepsilon \in H^s_{0,L}(\mathcal{C}_\Omega)$, for $\varepsilon > 0$ small enough.

Lemma 4.3. *There is $z \in \tilde{\mathcal{H}} \backslash \{0\}$ nonnegative and $\Lambda^* > 0$ such that for $(\lambda, \mu) \in C_{\Lambda^*}$.*

$$\sup_{t \geq 0} \mathcal{I}_{\lambda,\mu}(tz) < c_\infty,$$

where c_∞ is given in Lemma 3.3. In particular, $\alpha^-_{\lambda,\mu} < c_\infty$ for all $(\lambda, \mu) \in C_{\Lambda^*}$.

Proof. By an argument similar to that of the proof of [[3], formula (3.26)], we get

$$\begin{aligned} \left\|\phi \mathcal{W}_\varepsilon\right\|^2_{H^s_{0,L}(C_\Omega)} &= k_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \mathcal{W}_\varepsilon|^2 dx dy + O(1) \\ &= \varepsilon^{2s-N} k_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \mathcal{W}(x,y)|^2 dx dy + O(1). \end{aligned} \tag{4.11}$$

We notice that

$$\begin{aligned} \left\|\phi U_\varepsilon\right\|^{2^*_s}_{2^*_s} &= \int_\Omega |\phi U_\varepsilon|^{2^*_s} dx = \int_\Omega \frac{\phi(x)^{2^*_s}}{(\varepsilon^2 + |x|^2)^N} dx, \\ \|U_\varepsilon\|^{2^*_s}_{2^*_s} &= \int_{\mathbb{R}^N} \frac{1}{(\varepsilon^2 + |x|^2)^N} dx = \varepsilon^{-N} \|U\|^{2^*_s}_{2^*_s} \end{aligned}$$

Then, one has that

$$\left\|\phi U_\varepsilon\right\|^{2^*_s}_{2^*_s} - \varepsilon^{-N}\|U\|^{2^*_s}_{2^*_s} = \int_\Omega \frac{\phi^{2^*_s}(x) - 1}{(\varepsilon^2 + |x|^2)^N} dx - \int_{\mathbb{R}^N \backslash \Omega} \frac{dx}{(\varepsilon^2 + |x|^2)^N},$$

which yields

$$\begin{aligned} |\left\|\phi U_\varepsilon\right\|^{2^*_s}_{2^*_s} - \varepsilon^{-N}\|U\|^{2^*_s}_{2^*_s}| &\leq \int_{\Omega \backslash B(0;\rho)} \frac{1}{(\varepsilon^2 + |x|^2)^N} dx + \int_{\mathbb{R}^N \backslash \Omega} \frac{dx}{(\varepsilon^2 + |x|^2)^N} \\ &= \int_{\mathbb{R}^N \backslash B(0;\rho)} \frac{dx}{(\varepsilon^2 + |x|^2)^N} \leq \int_{\mathbb{R}^N \backslash B(0;\rho)} \frac{dx}{|x|^{2N}} = C_3. \end{aligned}$$

This implies that

$$1 - C_3 \varepsilon^N \|U\|^{-2^*_s}_{2^*_s} \leq \varepsilon^N \left\|\phi U_\varepsilon\right\|^{2^*_s}_{2^*_s} \|U\|^{-2^*_s}_{2^*_s} \leq 1 + C_3 \varepsilon^N \|U\|^{-2^*_s}_{2^*_s}$$

Taking ε so small that $C_3 \varepsilon^N \|U\|^{-2^*_s}_{2^*_s} < 1$, since $2/2^*_s = (N-2s)/N < 1$, we obtain

$$\begin{aligned} &1 - \varepsilon^N C_3 \|U\|^{-2^*_s}_{2^*_s} \leq \left(1 - \varepsilon^N C_3 \|U\|^{-2^*_s}_{2^*_s}\right)^{2/2^*_s} \leq \varepsilon^{N-2s} \left\|\phi U_\varepsilon\right\|^2_{2^*_s} \|U\|^{-2}_{2^*_s} \\ &\leq \left(1 + \varepsilon^N C_3 \|U\|^{-2^*_s}_{2^*_s}\right)^{2/2^*_s} \leq 1 + \varepsilon^N C_3 \|U\|^{-2^*_s}_{2^*_s}. \end{aligned}$$

Hence $\left\|\phi U_\varepsilon\right\|^2_{2^*_s} = \varepsilon^{2s-N}\|U\|^2_{2^*_s} + O\left(\varepsilon^{2s}\right)$. Since $\mathcal{W} = E_s(U)$ optimizes (2.7), by (4.11) we have

$$\frac{\left\|\phi \mathcal{W}_\varepsilon\right\|^2_{H^s_{0,L}(C_\Omega)}}{\left\|\phi U_\varepsilon\right\|^2_{2^*_s}} = \frac{\varepsilon^{2s-N} k_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \mathcal{W}(x,y)|^2 dx dy + O(1)}{\varepsilon^{2s-N}\|U\|^2_{2^*_s} + O\left(\varepsilon^{2s}\right)} \tag{4.12}$$

$$= \frac{k_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \mathcal{W}(x,y)|^2 dxdy}{\|U\|^2_{2^*_s}} \left(1 + O\left(\varepsilon^{N-2s}\right)\right)$$
$$= k_s \mathcal{S}(s,N) + O\left(\varepsilon^{N-2s}\right).$$

Now we consider the function $J : \tilde{\mathcal{H}} \to \mathbb{R}$ defined by

$$J(z) := \frac{1}{2}\|z\|^2_{\tilde{\mathcal{H}}} - \frac{1}{2^*_s} Q(z).$$

Set $w_{0,1} := \phi \mathcal{W}_\varepsilon, w_{0,2} := \tau_0 \phi \mathcal{W}_\varepsilon$ and $z_0 := (w_{0,1}, w_{0,2}) \in \tilde{\mathcal{H}}$. Notice that $J(0) = 0$, $J(tz_0) > 0$ for $t > 0$ small and $J(tz_0) < 0$ for $t > 0$ large. The map $t \mapsto J(tz_0)$ maximizes at

$$t_0 := \left(\frac{\|z_0\|^2_{\tilde{\mathcal{H}}}}{Q(z_0)}\right)^{\frac{1}{2^*_s - 2}}. \tag{4.13}$$

Then from (2.12), (4.12) and (4.13), we conclude that

$$\begin{aligned}
\sup_{t\ge 0} J(tz_0) &= J(t_0 z_0) = \left(\frac{1}{2} - \frac{1}{2^*_s}\right) \frac{\|z_0\|_{\tilde{\mathcal{H}}}^{\frac{22^*_s}{2^*_s-2}}}{(Q(z_0))^{\frac{2}{2^*_s-2}}} \\
&= \frac{s}{N}\left[\frac{\left(1+\tau_0^2\right) k_s \int_{C_\Omega} y^{1-2s} |\nabla(\phi \mathcal{W}_\varepsilon)|^2 dxdy}{\left(\eta_1 \tau_0^{\beta_1} + \eta_2 \tau_0^{\beta_2}\right)\left(\int_\Omega |\phi U_\varepsilon|^{2^*_s} dx\right)^{\frac{2}{2^*_s}}}\right]^{\frac{2^*_s}{2^*_s-2}} \\
&= \frac{s}{N}\left[\frac{f(\tau_0) k_s \int_{C_\Omega} y^{1-2s} |\nabla(\phi \mathcal{W}_\varepsilon)|^2}{\left(\int_\Omega |\phi U_\varepsilon|^{2^*_s} dx\right)^{\frac{2}{2s}}}\right]^{\frac{N}{2s}} \\
&= \frac{s}{N}\left[f(\tau_0)\left(k_s S(s,N) + O\left(\varepsilon^{N-2s}\right)\right)\right]^{\frac{N}{2s}} \\
&= \frac{s}{N}\left(k_s \mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{N}{2s}} + O\left(\varepsilon^{N-2s}\right).
\end{aligned} \tag{4.14}$$

We now choose $\delta_1 > 0$ so small that, for all $(\lambda,\mu) \in C_{\delta_1}$, we get

$$c_\infty = \frac{s}{N}(k_s \mathcal{S}_{\eta,\alpha,\beta})^{\frac{N}{2s}} - C_0(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}) > 0.$$

By the definition of $\mathcal{I}_{\lambda,\mu}$ and z_0, we have

$$\mathcal{I}_{\lambda,\mu}(tz_0) \le \frac{t^2}{2}\|z_0\|^2_{\tilde{\mathcal{H}}}, \quad \text{for all } t \ge 0 \text{ and } \lambda,\mu > 0,$$

which implies that there exists $t_0 \in (0,1)$ satisfying

$$\sup_{t\in[0,t_0]} \mathcal{I}_{\lambda,\mu}(tz_0) < c_\infty, \quad \text{for all } (\lambda,\mu) \in C_{\delta_1}.$$

Hence, from (4.14) we see that

$$\begin{aligned}\sup_{t\geq t_0}\mathcal{I}_{\lambda,\mu}(tz_0)&=\sup_{t\geq t_0}\left(J(tz_0)-\frac{t^q}{q}K(tz_0)\right)\\&\leq\frac{s}{N}\left(k_s\mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{N}{2s}}+O\left(\varepsilon^{N-2s}\right)-\frac{t_0^q}{q}\left(\lambda+\mu\tau_0^q\right)\int_{B(0;\rho)}\frac{|U_\varepsilon|^q}{|x|^\gamma}dx.\end{aligned}\tag{4.15}$$

Letting $0<\varepsilon\leq\rho$, we have

$$\begin{aligned}\int_{B(0;\rho)}\frac{|U_\varepsilon|^q}{|x|^\gamma}dx&=\int_{B(0;\rho)}\frac{1}{|x|^\gamma\left(\varepsilon^2+|x|^2\right)^{\frac{(N-2s)q}{2}}}dx\\&=\int_0^\rho\frac{r^{N-1}}{r^\gamma\varepsilon^{(N-2s)q}\left[1+\left(\frac{r}{\varepsilon}\right)^2\right]^{\frac{(N-2s)q}{2}}}dr\\&=\varepsilon^{N-\gamma+(2s-N)q}\int_0^{\frac{\rho}{\varepsilon}}\frac{r^{N-1}}{r^\gamma\left(1+r^2\right)^{\frac{(N-2s)q}{2}}}dr\\&=\varepsilon^{N-\gamma+2sq-qN}\int_0^1\frac{r^{N-1}}{r^\gamma\left(1+r^2\right)^{\frac{(N-2s)q}{2}}}dr\\&\quad+\varepsilon^{N-\gamma+2sq-qN}\int_1^{\frac{\rho}{\varepsilon}}\frac{r^{N-1}}{r^\gamma\left(1+r^2\right)^{\frac{(N-2s)q}{2}}}dr.\end{aligned}\tag{4.16}$$

From (4.16), we get

$$\frac{t_0^q}{q}\int_{B(0;\rho)}\frac{|U_\varepsilon|^q}{|x|^\gamma}dx\geq\begin{cases}C_3\varepsilon^{N-\gamma+sq-\frac{qN}{2}}, & \gamma>N-(N-2s)q,\\ C_4\varepsilon^{\frac{qN}{2}-sq}|\ln\varepsilon|, & \gamma=N-(N-2s)q,\\ C_5\varepsilon^{\frac{qN}{2}-sq}, & \gamma<N-(N-2s)q,\end{cases}\tag{4.17}$$

where $C_i>0(i=3,4,5)$ are positive constants (C_i independent of ε).
The case of $\gamma>N-(N-2s)q$, combining (4.14) with (4.17), one has

$$\sup_{t\geq0}\mathcal{I}_{\lambda,\mu}(tz_0)\leq\frac{s}{N}\left(k_s\mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{N}{2s}}+O\left(\varepsilon^{N-2s}\right)+C_2\varepsilon^{N-2s}-C_3\left(\lambda+\mu\tau_0^q\right)\varepsilon^{N-\gamma+sq-\frac{qN}{2}}.$$

Let $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}=\varepsilon^{N-2s}$, that is, $\varepsilon=\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{1}{N-2s}}$, then we can choose $\delta_1>0$ such that

$$\begin{aligned}&O\left(\varepsilon^{N-2s}\right)+C_2\varepsilon^{N-2s}-C_3\left(\lambda+\mu\tau_0^q\right)\varepsilon^{N-\gamma+sq-\frac{qN}{2}}\\=&O\left(\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{N-2s}\right)+C_2\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)-C_3\left(\lambda+\mu\tau_0^q\right)\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{2N-2\gamma+2sq-qN}{2(N-2s)}}\\<&-C_0\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right),\end{aligned}$$

for all $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,\delta_1)$. Then, for $(\lambda,\mu)\in C_{\delta_1}$, one gets

$$\sup_{t\geq0}\mathcal{I}_{\lambda,\mu}(tz_0)<c_\infty.$$

The case of $\gamma = N - (N-2s)q$, it follows from (4.14) and (4.17) that

$$\sup_{t\geq 0} \mathcal{I}_{\lambda,\mu}(tz_0) \leq \frac{s}{N}\left(k_s \mathcal{S}_{\eta,\alpha,\beta}\right)^{\frac{N}{2s}} + O\left(\varepsilon^{N-2s}\right) + C_2\varepsilon^{N-2s} - C_4\left(\lambda + \mu\tau_0^q\right)\varepsilon^{\frac{qN}{2}-sq}|\ln\varepsilon|.$$

Let $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} = \varepsilon^{N-2s}$, that is, $\varepsilon = \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{1}{N-2s}}$, choosing $\delta_2 > 0$ such that for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \delta_2)$, then one has

$$\begin{aligned}
&O\left(\varepsilon^{N-2s}\right) + C_2\varepsilon^{N-2s} - C_4\left(\lambda + \mu\tau_0^q\right)\varepsilon^{\frac{qN}{2}-sq}|\ln\varepsilon| \\
=&O\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right) + C_2\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right) - C_4\left(\lambda + \mu\tau_0^q\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{q}{2}}|\ln\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)| \\
<&-C_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right).
\end{aligned}$$

Consequently, for $(\lambda,\mu) \in \mathcal{C}_{\delta_2}$, we obtain

$$\sup_{t\geq 0} \mathcal{I}_{\lambda,\mu}(tz_0) < c_\infty.$$

If we set $\Lambda^* = \min\left\{\delta_1, \rho^{N-2s}, \delta_2\right\} > 0$, then for $(\lambda,\mu) \in \mathcal{C}_{\Lambda^*}$,

$$\sup_{t\geq 0} \mathcal{I}_{\lambda,\mu}(tz_0) < c_\infty. \tag{4.18}$$

Finally, we prove that $\alpha^-_{\lambda,\mu} < c_\infty$ for all $(\lambda,\mu) \in \mathcal{C}_{\Lambda^*}$. Recall that

$$z_0 = (w_{0,1}, w_{0,2}) = (\phi\mathcal{W}_\varepsilon, \tau_0\phi\mathcal{W}_\varepsilon).$$

Since $Q(z_0) > 0$, by Theorem 2.2 there exists $t_0 > 0$ such that $t_0 z_0 \in \mathcal{N}^-_{\lambda,\mu}$. By the definition of $\alpha^-_{\lambda,\mu}$and (4.18), we conclude that

$$\alpha^-_{\lambda,\mu} \leq \mathcal{I}_{\lambda,\mu}(t_0 z_0) \leq \sup_{t\geq 0} \mathcal{I}_{\lambda,\mu}(tz_0) < c_\infty$$

for all $(\lambda,\mu) \in \mathcal{C}_{\Lambda^*}$. □

Let Λ^* be as in Lemma 4.3. We prove the existence a local minimizer for $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$.

Proposition 4.3. *Let $\Lambda^* > 0$ be as in Lemma 4.3 and set*

$$\Lambda_2 := \min\left\{\Lambda^*, (\frac{q}{2})^{\frac{2}{2-q}}\Lambda_1\right\}.$$

For $(\lambda,\mu) \in \mathcal{C}_{\Lambda_2}, \mathcal{I}_{\lambda,\mu}$ has a minimizer z^-in $\mathcal{N}^-_{\lambda,\mu}$with $\mathcal{I}_{\lambda,\mu}(z^-) = \alpha^-_{\lambda,\mu}$. Furthermore, z^- is a positive solution of (2.3).

Proof. By (ii) of Proposition 4.1, there is a $(PS)_{\alpha^-_{\lambda,\mu}}$sequence $\{z_n\} \subset \mathcal{N}^-_{\lambda,\mu}$for $\mathcal{I}_{\lambda,\mu}$ for all $(\lambda,\mu) \in \mathcal{C}_{(\frac{q}{2})^{\frac{2}{2-q}}\Lambda_1}$. By Lemmas 3.3 and 4.3 and (ii) of Theorem 2.1, for all $(\lambda,\mu) \in \mathcal{C}_{\Lambda^*}, \mathcal{I}_{\lambda,\mu}$ satisfies the $(PS)_{\alpha^-_{\lambda,\mu}}$condition and $\alpha^-_{\lambda,\mu} > 0$. Then, there exists a subsequence still denoted by $\{z_n\}$ and $z^- = \left(w_1^-(x,y), w_2^-(x,y)\right) \in \mathcal{N}^-_{\lambda,\mu}$such that $z_n \to z^-$strongly in $\tilde{\mathcal{H}}$ and $\mathcal{I}_{\lambda,\mu}(z^-) = \alpha^-_{\lambda,\mu} > 0$, for all $(\lambda,\mu) \in \mathcal{C}_{\Lambda_2}$. Arguing as in the proof of Proposition 4.2, for $(\lambda,\mu) \in \mathcal{C}_{\Lambda_2}$, we obtain that $z^-$is a positive solution of (2.3). □

5. The proof of Theorem 1.1

By Proposition 4.2, for $(\lambda, \mu) \in C_{\Lambda_1}$, system (2.3) admits a positive solution $z^+ \in \mathcal{N}^+_{\lambda,\mu}$. By Proposition 4.3, a positive solution $z^- \in \mathcal{N}^-_{\lambda,\mu}$exists for all $(\lambda, \mu) \in C_{\Lambda_2}$. Furthermore, since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, then $z^+, z^-$are distinct positive solutions of system (2.3). In turn, $(u^{\pm}(x), v^{\pm}(x)) = \left(w_1^{\pm}(x,0), w_2^{\pm}(x,0)\right)$ are distinct positive solutions of (1.1).

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