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Tripled Fixed Point Approaches and Hyers-Ulam Stability With Applications

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Abstract. In this paper, we present tripling fixed point results for extended contractive mappings in the context of a generalized metric space. Many publications in the literature are improved, unified, and generalized by our theoretical results. Furthermore, the Ulam-Hyers stability problem for the tripled fixed point problem in vector-valued metric spaces has been examined as a stability analysis for fixed point approaches. Finally, as a type of application to support our research, the theoretical conclusions are used to explore the existence and uniqueness of solutions to a periodic boundary value problem.

1. Introduction

The existence and uniqueness of solutions to various types of integral and integro-differential equations, as well as methods for obtaining explicit or approximative solutions for these sorts of equations, are becoming increasingly important. A lot of writers have recently considered the problem of the existence of a fixed point (FP) for mappings with specific conditions within the context of generalized metric spaces, and some applications to delay integro-differential equations

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arising in infectious diseases from biomathematics, population growth, integral equations, and matrix equations are presented; for more details, see [1–3].

The solution to the given problem is similar to determining the FP of a specific self-mapping or non-self-mapping when it is expressed using FP approaches. FP theory is one of the most significant sections of nonlinear analysis due to its numerous applications in major fields including operational equations, fractal theory, optimization theory, etc.

In 1964, Perov [4] introduced the concept of vector-valued metrics for single-valued mappings, while in 2004, Petruşel [5] introduced the same idea for multi-valued mappings as a generalization of the Banach FP theorem [6]. It is useful to look at a more thorough concept called tripled fixed points (TFPs, for short) while investigating the existence of FPs for an operator. In 1987, Guo and Lakshmikantham [7] presented the concept of a coupled FP for continuous and discontinuous operators in reference to triple quasi solutions of an initial value problem for ordinary differential equations. Several authors concentrated on these ideas and utilized the findings to discuss if certain problems have solutions; for more information, see [8–21].

The stability problem of functional equations originated from a question of Stanisław Ulam [22], posed in 1940, concerning the stability of group homomorphisms. In 1941, Donald H. Hyers [23] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces in the case of additive mappings. Thus, the chronological contribution to the problem of stability is Ulam-Hyers (HU). For more details, see [24–27].

Throughout this paper, we denote $M_{m,m}(\mathbb{R}^+)$, 0 and I by the set of all $m \times m$ matrices with components in $\mathbb{R}^+$, zero and identity matrices, respectively. Further, $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

A converging matrix is one whose power sequence converges to the null matrix. The spectral radius is the maximum modulus of the eigenvaues. Thus, if it is less than one ,then all the eigenvalues have modulus less than one. The fact that convergent matrices are stable in a discrete context should be highlighted. This means that for every set of finite initial conditions, a discrete linear system whose dynamics are defined by convergent matrices is globally asymptotically stable to the zero equilibrium. The convergence of matrices is quite similar to the convergence of sequences or vectors, i.e., let $Q \in M_{m\times m}(\mathbb{R}^+)$, a matrix Q is said to be convergent if and only if $\lim_{n\to\infty} Q^n = 0$, for more details in this regards; see [28].

The following are some examples of convergent matrices to zero:

Example 1.1. *The matrix*

$$Q = \begin{pmatrix} \ell^1 & \ell^1 \\ \ell^2 & \ell^2 \end{pmatrix} \text{ or } Q = \begin{pmatrix} \ell^1 & \ell^2 \\ \ell^1 & \ell^2 \end{pmatrix},$$

in $M_{2,2}(\mathbb{R}^+)$ *with* $\ell^1 + \ell^2 < 1$ *is convergent.*

Example 1.2. *Any matrix in* $M_{2,2}(\mathbb{R}^+)$ *in the shape*

$$Q = \begin{pmatrix} \ell^1 & \ell^2 \\ 0 & \ell^3 \end{pmatrix},$$

is convergent, provided that $\max\{\ell^1, \ell^3\} < 1$.

Example 1.3. *The matrix*

$$Q = \begin{pmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Gamma_m \end{pmatrix}_{m\times m},$$

in $M_{m,m}(\mathbb{R}^+)$ *is convergent, provided that* $\max\{\Gamma_j : 1 \le j \le m\} < 1$.

The following is an example of a non-convergent matrix:

Example 1.4. *If we consider* $\ell^1 + \ell^2 \ge 1$ *in the matrix*

$$Q = \begin{pmatrix} \ell^1 & \ell^2 \\ \ell^3 & \ell^4 \end{pmatrix},$$

in $M_{2,2}(\mathbb{R}^+)$*, then, it is not convergent to zero.*

The definition of addition and multiplication on $\mathbb{R}^m$ is described as follows: Assume that $\ell, \ell^* \in \mathbb{R}^m$, where $\ell = (\ell_1, \ell_2, ..., \ell_m)$ and $\ell^* = \left(\ell_1^*, \ell_2^*, ..., \ell_m^*\right)$, then

$$\ell + \ell^* = \left(\ell_1 + \ell_1^*, \ell_2 + \ell_2^*, ..., \ell_m + \ell_m^*\right),$$

and

$$\ell.\ell^* = \left(\ell_1.\ell_1^*, \ell_2.\ell_2^*, ..., \ell_m.\ell_m^*\right).$$

It should be noted that, for $u \in \mathbb{R}$, $\ell_i \le \ell_i^*$ (resp. $\ell_i < \ell_i^*$) for each $1 \le i \le m$, also $\ell_i \le u$ (resp. $\ell_i < u$) for all $1 \le i \le m$, respectively. This subject has been thoroughly researched in academic works [29,30].

Definition 1.1. *[4] Let* Ω *be a non-empty set. We say that the mapping* $\omega : \Omega^2 \to \mathbb{R}^m$ *is a vector-valued metric on* Ω*, if the following hypotheses are true:*

(ω_1) $\omega\left(\ell^1, \ell^2\right) \ge 0$, $\omega\left(\ell^1, \ell^2\right) = 0$ *if and only if* $\ell^1 = \ell^2$,
(ω_2) $\omega\left(\ell^1, \ell^2\right) = \omega\left(\ell^2, \ell^1\right)$,
(ω_3) $\omega\left(\ell^1, \ell^2\right) \le \omega\left(\ell^1, \ell^3\right) + \omega\left(\ell^3, \ell^2\right)$,

for all $\ell^1, \ell^2, \ell^3 \in \Omega$. *If* $\ell^1, \ell^2 \in \mathbb{R}^m$ *where* $\ell^1 = \left(\ell_1^1, ..., \ell_m^1\right)$ *and* $\ell^2 = \left(\ell_1^2, ..., \ell_m^2\right)$*, then* $\ell^1 \le \ell^2$ *if and only if* $\ell_i^1 \le \ell_i^2$ *for* $1 \le i \le m$*, and* (Ω, ω) *is called a generalized metric space (GMS, for short).*

Lemma 1.1. *[31, 32] Assume that* Q *is a square matrix of nonnegative numbers. Then, the following statements are equivalent:*

(1) $Q^n \to 0$ *as* $n \to \infty$.
(2) $I - Q$ *is non-singular and*

$$(I - Q)^{-1} = I + Q + Q^2 + ...$$

(3) $|\hbar| < 1$ *for each* $\hbar \in \mathbb{C}$*, with* $(Q - \hbar I)$ *is convergent to 0.*

(4) *$I-Q$ is non-singular and $(I-Q)^{-1}$ has non-negative elements.*

In 1964, Perov [4] presented the following theorem, which was called the Perove FP theorem:

Theorem 1.1. *Let (Ω,ω) be a complete GMS and $\mathfrak{R}:\Omega\to\Omega$ be a given operator satisfying*

$$\omega\left(\mathfrak{R}\left(\vartheta\right),\mathfrak{R}\left(\ell\right)\right)\leq Q\omega\left(\vartheta,\ell\right),\ \text{for all}\,\vartheta,\ell\in\Omega,$$

where $Q\in M_{m,m}(\mathbb{R})$ is a convergent matrix to zero. Then

(p_1) *$\{\vartheta^*\}=Fix\{\mathfrak{R}\}$;*

(p_2) *the approximation sequence $\{\{\vartheta_n\}_{n\in\mathbb{N}}:\vartheta_n=\mathfrak{R}^n\left(\vartheta_0\right)\}$ is convergent and $\vartheta_n\to\vartheta^*$ as $n\to\infty$;*

(p_3) *the estimation below is true*

$$\omega\left(\vartheta_n,\vartheta^*\right)\leq Q^n\left(I-Q\right)^{-1}\omega\left(\vartheta_0,\vartheta_1\right);$$

(p_4) *let $\mathfrak{I}:\Omega\to\Omega$ be an operator and there is $\xi\in\left(\mathbb{R}^m_+\right)^*$ with $\omega\left(\mathfrak{R}\left(\vartheta\right),\mathfrak{I}\left(\vartheta\right)\right)\leq\xi$ for all $\vartheta\in\Omega$. If $\wp\in Fix(\mathfrak{I})$, then*

$$\omega\left(\wp,\vartheta^*\right)\leq(I-Q)^{-1}\xi;$$

(p_5) *if $\mathfrak{I}:\Omega\to\Omega$ is an operator and there is $\xi\in\left(\mathbb{R}^m_+\right)^*$ with $\omega\left(\mathfrak{R}\left(\vartheta\right),\mathfrak{I}\left(\vartheta\right)\right)\leq\xi$ for all $\vartheta\in\Omega$, then for the sequence $\ell_n=\mathfrak{R}^n\left(\vartheta_0\right)$, we have the following estimation:*

$$\omega\left(\ell_n,\vartheta^*\right)\leq(I-Q)^{-1}\xi+Q^n(I-Q)^{-1}\left(\omega\left(\ell_0,\vartheta^*\right)\right).$$

Definition 1.2. *Let $z:\Omega\to\Omega$ be an operator defined on a GMS (Ω,ω). The equation of FP*

$$\upsilon=z\left(\upsilon\right),\ \upsilon\in\Omega, \tag{1.1}$$

is called generalized Hyers-Ulam (GHU) stable if there is an increasing function $\varphi:\mathbb{R}^m_+\to\mathbb{R}^m_+$ and continuous in 0 with $\varphi(0)=0$ such that for any $\epsilon=(\epsilon_1,\epsilon_2,...\epsilon_m)$ with $\epsilon_i>0$ for $i\in\{1,...,m\}$ and any solution η^* of the equation

$$\omega\left(\eta^*,z\left(\eta^*\right)\right)\leq\epsilon,$$

there is a solution υ^* of (1.1) such that

$$\omega\left(\upsilon^*,\eta^*\right)\leq\varphi\left(\epsilon\right).$$

Moreover, the equation of FP (1.1) is called HU stable if $\varphi\left(\ell\right)=Q\ell$, where $\ell\in\mathbb{R}^m_+$ and $Q\in M_{m,m}(\mathbb{R}^+)$.

The result below is considered a direct consequence of Perov's FP theorem.

Theorem 1.2. *Let (Ω,ω) be a complete GMS and $z:\Omega\to\Omega$ be a given operator. If there is a convergent matrix $Q\in M_{m,m}(\mathbb{R})$ to zero, then the FP equation (1.1) is HU stable, provided that the inequality below holds*

$$\omega\left(z\left(\upsilon\right),z\left(\rho\right)\right)\leq Q\omega\left(\upsilon,\rho\right),\ \forall\upsilon,\rho\in\Omega.$$

Assume that (Ω, ω) is a metric space. We shall take into consideration the following operational equation system (OES, for simplicity):

$$\begin{cases} \upsilon = \Re(\upsilon, \rho, \varrho) \\ \rho = \Re(\rho, \varrho, \upsilon) \\ \varrho = \Re(\varrho, \upsilon, \rho) \end{cases},$$

where $\Re : \Omega^3 \to \Omega$ is a given operator and the solution $(\upsilon, \rho, \varrho) \in \Omega^3$ is called a TFP of Ω [7].

There is no doubt that the study of FPs on a generalized matrix space to study the convergence and divergence of matrices is one of the topics that has a long history in mathematical analysis due to its frequent application. In this paper, we will study this direction by presenting the existence and uniqueness of TFP for two contractive-type operators in GMSs. Since the stability study of such points opens many applications for many authors in this direction, HU stability for the TFP of two contractive-type operators in GMSs is investigated. In order for the theoretical results to become more efficient and effective, there is a need for supporting applications, so we apply the theoretical results to obtain the existence and uniqueness of solutions to periodic boundary value problems (PBVPs, for simplicity).

2. Main Results

In this section, we examine the results for the existence, uniqueness, data dependence, and HU stability of single-valued operators for the TFP. We start this part with the following definition:

Definition 2.1. *Let $\Re_i : \Omega^3 \to \Omega$, $(i = 1, 2, 3)$ be operators defined on a GMS (Ω, ω). We say that the OES*

$$\begin{cases} \upsilon = \Re_1(\upsilon, \rho, \varrho) \\ \rho = \Re_2(\upsilon, \rho, \varrho) \\ \varrho = \Re_3(\upsilon, \rho, \varrho) \end{cases}, \tag{2.1}$$

is HU stable if there exist $r_j > 0$, $(i = 1, 2, 3, 4, 5, 6, 7, 8, 9)$ such that for each $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ and for a solution $(\ell_1, \ell_2, \ell_3) \in \Omega^3$ to the inequalities

$$\begin{cases} \omega(\ell_1, \Re_1(\ell_1, \ell_2, \ell_3)) \leq \epsilon_1 \\ \omega(\ell_2, \Re_2(\ell_1, \ell_2, \ell_3)) \leq \epsilon_2 \\ \omega(\ell_3, \Re_3(\ell_1, \ell_2, \ell_3)) \leq \epsilon_3 \end{cases},$$

there is a solution $(\upsilon^*, \rho^*, \varrho^*) \in \Omega^3$ of (1.1) such that

$$\begin{cases} \omega(\ell_1, \upsilon^*) \leq r_1\epsilon_1 + r_2\epsilon_2 + r_3\epsilon_3 \\ \omega(\ell_2, \rho^*) \leq r_4\epsilon_1 + r_5\epsilon_2 + r_6\epsilon_3 \\ \omega(\ell_3, \varrho^*) \leq r_7\epsilon_1 + r_8\epsilon_2 + r_9\epsilon_3 \end{cases}.$$

Now, we state and prove our main theorem.

Theorem 2.1. *Let (Ω,ω) be a complete GMS and $\mathfrak{R}_1,\mathfrak{R}_2,\mathfrak{R}_3:\Omega^3\to\Omega$ be such that*

$$\begin{cases}\omega\left(\mathfrak{R}_1(\upsilon,\rho,\varrho),\mathfrak{R}_1(\ell_1,\ell_2,\ell_3)\right)\le a_1\omega(\upsilon,\ell_1)+a_2\omega(\rho,\ell_2)+a_3\omega(\varrho,\ell_3)\\ \omega\left(\mathfrak{R}_2(\upsilon,\rho,\varrho),\mathfrak{R}_2(\ell_1,\ell_2,\ell_3)\right)\le a_4\omega(\upsilon,\ell_1)+a_5\omega(\rho,\ell_2)+a_6\omega(\varrho,\ell_3)\\ \omega\left(\mathfrak{R}_3(\upsilon,\rho,\varrho),\mathfrak{R}_3(\ell_1,\ell_2,\ell_3)\right)\le a_7\omega(\upsilon,\ell_1)+a_8\omega(\rho,\ell_2)+a_9\omega(\varrho,\ell_3)\end{cases},$$

for all $(\upsilon,\rho,\varrho),(\ell_1,\ell_2,\ell_3)\in\Omega^3$. If the matrix $Q=\begin{pmatrix}a_1&a_2&a_3\\a_4&a_5&a_6\\a_7&a_8&a_9\end{pmatrix}$ is convergent to zero, then

($\heartsuit_1$) *there is a unique element $(\upsilon^*,\rho^*,\varrho^*)\in\Omega^3$ such that*

$$\begin{cases}\upsilon^*=\mathfrak{R}_1(\upsilon^*,\rho^*,\varrho^*)\\ \rho^*=\mathfrak{R}_2(\upsilon^*,\rho^*,\varrho^*)\\ \varrho^*=\mathfrak{R}_3(\upsilon^*,\rho^*,\varrho^*)\end{cases};$$

($\heartsuit_2$) *for each $n\in\mathbb{N}$, $\lim_{n\to\infty}\left(\mathfrak{R}_1^n(\upsilon,\rho,\varrho),\mathfrak{R}_2^n(\upsilon,\rho,\varrho),\mathfrak{R}_3^n(\upsilon,\rho,\varrho)\right)=(\upsilon^*,\rho^*,\varrho^*)$, where*

$$\begin{cases}\mathfrak{R}_1^{n+1}(\upsilon,\rho,\varrho)=\mathfrak{R}_1^n\left(\mathfrak{R}_1(\upsilon^*,\rho^*,\varrho^*),\mathfrak{R}_2(\upsilon^*,\rho^*,\varrho^*),\mathfrak{R}_3(\upsilon^*,\rho^*,\varrho^*)\right)\\ \mathfrak{R}_2^{n+1}(\upsilon,\rho,\varrho)=\mathfrak{R}_2^n\left(\mathfrak{R}_1(\upsilon^*,\rho^*,\varrho^*),\mathfrak{R}_2(\upsilon^*,\rho^*,\varrho^*),\mathfrak{R}_3(\upsilon^*,\rho^*,\varrho^*)\right)\\ \mathfrak{R}_3^{n+1}(\upsilon,\rho,\varrho)=\mathfrak{R}_3^n\left(\mathfrak{R}_1(\upsilon^*,\rho^*,\varrho^*),\mathfrak{R}_2(\upsilon^*,\rho^*,\varrho^*),\mathfrak{R}_3(\upsilon^*,\rho^*,\varrho^*)\right)\end{cases},\ n\ge 0;$$

($\heartsuit_3$) *the estimation below holds*

$$\begin{pmatrix}\omega\left(\mathfrak{R}_1^n(\upsilon_0,\rho_0,\varrho_0),\upsilon^*\right)\\ \omega\left(\mathfrak{R}_2^n(\upsilon_0,\rho_0,\varrho_0),\rho^*\right)\\ \omega\left(\mathfrak{R}_3^n(\upsilon_0,\rho_0,\varrho_0),\varrho^*\right)\end{pmatrix}\le Q^n(I-Q)^{-1}\begin{pmatrix}\omega\left(\upsilon_0,\mathfrak{R}_1(\upsilon_0,\rho_0,\varrho_0)\right)\\ \omega\left(\rho_0,\mathfrak{R}_1(\upsilon_0,\rho_0,\varrho_0)\right)\\ \omega\left(\varrho_0,\mathfrak{R}_3(\upsilon_0,\rho_0,\varrho_0)\right)\end{pmatrix};$$

($\heartsuit_4$) *assume that the operators $\mathfrak{I}_1,\mathfrak{I}_2,\mathfrak{I}_3:\Omega^3\to\Omega$ such that there exist $\xi_1,\xi_2,\xi_3>0$ with*

$$\begin{cases}\omega\left(\mathfrak{R}_1(\upsilon,\rho,\varrho),\mathfrak{I}_1(\upsilon,\rho,\varrho)\right)\le\xi_1\\ \omega\left(\mathfrak{R}_2(\upsilon,\rho,\varrho),\mathfrak{I}_2(\upsilon,\rho,\varrho)\right)\le\xi_2\\ \omega\left(\mathfrak{R}_3(\upsilon,\rho,\varrho),\mathfrak{I}_3(\upsilon,\rho,\varrho)\right)\le\xi_3\end{cases},$$

for all $(\upsilon,\rho,\varrho)\in\Omega^3$. If $(\wp,\hbar,\theta)\in\Omega^3$ such that

$$\begin{cases}\wp=\mathfrak{I}_1(\wp,\hbar,\theta)\\ \hbar=\mathfrak{I}_2(\wp,\hbar,\theta)\\ \theta=\mathfrak{I}_3(\wp,\hbar,\theta)\end{cases},$$

then

$$\begin{pmatrix}\omega(\wp,\upsilon^*)\\ \omega(\hbar,\rho^*)\\ \omega(\theta,\varrho^*)\end{pmatrix}\le(I-Q)^{-1}\xi,$$

where $\xi=\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}$;

($\heartsuit_5$) *by combining* $(\heartsuit_2)-(\heartsuit_4)$, *we have*

$$\begin{pmatrix} \omega\left(\mathfrak{R}_1^n(v_0,\rho_0,\varrho_0),v^*\right) \\ \omega\left(\mathfrak{R}_2^n(v_0,\rho_0,\varrho_0),\rho^*\right) \\ \omega\left(\mathfrak{R}_3^n(v_0,\rho_0,\varrho_0),\varrho^*\right) \end{pmatrix} \leq (I-Q)^{-1}\xi + Q^n(I-Q)^{-1}\begin{pmatrix} \omega\left(v_0,\mathfrak{R}_1^n(v_0,\rho_0,\varrho_0)\right) \\ \omega\left(\rho_0,\mathfrak{R}_1^n(v_0,\rho_0,\varrho_0)\right) \\ \omega\left(\varrho_0,\mathfrak{R}_3^n(v_0,\rho_0,\varrho_0)\right) \end{pmatrix};$$

($\heartsuit_6$) *the OES*

$$\begin{cases} v = \mathfrak{R}_1(v,\rho,\varrho) \\ \rho = \mathfrak{R}_2(v,\rho,\varrho) \ , \\ \varrho = \mathfrak{R}_3(v,\rho,\varrho) \end{cases} \tag{2.2}$$

is HU stable.

Proof. $(\heartsuit_1)-(\heartsuit_2)$ Define the operator $\mathfrak{R}:\Omega^3\to\Omega^3$ by

$$\mathfrak{R}(v,\rho,\varrho) = \begin{pmatrix} \mathfrak{R}_1(v,\rho,\varrho) \\ \mathfrak{R}_2(v,\rho,\varrho) \\ \mathfrak{R}_3(v,\rho,\varrho) \end{pmatrix} = \left(\mathfrak{R}_1(v,\rho,\varrho),\mathfrak{R}_2(v,\rho,\varrho),\mathfrak{R}_3(v,\rho,\varrho)\right).$$

Consider $\Lambda=\Omega^3$ and let $\widehat{\omega}:\Lambda\times\Lambda\to\mathbb{R}_+\times\mathbb{R}_+$ be described as

$$\widehat{\omega}\left(\omega(v,\rho),\omega(\ell_1,\ell_2)\right) = \begin{pmatrix} \omega(v,\ell_1) \\ \omega(\rho,\ell_2) \end{pmatrix}.$$

Then, we get

$$\begin{aligned} \widehat{\omega}\left(\mathfrak{R}(v,\rho,\varrho),\mathfrak{R}(\ell_1,\ell_2,\ell_3)\right) &= \widehat{\omega}\left(\begin{pmatrix} \mathfrak{R}_1(v,\rho,\varrho) \\ \mathfrak{R}_2(v,\rho,\varrho) \\ \mathfrak{R}_3(v,\rho,\varrho) \end{pmatrix},\begin{pmatrix} \mathfrak{R}_1(\ell_1,\ell_2,\ell_3) \\ \mathfrak{R}_2(\ell_1,\ell_2,\ell_3) \\ \mathfrak{R}_3(\ell_1,\ell_2,\ell_3) \end{pmatrix}\right) \\ &= \begin{pmatrix} \omega\left(\mathfrak{R}_1(v,\rho,\varrho),\mathfrak{R}_1(\ell_1,\ell_2,\ell_3)\right) \\ \omega\left(\mathfrak{R}_2(v,\rho,\varrho),\mathfrak{R}_2(\ell_1,\ell_2,\ell_3)\right) \\ \omega\left(\mathfrak{R}_3(v,\rho,\varrho),\mathfrak{R}_3(\ell_1,\ell_2,\ell_3)\right) \end{pmatrix} \\ &\leq \begin{pmatrix} a_1\omega(v,\ell_1)+a_2\omega(\rho,\ell_2)+a_3\omega(\varrho,\ell_3) \\ a_4\omega(v,\ell_1)+a_5\omega(\rho,\ell_2)+a_6\omega(\varrho,\ell_3) \\ a_7\omega(v,\ell_1)+a_8\omega(\rho,\ell_2)+a_9\omega(\varrho,\ell_3) \end{pmatrix} \\ &= \left(\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}\right)\begin{pmatrix} \omega(v,\ell_1) \\ \omega(\rho,\ell_2) \\ \omega(\varrho,\ell_3) \end{pmatrix} \\ &= Q.\widehat{\omega}\left((v,\rho,\varrho),(\ell_1,\ell_2,\ell_3)\right). \end{aligned}$$

Put $(v,\rho,\varrho)=\vartheta$ and $(\ell_1,\ell_2,\ell_3)=\ell$, we obtain

$$\widehat{\omega}\left(\mathfrak{R}(\vartheta),\mathfrak{R}(\ell)\right) \leq Q.\widehat{\omega}(\vartheta,\ell).$$

Based on Theorem 1.1 (p_1), there is a unique element $(v^*,\rho^*,\varrho^*)\in\Omega^3$ such that

$$(v^*,\rho^*,\varrho^*) = \mathfrak{R}(v^*,\rho^*,\varrho^*),$$

yields,

$$\begin{cases} \upsilon^* = \mathfrak{R}_1(\upsilon^*, \rho^*, \varrho^*) \\ \rho^* = \mathfrak{R}_2(\upsilon^*, \rho^*, \varrho^*) \\ \varrho^* = \mathfrak{R}_3(\upsilon^*, \rho^*, \varrho^*) \end{cases}.$$

In addition, for $\vartheta \in \Omega^3$, we get $\mathfrak{R}^n(\vartheta) \to \vartheta^*$ as $n \to \infty$, where

$$\begin{aligned} \vartheta &= \mathfrak{R}^0(\vartheta),\ \mathfrak{R}^1\vartheta = \mathfrak{R}(\upsilon, \rho, \varrho) = (\mathfrak{R}_1(\upsilon, \rho, \varrho), \mathfrak{R}_2(\upsilon, \rho, \varrho), \mathfrak{R}_3(\upsilon, \rho, \varrho)), \\ \mathfrak{R}^2\vartheta &= \mathfrak{R}(\mathfrak{R}_1(\upsilon, \rho, \varrho), \mathfrak{R}_2(\upsilon, \rho, \varrho), \mathfrak{R}_3(\upsilon, \rho, \varrho)) = \left(\mathfrak{R}_1^2(\upsilon, \rho, \varrho), \mathfrak{R}_2^2(\upsilon, \rho, \varrho), \mathfrak{R}_3^2(\upsilon, \rho, \varrho)\right), \end{aligned}$$

and in general, we can write

$$\begin{aligned} \mathfrak{R}_1^{n+1}(\upsilon, \rho, \varrho) &= \mathfrak{R}_1^n(\mathfrak{R}_1(\upsilon, \rho, \varrho), \mathfrak{R}_2(\upsilon, \rho, \varrho), \mathfrak{R}_3(\upsilon, \rho, \varrho)), \\ \mathfrak{R}_2^{n+1}(\upsilon, \rho, \varrho) &= \mathfrak{R}_2^n(\mathfrak{R}_1(\upsilon, \rho, \varrho), \mathfrak{R}_2(\upsilon, \rho, \varrho), \mathfrak{R}_3(\upsilon, \rho, \varrho)), \\ \mathfrak{R}_3^{n+1}(\upsilon, \rho, \varrho) &= \mathfrak{R}_3^n(\mathfrak{R}_1(\upsilon, \rho, \varrho), \mathfrak{R}_2(\upsilon, \rho, \varrho), \mathfrak{R}_3(\upsilon, \rho, \varrho)). \end{aligned}$$

It follows that

$$\mathfrak{R}^n(\vartheta) = \left(\mathfrak{R}_1^n(\vartheta), \mathfrak{R}_2^n(\vartheta), \mathfrak{R}_3^n(\vartheta)\right) \to \vartheta^* = (\upsilon^*, \rho^*, \varrho^*) \text{ as } n \to \infty,\ \forall \vartheta \in \Omega^3.$$

This implies that

$$\begin{cases} \mathfrak{R}_1^n(\upsilon, \rho, \varrho) \to \upsilon^* \\ \mathfrak{R}_2^n(\upsilon, \rho, \varrho) \to \rho^* \\ \mathfrak{R}_3^n(\upsilon, \rho, \varrho) \to \varrho^* \end{cases}, \text{ as } n \to \infty.$$

($\heartsuit_3$) According to Theorem 1.1 (p_3), we successively have

$$\begin{aligned} &\begin{pmatrix} \omega\left(\mathfrak{R}_1^n(\upsilon_0, \rho_0, \varrho_0), \upsilon^*\right) \\ \omega\left(\mathfrak{R}_2^n(\upsilon_0, \rho_0, \varrho_0), \rho^*\right) \\ \omega\left(\mathfrak{R}_3^n(\upsilon_0, \rho_0, \varrho_0), \varrho^*\right) \end{pmatrix} \\ &= \widehat{\omega}(\mathfrak{R}^n(\upsilon_0, \rho_0, \varrho_0), (\upsilon^*, \rho^*, \varrho^*)) \\ &\leq Q^n(I-Q)^{-1}\widehat{\omega}((\upsilon_0, \rho_0, \varrho_0), (\mathfrak{R}_1(\upsilon_0, \rho_0, \varrho_0), \mathfrak{R}_2(\upsilon_0, \rho_0, \varrho_0), \mathfrak{R}_3(\upsilon_0, \rho_0, \varrho_0))) \\ &= Q^n(I-Q)^{-1}\begin{pmatrix} \omega(\upsilon_0, \mathfrak{R}_1(\upsilon_0, \rho_0, \varrho_0)) \\ \omega(\rho_0, \mathfrak{R}_1(\upsilon_0, \rho_0, \varrho_0)) \\ \omega(\varrho_0, \mathfrak{R}_3(\upsilon_0, \rho_0, \varrho_0)) \end{pmatrix}. \end{aligned}$$

($\heartsuit_4$) Let $\mathfrak{I} : \Omega^3 \to \Omega^3$ be such that

$$\mathfrak{I}(\upsilon, \rho, \varrho) = \begin{pmatrix} \mathfrak{I}_1(\upsilon, \rho, \varrho) \\ \mathfrak{I}_2(\upsilon, \rho, \varrho) \\ \mathfrak{I}_3(\upsilon, \rho, \varrho) \end{pmatrix},$$

and

$$\widehat{\omega}\left(\mathfrak{R}\left(\upsilon,\rho,\varrho\right),\mathfrak{I}\left(\upsilon,\rho,\varrho\right)\right) = \widehat{\omega}\left(\left(\begin{array}{c}\mathfrak{R}_1\left(\upsilon,\rho,\varrho\right)\\ \mathfrak{R}_2\left(\upsilon,\rho,\varrho\right)\\ \mathfrak{R}_3\left(\upsilon,\rho,\varrho\right)\end{array}\right),\left(\begin{array}{c}\mathfrak{I}_1\left(\upsilon,\rho,\varrho\right)\\ \mathfrak{I}_2\left(\upsilon,\rho,\varrho\right)\\ \mathfrak{I}_3\left(\upsilon,\rho,\varrho\right)\end{array}\right)\right) = \left(\begin{array}{c}\omega\left(\mathfrak{R}_1\left(\upsilon,\rho,\varrho\right),\mathfrak{I}_1\left(\upsilon,\rho,\varrho\right)\right)\\ \omega\left(\mathfrak{R}_2\left(\upsilon,\rho,\varrho\right),\mathfrak{I}_2\left(\upsilon,\rho,\varrho\right)\right)\\ \omega\left(\mathfrak{R}_3\left(\upsilon,\rho,\varrho\right),\mathfrak{I}_3\left(\upsilon,\rho,\varrho\right)\right)\end{array}\right)\leq\xi. \tag{2.3}$$

Then by Theorem 1.1 (p_4), we conclude that

$$\widehat{\omega}\left(\left(\upsilon^*,\rho^*,\varrho^*\right),\left(\wp,\hbar,\theta\right)\right)\leq(I-Q)^{-1}\xi.$$

$(\heartsuit_5)$ It follows from (2.3) that

$$\widehat{\omega}\left(\mathfrak{R}\left(\upsilon,\rho,\varrho\right),\mathfrak{I}\left(\upsilon,\rho,\varrho\right)\right)\leq\xi,$$

and since $\mathfrak{I}^n\left(\upsilon,\rho,\varrho\right)=\mathfrak{I}\left(\mathfrak{I}^{n-1}\left(\left(\upsilon,\rho,\varrho\right)\right)\right)$ for $(\upsilon,\rho,\varrho)\in\Omega^3$, we can iteratively write

$$\begin{aligned}\widehat{\omega}\left(\mathfrak{I}^n\left(\upsilon_0,\rho_0,\varrho_0\right),\left(\upsilon^*,\rho^*,\varrho^*\right)\right) &\leq \widehat{\omega}\left(\mathfrak{I}^n\left(\upsilon_0,\rho_0,\varrho_0\right),\mathfrak{R}^n\left(\upsilon_0,\rho_0,\varrho_0\right)\right)+\widehat{\omega}\left(\mathfrak{R}^n\left(\upsilon_0,\rho_0,\varrho_0\right),\left(\upsilon^*,\rho^*,\varrho^*\right)\right)\\ &= \widehat{\omega}\left(\mathfrak{I}^n\left(\upsilon_0,\rho_0,\varrho_0\right),\mathfrak{R}^n\left(\upsilon_0,\rho_0,\varrho_0\right)\right)\\ &\quad +Q^n(I-Q)^{-1}\widehat{\omega}\left(\mathfrak{R}\left(\upsilon_0,\rho_0,\varrho_0\right),\left(\upsilon_0,\rho_0,\varrho_0\right)\right).\end{aligned} \tag{2.4}$$

On the other hand, we get

$$\begin{aligned}\widehat{\omega}\left(\mathfrak{I}^n\left(\upsilon_0,\rho_0,\varrho_0\right),\mathfrak{R}^n\left(\upsilon_0,\rho_0,\varrho_0\right)\right) &= \widehat{\omega}\Big(\mathfrak{I}\left(\mathfrak{I}^{n-1}\left(\upsilon_0,\rho_0,\varrho_0\right)\right),\mathfrak{R}\left(\mathfrak{R}^{n-1}\left(\upsilon_0,\rho_0,\varrho_0\right)\right)\Big)\\ &\leq \widehat{\omega}\Big(\mathfrak{I}\left(\mathfrak{I}^{n-1}\left(\upsilon_0,\rho_0,\varrho_0\right)\right),\mathfrak{R}\left(\mathfrak{I}^{n-1}\left(\upsilon_0,\rho_0,\varrho_0\right)\right)\Big)\\ &\quad +\widehat{\omega}\Big(\mathfrak{R}\left(\mathfrak{I}^{n-1}\left(\upsilon_0,\rho_0,\varrho_0\right)\right),\mathfrak{R}\left(\mathfrak{R}^{n-1}\left(\upsilon_0,\rho_0,\varrho_0\right)\right)\Big)\\ &\leq \xi+Q\widehat{\omega}\left(\mathfrak{I}^{n-1}\left(\upsilon_0,\rho_0,\varrho_0\right),\mathfrak{R}^{n-1}\left(\upsilon_0,\rho_0,\varrho_0\right)\right)\\ &\leq \xi+Q\Big(\xi+\widehat{\omega}\left(\mathfrak{I}^{n-2}\left(\upsilon_0,\rho_0,\varrho_0\right),\mathfrak{R}^{n-2}\left(\upsilon_0,\rho_0,\varrho_0\right)\right)\Big)\\ &\leq \cdots\\ &\leq \xi\left(I+Q+Q^2+...+Q^n+...\right)=\xi(I-Q)^{-1}.\end{aligned} \tag{2.5}$$

Applying (2.5) in (2.4), we have

$$\widehat{\omega}\left(\mathfrak{I}^n\left(\upsilon_0,\rho_0,\varrho_0\right),\left(\upsilon^*,\rho^*,\varrho^*\right)\right)\leq\xi(I-Q)^{-1}+Q^n(I-Q)^{-1}\widehat{\omega}\left(\mathfrak{R}\left(\upsilon_0,\rho_0,\varrho_0\right),\left(\upsilon_0,\rho_0,\varrho_0\right)\right).$$

$(\heartsuit_5)$ It follows from $(\heartsuit_1)$ and $(\heartsuit_2)$ that there is a unique element $(\upsilon^*,\rho^*,\varrho^*)\in\Omega^3$ such that $(\upsilon^*,\rho^*,\varrho^*)$ is a solution to (2.2) and $\lim_{n\to\infty}\left(\mathfrak{R}_1^n\left(\upsilon,\rho,\varrho\right)\right)=\left(\upsilon^*,\rho^*,\varrho^*\right)$. Let $\epsilon_1,\epsilon_2,\epsilon_3>0$ and $(\ell_1,\ell_2,\ell_3)\in\Omega^3$ such that

$$\left\{\begin{array}{l}\omega\left(\ell_1,\mathfrak{R}_1\left(\ell_1,\ell_2,\ell_3\right)\right)\leq\epsilon_1\\ \omega\left(\ell_2,\mathfrak{R}_2\left(\ell_1,\ell_2,\ell_3\right)\right)\leq\epsilon_2\\ \omega\left(\ell_3,\mathfrak{R}_3\left(\ell_1,\ell_2,\ell_3\right)\right)\leq\epsilon_3\end{array}\right..$$

Hence, we have

$$
\begin{aligned}
& \widehat{\omega}\left(\left(\ell_1, \ell_2, \ell_3\right),\left(v^*, \rho^*, \varrho^*\right)\right) \\
\leq\ & \widehat{\omega}\left(\left(\ell_1, \ell_2, \ell_3\right),\left(\mathfrak{R}_1\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_2\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_3\left(\ell_1, \ell_2, \ell_3\right)\right)\right) \\
& +\widehat{\omega}\left(\left(\mathfrak{R}_1\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_2\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_3\left(\ell_1, \ell_2, \ell_3\right)\right),\left(v^*, \rho^*, \varrho^*\right)\right) \\
=\ & \widehat{\omega}\left(\left(\ell_1, \ell_2, \ell_3\right),\left(\mathfrak{R}_1\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_2\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_3\left(\ell_1, \ell_2, \ell_3\right)\right)\right) \\
& +\widehat{\omega}\left(\left(\mathfrak{R}_1\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_2\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_3\left(\ell_1, \ell_2, \ell_3\right)\right),\left(\mathfrak{R}_1\left(v^*, \rho^*, \varrho^*\right), \mathfrak{R}_2\left(v^*, \rho^*, \varrho^*\right), \mathfrak{R}_3\left(v^*, \rho^*, \varrho^*\right)\right)\right) \\
=\ & \begin{pmatrix} \omega\left(\ell_1, \mathfrak{R}_1\left(\ell_1, \ell_2, \ell_3\right)\right) \\ \omega\left(\ell_2, \mathfrak{R}_2\left(\ell_1, \ell_2, \ell_3\right)\right) \\ \omega\left(\ell_3, \mathfrak{R}_3\left(\ell_1, \ell_2, \ell_3\right)\right) \end{pmatrix} + \begin{pmatrix} \omega\left(\mathfrak{R}_1\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_1\left(v^*, \rho^*, \varrho^*\right)\right) \\ \omega\left(\mathfrak{R}_2\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_2\left(v^*, \rho^*, \varrho^*\right)\right) \\ \omega\left(\mathfrak{R}_3\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}_3\left(v^*, \rho^*, \varrho^*\right)\right) \end{pmatrix} \\
\leq\ & \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} + \widehat{\omega}\left(\mathfrak{R}\left(\ell_1, \ell_2, \ell_3\right), \mathfrak{R}\left(v^*, \rho^*, \varrho^*\right)\right) \\
\leq\ & \epsilon + Q\widehat{\omega}\left(\left(\ell_1, \ell_2, \ell_3\right),\left(v^*, \rho^*, \varrho^*\right)\right).
\end{aligned}
$$

Because $(I-Q)$ is invertible and $(I-Q)^{-1}$ has positive elements, we can write

$$\widehat{\omega}\left(\left(\ell_1, \ell_2, \ell_3\right),\left(v^*, \rho^*, \varrho^*\right)\right) \leq (I-Q)^{-1}\epsilon,$$

or, equivalently

$$\begin{pmatrix} \omega\left(\ell_1, v^*\right) \\ \omega\left(\ell_2, \rho^*\right) \\ \omega\left(\ell_3, \varrho^*\right) \end{pmatrix} \leq (I-Q)^{-1}\epsilon.$$

Set $(I-Q)^{-1} = \begin{pmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{pmatrix}$, then, we get

$$\begin{pmatrix} \omega\left(\ell_1, v^*\right) \\ \omega\left(\ell_2, \rho^*\right) \\ \omega\left(\ell_3, \varrho^*\right) \end{pmatrix} \leq \begin{pmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix},$$

which implies that

$$\begin{cases} \omega\left(\ell_1, v^*\right) \leq r_1\epsilon_1 + r_2\epsilon_2 + r_3\epsilon_3 \\ \omega\left(\ell_2, \rho^*\right) \leq r_4\epsilon_1 + r_5\epsilon_2 + r_6\epsilon_3 \\ \omega\left(\ell_3, \varrho^*\right) \leq r_7\epsilon_1 + r_8\epsilon_2 + r_9\epsilon_3 \end{cases},$$

this proves that the OES (2.2) is HU stable. □

Remark 2.1. *Our theorem converts some nice papers [33–37] in coupled fixed point theorems to TFP theorems by setting* (Ω, ω) *is a metric space and the operator* $\mathfrak{R} : \Omega^3 \to \Omega$ *is defined by* $\mathfrak{R}_1(v, \rho, \varrho) = \mathfrak{R}(v, \rho, \varrho)$, $\mathfrak{R}_2(\rho, \varrho, v) = \mathfrak{R}(\rho, \varrho, v)$ *and* $\mathfrak{R}_3(\varrho, v, \rho) = \mathfrak{R}(\varrho, v, \rho)$.

3. Supportive Application

In fact, this part serves as the pillar of our paper, where we shall apply Theorem 1.2 to discuss the existence, uniqueness, and HU stability of the solution to a periodic boundary value problem (PBVP).

Consider the PBVP

$$\begin{cases} \upsilon' = \wp_1(\tau,\upsilon) + \wp_2(\tau,\rho) + \wp_3(\tau,\varrho) \\ \rho' = \wp_1(\tau,\rho) + \wp_2(\tau,\varrho) + \wp_3(\tau,\upsilon) \\ \varrho' = \wp_1(\tau,\varrho) + \wp_2(\tau,\upsilon) + \wp_3(\tau,\rho) \\ \upsilon(0) = \upsilon(\varsigma),\ \rho(0) = \upsilon(\varsigma),\ \upsilon(0) = \upsilon(\varsigma) \end{cases}, \tag{3.1}$$

where $\varsigma > 0$ and $\wp_1$, $\wp_2$ and $\wp_3$ are continuous functions and fulfill the following hypothesis:

(H) there exist $\hbar_1, \hbar_2, \hbar_3, \eta_1, \eta_2, \eta_3 > 0$ such that, for each $\upsilon, \rho, \rho \in \mathbb{R}$, $\upsilon \geq \rho$, $\rho \geq \varrho$ and $\varrho \geq \upsilon$,

$$\begin{aligned} 0 &\leq (\wp_1(\tau,\upsilon) + \hbar_1\upsilon) - (\wp_1(\tau,\rho) + \hbar_1\rho) \leq \eta_1(\upsilon - \rho), \\ -\eta_2(\rho - \varrho) &\leq (\wp_2(\tau,\rho) - \hbar_2\rho) - (\wp_2(\tau,\varrho) - \hbar_2\varrho) \leq 0, \\ 0 &\leq (\wp_3(\tau,\varrho) + \hbar_3\varrho) - (\wp_3(\tau,\upsilon) + \hbar_3\upsilon) \leq \eta_3(\varrho - \upsilon), \end{aligned}$$

where $Q = \begin{pmatrix} \frac{\eta_1}{\hbar_1+\hbar_2+\hbar_3} & \frac{\eta_2}{\hbar_1+\hbar_2+\hbar_3} & \frac{\eta_3}{\hbar_1+\hbar_2+\hbar_3} \\ \frac{\eta_2}{\hbar_1+\hbar_2+\hbar_3} & \frac{\eta_3}{\hbar_1+\hbar_2+\hbar_3} & \frac{\eta_1}{\hbar_1+\hbar_2+\hbar_3} \\ \frac{\eta_3}{\hbar_1+\hbar_2+\hbar_3} & \frac{\eta_1}{\hbar_1+\hbar_2+\hbar_3} & \frac{\eta_2}{\hbar_1+\hbar_2+\hbar_3} \end{pmatrix}$ is a matrix convergent to zero.

To facilitate our study, we discuss the existence of a solution to the periodic problem below.

$$\begin{cases} \upsilon' + \hbar_1\upsilon - \hbar_2\rho + \hbar_3\varrho = \wp_1(\tau,\upsilon) + \wp_2(\tau,\rho) + \wp_3(\tau,\varrho) + \hbar_1\upsilon - \hbar_2\rho + \hbar_3\varrho, \\ \rho' + \hbar_1\rho - \hbar_2\varrho + \hbar_3\upsilon = \wp_1(\tau,\rho) + \wp_2(\tau,\varrho) + \wp_3(\tau,\upsilon) + \hbar_1\rho - \hbar_2\varrho + \hbar_3\upsilon \\ \varrho' + \hbar_1\varrho - \hbar_2\upsilon + \hbar_3\rho = \wp_1(\tau,\varrho) + \wp_2(\tau,\upsilon) + \wp_3(\tau,\rho) + \hbar_1\varrho - \hbar_2\upsilon + \hbar_3\rho \end{cases}, \tag{3.2}$$

with the periodic conditions

$$\upsilon(0) = \upsilon(\varsigma), \rho(0) = \upsilon(\varsigma), \upsilon(0) = \upsilon(\varsigma). \tag{3.3}$$

This system is equivalent to

$$\begin{aligned} \upsilon(\tau) = &\int_0^\varsigma \Theta_1(\tau,\delta)\left[\wp_1(\delta,\upsilon) + \wp_2(\delta,\rho) + \wp_3(\delta,\varrho) + \hbar_1\upsilon - \hbar_2\rho + \hbar_3\varrho\right] \\ &+ \Theta_2(\tau,\delta)\left[\wp_1(\delta,\rho) + \wp_2(\delta,\varrho) + \wp_3(\delta,\upsilon) + \hbar_1\rho - \hbar_2\varrho + \hbar_3\upsilon\right] \\ &+ \Theta_3(\tau,\delta)\left[\wp_1(\delta,\varrho) + \wp_2(\delta,\upsilon) + \wp_3(\delta,\rho) + \hbar_1\varrho - \hbar_2\upsilon + \hbar_3\rho\right] d\delta, \end{aligned}$$

$$\begin{aligned} \rho(\tau) = &\int_0^\varsigma \Theta_1(\tau,\delta)\left[\wp_1(\delta,\rho) + \wp_2(\delta,\varrho) + \wp_3(\delta,\upsilon) + \hbar_1\rho - \hbar_2\varrho + \hbar_3\upsilon\right] \\ &+ \Theta_2(\tau,\delta)\left[\wp_1(\delta,\varrho) + \wp_2(\delta,\upsilon) + \wp_3(\delta,\rho) + \hbar_1\varrho - \hbar_2\upsilon + \hbar_3\rho\right] \\ &+ \Theta_3(\tau,\delta)\left[\wp_1(\delta,\upsilon) + \wp_2(\delta,\rho) + \wp_3(\delta,\varrho) + \hbar_1\upsilon - \hbar_2\rho + \hbar_3\varrho\right] d\delta, \end{aligned}$$

$$\begin{aligned}\varrho(\tau) &= \int_0^{\varsigma} \Theta_1(\tau,\delta)\left[\wp_1(\delta,\varrho)+\wp_2(\delta,\upsilon)+\wp_3(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right] \\ &\quad +\Theta_2(\tau,\delta)\left[\wp_1(\delta,\upsilon)+\wp_2(\delta,\rho)+\wp_3(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right] \\ &\quad +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\rho)+\wp_2(\delta,\varrho)+\wp_3(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right] d\delta,\end{aligned}$$

where the continuous Green's functions Θ_1, Θ_2 and Θ_3 are described as

$$\begin{aligned}\Theta_1(\tau,\delta) &= \Theta_3(\tau,\delta) = \begin{cases} \frac{1}{3}\left(\frac{e^{\Upsilon_1(\tau-\delta)}}{1-e^{\Upsilon_1\varsigma}}+\frac{e^{\Upsilon_2(\tau-\delta)}}{1-e^{\Upsilon_2\varsigma}}+\frac{e^{\Upsilon_3(\tau-\delta)}}{1-e^{\Upsilon_3\varsigma}}\right), & 0\le\delta<\tau\le\varsigma, \\ \frac{1}{3}\left(\frac{e^{\Upsilon_1(\tau+\varsigma-\delta)}}{1-e^{\Upsilon_1\varsigma}}+\frac{e^{\Upsilon_2(\tau+\varsigma-\delta)}}{1-e^{\Upsilon_2\varsigma}}+\frac{e^{\Upsilon_3(\tau+\varsigma-\delta)}}{1-e^{\Upsilon_3\varsigma}}\right), & 0\le\tau<\delta\le\varsigma, \end{cases} \\ \Theta_2(\tau,\delta) &= \begin{cases} \frac{1}{3}\left(\frac{e^{\Upsilon_2(\tau-\delta)}}{1-e^{\Upsilon_2\varsigma}}-\frac{e^{\Upsilon_3(\tau-\delta)}}{1-e^{\Upsilon_3\varsigma}}-\frac{e^{\Upsilon_1(\tau-\delta)}}{1-e^{\Upsilon_1\varsigma}}\right), & 0\le\delta<\tau\le\varsigma, \\ \frac{1}{3}\left(\frac{e^{\Upsilon_2(\tau+\varsigma-\delta)}}{1-e^{\Upsilon_2\varsigma}}-\frac{e^{\Upsilon_3(\tau+\varsigma-\delta)}}{1-e^{\Upsilon_3\varsigma}}-\frac{e^{\Upsilon_1(\tau+\varsigma-\delta)}}{1-e^{\Upsilon_1\varsigma}}\right), & 0\le\tau<\delta\le\varsigma, \end{cases}\end{aligned}$$

$\Upsilon_1 = -(\hbar_1+\hbar_2+\hbar_3)$, and $\Upsilon_2 = \Upsilon_3 = (\hbar_1-\hbar_2+\hbar_3)$.

Now, by choosing $\hbar_1$, $\hbar_2$ and $\hbar_3$ suitably, we need to guarantee that $\Theta_1(\tau,\delta), \Theta_3(\tau,\delta) \ge 0$, for $0\le\tau,\delta\le\varsigma$ and $\Theta_2(\tau,\delta)\le 0$, for $0\le\tau,\delta\le\varsigma$. For this, we present the following lemma:

Lemma 3.1. *If*

$$\ln\left(\frac{2e-1}{e}\right) \le (\hbar_1-\hbar_2+\hbar_3)\,\varsigma, \tag{3.4}$$

$$(\hbar_1+\hbar_2+\hbar_3)\,\varsigma \le 1. \tag{3.5}$$

Then $\Theta_1(\tau,\delta), \Theta_3(\tau,\delta)\ge 0$, *for* $0\le\tau,\delta\le\varsigma$ *and* $\Theta_2(\tau,\delta)\le 0$, *for* $0\le\tau,\delta\le\varsigma$.

Proof. Since $\Upsilon_1<0$ and by (3.4), $\Upsilon_2>0$ and $\Upsilon_2>0$. Hence, $\Theta_2(\tau,\delta)\le 0$, for $0\le\tau,\delta\le\varsigma$.

On the other hand, from (3.4) and (3.5), one can obtain that

$$-\frac{e^{\Upsilon_1(\tau-\delta)}}{1-e^{\Upsilon_1\varsigma}} \le \frac{e}{1-e} \le \frac{e^{\Upsilon_2(\tau-\delta)}}{1-e^{\Upsilon_2\varsigma}} \text{ and } -\frac{e^{\Upsilon_3(\tau-\delta)}}{1-e^{\Upsilon_3\varsigma}} \le \frac{e}{1-e} \le \frac{e^{\Upsilon_2(\tau-\delta)}}{1-e^{\Upsilon_2\varsigma}}.$$

Therefore, under the assumption, we have $\Theta_1(\tau,\delta), \Theta_3(\tau,\delta)\ge 0$, for $0\le\tau,\delta\le\varsigma$. □

Assume that $\Omega = C(U,\mathbb{R})$ is the set of real valued continuous function on the interval U. Describe a distance $\omega : \Omega\times\Omega\to\mathbb{R}$ as

$$\omega(\upsilon,\rho) = \sup_{\tau\in U}\left|\upsilon(\tau)-\rho(\tau)\right|, \text{ for } \upsilon,\rho\in\Omega.$$

Clearly, (Ω,ω) is a metric space. For $\upsilon,\rho,\varrho,\upsilon^*,\rho^*,\varrho^*\in\Omega$, consider

$$\widehat{\omega}\left((\upsilon,\rho,\varrho),(\upsilon^*,\rho^*,\varrho^*)\right) = \begin{pmatrix}\omega(\upsilon,\upsilon^*)\\ \omega(\rho,\rho^*)\\ \omega(\varrho,\varrho^*)\end{pmatrix}.$$

Also, let us define $\mho : \Omega^3 \to \Omega$ by

$$\begin{aligned}\mho(\upsilon,\rho,\varrho)(\tau) &= \int\limits_0^{\varsigma} \Theta_1(\tau,\delta)\left[\wp_1(\delta,\upsilon)+\wp_2(\delta,\rho)+\wp_3(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right] \\ &\quad +\Theta_2(\tau,\delta)\left[\wp_1(\delta,\rho)+\wp_2(\delta,\varrho)+\wp_3(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right] \\ &\quad +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\varrho)+\wp_2(\delta,\upsilon)+\wp_3(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right] d\delta.\end{aligned}$$

Obviously, if $(\upsilon,\rho,\varrho)\in\Omega^3$ is a TFP of ∇, then we get

$$\upsilon(\tau)=\mho(\upsilon,\rho,\varrho)(\tau),\ \rho(\tau)=\mho(\rho,\varrho,\upsilon)(\tau)\ \text{ and } \varrho(\tau)=\mho(\varrho,\upsilon,\rho)(\tau)\ \text{ for all } \tau\in U.$$

Thus, (υ,ρ,ϱ) is a solution of the problem (3.2)-(3.3). To facilitate our task, we state the following definition:

Definition 3.1. *The system*

$$\begin{cases} \upsilon(\tau)=\int\limits_0^{\varsigma}\Theta_1(\tau,\delta)\left[\wp_1(\delta,\upsilon)+\wp_2(\delta,\rho)+\wp_3(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right] \\ \quad +\Theta_2(\tau,\delta)\left[\wp_1(\delta,\rho)+\wp_2(\delta,\varrho)+\wp_3(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right] \\ \quad +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\varrho)+\wp_2(\delta,\upsilon)+\wp_3(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right]d\delta \\ \rho(\tau)=\int\limits_0^{\varsigma}\Theta_1(\tau,\delta)\left[\wp_1(\delta,\rho)+\wp_2(\delta,\varrho)+\wp_3(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right] \\ \quad +\Theta_2(\tau,\delta)\left[\wp_1(\delta,\varrho)+\wp_2(\delta,\upsilon)+\wp_3(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right] \\ \quad +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\upsilon)+\wp_2(\delta,\rho)+\wp_3(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right]d\delta \\ \varrho(\tau)=\int\limits_0^{\varsigma}\Theta_1(\tau,\delta)\left[\wp_1(\delta,\varrho)+\wp_2(\delta,\upsilon)+\wp_3(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right] \\ \quad +\Theta_2(\tau,\delta)\left[\wp_1(\delta,\upsilon)+\wp_2(\delta,\rho)+\wp_3(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right] \\ \quad +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\rho)+\wp_2(\delta,\varrho)+\wp_3(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right]d\delta \end{cases}, \tag{3.6}$$

is called HU stable if there are $r_j>0$, $(i=1,2,3,4,5,6,7,8,9)$ *so that for each* $\epsilon_1,\epsilon_2,\epsilon_3>0$ *and each solution* $(\ell_1,\ell_2,\ell_3)\in\Omega^3$ *of the following system*

$$\begin{cases} \Big|\ell_1(\tau)-\int\limits_0^{\varsigma}\Theta_1(\tau,\delta)\left[\wp_1(\delta,\ell_1)+\wp_2(\delta,\ell_2)+\wp_3(\delta,\ell_3)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right] \\ \quad +\Theta_2(\tau,\delta)\left[\wp_1(\delta,\ell_2)+\wp_2(\delta,\ell_3)+\wp_3(\delta,\ell_1)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right] \\ +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\ell_3)+\wp_2(\delta,\ell_1)+\wp_3(\delta,\ell_2)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right]d\delta\Big|\le\epsilon_1 \\ \Big|\ell_2(\tau)-\int\limits_0^{\varsigma}\Theta_1(\tau,\delta)\left[\wp_1(\delta,\ell_2)+\wp_2(\delta,\ell_3)+\wp_3(\delta,\ell_1)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right] \\ \quad +\Theta_2(\tau,\delta)\left[\wp_1(\delta,\ell_3)+\wp_2(\delta,\ell_1)+\wp_3(\delta,\ell_2)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right] \\ +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\ell_1)+\wp_2(\delta,\ell_2)+\wp_3(\delta,\ell_3)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right]d\delta\Big|\le\epsilon_2 \\ \Big|\ell_3(\tau)-\int\limits_0^{\varsigma}\Theta_1(\tau,\delta)\left[\wp_1(\delta,\ell_3)+\wp_2(\delta,\ell_1)+\wp_3(\delta,\ell_2)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right] \\ \quad +\Theta_2(\tau,\delta)\left[\wp_1(\delta,\ell_1)+\wp_2(\delta,\ell_2)+\wp_3(\delta,\ell_3)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right] \\ +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\ell_2)+\wp_2(\delta,\ell_3)+\wp_3(\delta,\ell_1)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right]d\delta\Big|\le\epsilon_3 \end{cases},$$

there is a solution $(\upsilon^*, \rho^*, \varrho^*) \in \Omega^3$ *of (3.6) so that*

$$\begin{cases} \omega\left(\ell_1(\tau), \upsilon^*(\tau)\right) \leq r_1\epsilon_1 + r_2\epsilon_2 + r_3\epsilon_3 \\ \omega\left(\ell_2(\tau), \rho^*(\tau)\right) \leq r_4\epsilon_1 + r_5\epsilon_2 + r_6\epsilon_3 \\ \omega\left(\ell_3(\tau), \varrho^*(\tau)\right) \leq r_7\epsilon_1 + r_8\epsilon_2 + r_9\epsilon_3 \end{cases}.$$

By introducing the subsequent theorem, we can now talk about the existence, uniqueness, and HU stability of the solution to a PBVP.

Theorem 3.1. *Under the hypothesis* (H)*, consider the problem (3.1) with* $\wp_1, \wp_2, \wp_3 \in C(U \times \mathbb{R}, \mathbb{R})$*. If (3.4) and (3.5) are true, then*

(T_1) *there is a unique solution* (ℓ_1, ℓ_2, ℓ_3) *for the PBVP (3.1), provided that*

$$\sup_{\tau \in I}\left(\int_0^{\varsigma}\left|\Theta_1(\tau,\delta) - \Theta_2(\tau,\delta) + \Theta_3(\tau,\delta)\right|\right)d\delta \leq \frac{1}{\hbar_1 + \hbar_2 + \hbar_3}. \tag{3.7}$$

(T_2) *if* $\wp_1^*, \wp_2^*, \wp_3^* \in C(U \times \mathbb{R}, \mathbb{R})$ *such that there is* $\xi_1, \xi_2, \xi_3 > 0$ *with*

$$\begin{cases} \left|\wp_1(\tau,\ell) - \wp_1^*(\tau,\ell)\right| \leq \xi_1 \\ \left|\wp_2(\tau,\ell) - \wp_2^*(\tau,\ell)\right| \leq \xi_2 \\ \left|\wp_3(\tau,\ell) - \wp_3^*(\tau,\ell)\right| \leq \xi_3 \end{cases},$$

for all $(\tau, \ell) \in U \times \mathbb{R}$*. Assume that* $(\wp, \hbar, \theta) \in \Omega^3$ *is a solution to the proposed problem (3.1) by taking* $\wp_1 = \wp_1^*$*,* $\wp_2 = \wp_2^*$ *and* $\wp_3 = \wp_3^*$*, then*

$$\widehat{\omega}\left((\ell_1,\ell_2,\ell_3),(\wp,\hbar,\theta)\right) = \begin{pmatrix} \omega(\ell_1,\wp) \\ \omega(\ell_2,\hbar) \\ \omega(\ell_3,\theta) \end{pmatrix} \leq (I-Q)^{-1}\xi.$$

where $\xi = \begin{pmatrix} (\xi_1+\xi_2+\xi_3)\left(\frac{1}{\hbar_1+\hbar_2+\hbar_3}\right) \\ (\xi_1+\xi_2+\xi_3)\left(\frac{1}{\hbar_1+\hbar_2+\hbar_3}\right) \\ (\xi_1+\xi_2+\xi_3)\left(\frac{1}{\hbar_1+\hbar_2+\hbar_3}\right) \end{pmatrix}$*, if and only if*

$$\sup_{\tau \in I}\left(\int_0^{\varsigma}\left|\Theta_1(\tau,\delta)\right|d\delta + \int_0^{\varsigma}\left|\Theta_2(\tau,\delta)\right|d\delta + \int_0^{\varsigma}\left|\Theta_3(\tau,\delta)\right|d\delta\right) \leq \frac{1}{\hbar_1+\hbar_2+\hbar_3}$$

(T_3) *the problem (3.6) is HU stable.*

Proof. (T_1)

$$\begin{aligned} & \omega\left(\mho(\ell_1,\ell_2,\ell_3), \mho(\upsilon,\rho,\varrho)\right) \\ = \; & \sup_{\tau\in I}\left|\mho(\ell_1,\ell_2,\ell_3) - \mho(\upsilon,\rho,\varrho)\right| \\ = \; & \sup_{\tau\in I}\int_0^{\varsigma}\Theta_1(\tau,\delta)\left[\wp_1(\delta,\ell_1) + \wp_2(\delta,\ell_2) + \wp_3(\delta,\ell_3) + \hbar_1\ell_1 - \hbar_2\ell_2 + \hbar_3\ell_3\right] \end{aligned}$$

$$
\begin{aligned}
& +\Theta_2(\tau,\delta)\left[\wp_1(\delta,\ell_2)+\wp_2(\delta,\ell_3)+\wp_3(\delta,\ell_1)+\hbar_1\ell_2-\hbar_2\ell_3+\hbar_3\ell_1\right] \\
& +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\ell_3)+\wp_2(\delta,\ell_1)+\wp_3(\delta,\ell_2)+\hbar_1\ell_3-\hbar_2\ell_1+\hbar_3\ell_2\right]d\delta \\
& -\int_0^{\varsigma}\Theta_1(\tau,\delta)\left[\wp_1(\delta,\upsilon)+\wp_2(\delta,\rho)+\wp_3(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right] \\
& +\Theta_2(\tau,\delta)\left[\wp_1(\delta,\rho)+\wp_2(\delta,\varrho)+\wp_3(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right] \\
& +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\varrho)+\wp_2(\delta,\upsilon)+\wp_3(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right]d\delta. \\
= \; & \sup_{\tau\in I}\int_0^{\varsigma}\Theta_1(\tau,\delta)\left[\left(\wp_1(\delta,\ell_1)+\wp_2(\delta,\ell_2)+\wp_3(\delta,\ell_3)+\hbar_1\ell_1-\hbar_2\ell_2+\hbar_3\ell_3\right)\right. \\
& \left.-\left(\wp_1(\delta,\upsilon)+\wp_2(\delta,\rho)+\wp_3(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right)\right] \\
& +\Theta_2(\tau,\delta)\left[\left(\wp_1(\delta,\ell_2)+\wp_2(\delta,\ell_3)+\wp_3(\delta,\ell_1)+\hbar_1\ell_2-\hbar_2\ell_3+\hbar_3\ell_1\right)\right. \\
& \left.-\left(\wp_1(\delta,\rho)+\wp_2(\delta,\varrho)+\wp_3(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right)\right] \\
& +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\ell_3)+\wp_2(\delta,\ell_1)+\wp_3(\delta,\ell_2)+\hbar_1\ell_3-\hbar_2\ell_1+\hbar_3\ell_2\right. \\
& \left.-\left(\wp_1(\delta,\varrho)+\wp_2(\delta,\upsilon)+\wp_3(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right)\right]d\delta,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \omega\left(\mho(\ell_1,\ell_2,\ell_3),\mho(\upsilon,\rho,\varrho)\right) \\
= \; & \sup_{\tau\in I}\int_0^{\varsigma}\Theta_1(\tau,\delta)\left[\left(\wp_1(\delta,\ell_1)+\wp_2(\delta,\ell_2)+\wp_3(\delta,\ell_3)+\hbar_1\ell_1-\hbar_2\ell_2+\hbar_3\ell_3\right)\right. \\
& \left.-\left(\wp_1(\delta,\upsilon)+\wp_2(\delta,\rho)+\wp_3(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right)\right] \\
& -\Theta_2(\tau,\delta)\left[\left(\wp_1(\delta,\rho)+\wp_2(\delta,\varrho)+\wp_3(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right)\right. \\
& \left.-\left(\wp_1(\delta,\ell_2)+\wp_2(\delta,\ell_3)+\wp_3(\delta,\ell_1)+\hbar_1\ell_2-\hbar_2\ell_3+\hbar_3\ell_1\right)\right] \\
& +\Theta_3(\tau,\delta)\left[\wp_1(\delta,\ell_3)+\wp_2(\delta,\ell_1)+\wp_3(\delta,\ell_2)+\hbar_1\ell_3-\hbar_2\ell_1+\hbar_3\ell_2\right. \\
& \left.-\left(\wp_1(\delta,\varrho)+\wp_2(\delta,\upsilon)+\wp_3(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right)\right]d\delta \\
\leq \; & \sup_{\tau\in I}\int_0^{\varsigma}\Theta_1(\tau,\delta)\left(\left[\eta_1(\ell_1-\upsilon)+\eta_2(\ell_2-\upsilon)+\eta_3(\ell_3-\upsilon)\right.\right. \\
& -\Theta_2(\tau,\delta)\left[\left[\eta_1(\rho-\ell_2)+\eta_2(\varrho-\ell_3)+\eta_3(\upsilon-\ell_1)\right]\right. \\
& +\Theta_3(\tau,\delta)\left(\left[\eta_1(\ell_3-\varrho)+\eta_2(\ell_1-\upsilon)+\eta_3(\ell_2-\rho)\right)\right)d\delta.
\end{aligned}
$$

Applying the condition (3.7), we have

$$
\begin{aligned}
& \omega\left(\mho(\ell_1,\ell_2,\ell_3),\mho(\upsilon,\rho,\varrho)\right) \\
\leq \; & \left[\eta_1(\ell_1-\upsilon)+\eta_2(\ell_2-\rho)+\eta_3(\ell_3-\varrho)\right]\sup_{\tau\in I}\int_0^{\varsigma}\left|\Theta_1(\tau,\delta)-\Theta_2(\tau,\delta)+\Theta_3(\tau,\delta)\right|d\delta
\end{aligned}
$$

$$\leq \frac{\eta_1}{\hbar_1+\hbar_2+\hbar_3}\omega\left(\ell_1,\upsilon\right)+\frac{\eta_2}{\hbar_1+\hbar_2+\hbar_3}\omega\left(\ell_2,\rho\right)+\frac{\eta_3}{\hbar_1+\hbar_2+\hbar_3}\omega\left(\ell_3,\varrho\right).$$

Similarly, one can obtain

$$\omega\left(\mho\left(\ell_2,\ell_3,\ell_1\right),\mho\left(\rho,\varrho,\upsilon\right)\right)\leq \frac{\eta_1}{\hbar_1+\hbar_2+\hbar_3}\omega\left(\ell_2,\rho\right)+\frac{\eta_2}{\hbar_1+\hbar_2+\hbar_3}\omega\left(\ell_3,\varrho\right)+\frac{\eta_3}{\hbar_1+\hbar_2+\hbar_3}\omega\left(\ell_1,\upsilon\right),$$

and

$$\omega\left(\mho\left(\ell_3,\ell_1,\ell_2\right),\mho\left(\varrho,\upsilon,\rho\right)\right)\leq \frac{\eta_1}{\hbar_1+\hbar_2+\hbar_3}\omega\left(\ell_3,\varrho\right)+\frac{\eta_2}{\hbar_1+\hbar_2+\hbar_3}\omega\left(\ell_1,\upsilon\right)+\frac{\eta_3}{\hbar_1+\hbar_2+\hbar_3}\omega\left(\ell_2,\rho\right).$$

Choose $a_1=\frac{\eta_1}{\hbar_1+\hbar_2+\hbar_3}$, $a_2=\frac{\eta_2}{\hbar_1+\hbar_2+\hbar_3}$ and $a_3=\frac{\eta_3}{\hbar_1+\hbar_2+\hbar_3}$, we get

$$\begin{aligned}\omega\left(\mho\left(\ell_1,\ell_2,\ell_3\right),\mho\left(\upsilon,\rho,\varrho\right)\right) &\leq a_1\omega\left(\ell_1,\upsilon\right)+a_2\omega\left(\ell_2,\rho\right)+a_3\omega\left(\ell_3,\varrho\right),\\ \omega\left(\mho\left(\ell_2,\ell_3,\ell_1\right),\mho\left(\rho,\varrho,\upsilon\right)\right) &\leq a_1\omega\left(\ell_2,\rho\right)+a_2\omega\left(\ell_3,\varrho\right)+a_3\omega\left(\ell_1,\upsilon\right),\end{aligned}$$

and

$$\omega\left(\mho\left(\ell_3,\ell_1,\ell_2\right),\mho\left(\varrho,\upsilon,\rho\right)\right)\leq a_1\omega\left(\ell_3,\varrho\right)+a_2\omega\left(\ell_1,\upsilon\right)+a_3\omega\left(\ell_2,\rho\right).$$

It follows that

$$\begin{aligned}\widehat{\omega}\left(\mho\left(\ell_1,\ell_2,\ell_3\right),\mho\left(\upsilon,\rho,\varrho\right)\right) &= \begin{pmatrix}\omega\left(\mho\left(\ell_1,\ell_2,\ell_3\right),\mho\left(\upsilon,\rho,\varrho\right)\right)\\ \omega\left(\mho\left(\ell_2,\ell_3,\ell_1\right),\mho\left(\rho,\varrho,\upsilon\right)\right)\\ \omega\left(\mho\left(\ell_3,\ell_1,\ell_2\right),\mho\left(\varrho,\upsilon,\rho\right)\right)\end{pmatrix}\\ &\leq \begin{pmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ a_7 & a_8 & a_9\end{pmatrix}.\begin{pmatrix}\omega\left(\ell_1,\upsilon\right)\\ \omega\left(\ell_2,\rho\right)\\ \omega\left(\ell_3,\varrho\right)\end{pmatrix}=Q.\widehat{\omega}\left(\left(\ell_1,\ell_2,\ell_3\right),\left(\upsilon,\rho,\varrho\right)\right),\end{aligned}$$

where Q is a matrix convergent to zero. Hence, the assumption of Theorem 2.1 is fulfilled. Then the PBVP (3.2)-(3.3) has a unique solution on U.

(T_2) Define an operator $\mho^*:\Omega^3\to\Omega$ by

$$\begin{aligned}\mho^*\left(\upsilon,\rho,\varrho\right)\left(\tau\right) &= \int_0^{\varsigma}\Theta_1\left(\tau,\delta\right)\left[\wp_1^*\left(\delta,\upsilon\right)+\wp_2^*\left(\delta,\rho\right)+\wp_3^*\left(\delta,\varrho\right)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right]\\ &+\Theta_2\left(\tau,\delta\right)\left[\wp_1^*\left(\delta,\rho\right)+\wp_2^*\left(\delta,\varrho\right)+\wp_3^*\left(\delta,\upsilon\right)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right]\\ &+\Theta_3\left(\tau,\delta\right)\left[\wp_1^*\left(\delta,\varrho\right)+\wp_2^*\left(\delta,\upsilon\right)+\wp_3^*\left(\delta,\rho\right)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right]d\delta.\end{aligned}$$

Then, we get

$$\begin{aligned}&\left|\mho\left(\upsilon,\rho,\varrho\right)\left(\tau\right)-\mho^*\left(\upsilon,\rho,\varrho\right)\left(\tau\right)\right|\\ =&\left|\int_0^{\varsigma}\Theta_1\left(\tau,\delta\right)\left[\wp_1\left(\delta,\upsilon\right)+\wp_2\left(\delta,\rho\right)+\wp_3\left(\delta,\varrho\right)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right]\right.\\ &+\Theta_2\left(\tau,\delta\right)\left[\wp_1\left(\delta,\rho\right)+\wp_2\left(\delta,\varrho\right)+\wp_3\left(\delta,\upsilon\right)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right]\\ &+\Theta_3\left(\tau,\delta\right)\left[\wp_1\left(\delta,\varrho\right)+\wp_2\left(\delta,\upsilon\right)+\wp_3\left(\delta,\rho\right)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right]d\delta\end{aligned}$$

$$-\int_0^{\varsigma}\Theta_1(\tau,\delta)\left[\wp_1^*(\delta,\upsilon)+\wp_2^*(\delta,\rho)+\wp_3^*(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right]$$
$$+\Theta_2(\tau,\delta)\left[\wp_1^*(\delta,\rho)+\wp_2^*(\delta,\varrho)+\wp_3^*(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right]$$
$$+\Theta_3(\tau,\delta)\left[\wp_1^*(\delta,\varrho)+\wp_2^*(\delta,\upsilon)+\wp_3^*(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right]d\delta\Big|$$
$$\leq \int_0^{\varsigma}\Big|\Theta_1(\tau,\delta)\left[\left[\wp_1(\delta,\upsilon)+\wp_2(\delta,\rho)+\wp_3(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right]\right.$$
$$\left.-\left[\wp_1^*(\delta,\upsilon)+\wp_2^*(\delta,\rho)+\wp_3^*(\delta,\varrho)+\hbar_1\upsilon-\hbar_2\rho+\hbar_3\varrho\right]\right]$$
$$+\Theta_2(\tau,\delta)\left[\left[\wp_1(\delta,\rho)+\wp_2(\delta,\varrho)+\wp_3(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right]\right.$$
$$\left.-\left[\wp_1^*(\delta,\rho)+\wp_2^*(\delta,\varrho)+\wp_3^*(\delta,\upsilon)+\hbar_1\rho-\hbar_2\varrho+\hbar_3\upsilon\right]\right]$$
$$+\Theta_3(\tau,\delta)\left[\left[\wp_1(\delta,\varrho)+\wp_2(\delta,\upsilon)+\wp_3(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right]\right.$$
$$\left.-\left[\wp_1^*(\delta,\varrho)+\wp_2^*(\delta,\upsilon)+\wp_3^*(\delta,\rho)+\hbar_1\varrho-\hbar_2\upsilon+\hbar_3\rho\right]\right]\Big|d\delta.$$

Using our assumption, we have

$$\begin{aligned}&\left|\mho(\upsilon,\rho,\varrho)(\tau)-\mho^*(\upsilon,\rho,\varrho)(\tau)\right|\\ \leq\ &(\xi_1+\xi_2+\xi_3)\times\sup_{\tau\in I}\left(\int_0^{\varsigma}\left|\Theta_1(\tau,\delta)\right|d\delta+\int_0^{\varsigma}\left|\Theta_2(\tau,\delta)\right|d\delta+\int_0^{\varsigma}\left|\Theta_3(\tau,\delta)\right|d\delta\right)\\ \leq\ &\frac{\xi_1+\xi_2+\xi_3}{\hbar_1+\hbar_2+\hbar_3}.\end{aligned}$$

Analogously, one can obtain that

$$\left|\mho(\rho,\varrho,\upsilon)(\tau)-\mho^*(\rho,\varrho,\upsilon)(\tau)\right|\leq\frac{\xi_1+\xi_2+\xi_3}{\hbar_1+\hbar_2+\hbar_3},$$

and

$$\left|\mho(\varrho,\upsilon,\rho)(\tau)-\mho^*(\varrho,\upsilon,\rho)(\tau)\right|\leq\frac{\xi_1+\xi_2+\xi_3}{\hbar_1+\hbar_2+\hbar_3}.$$

Now, the rest of proof follows immediately from Theorem 2.1 ($\heartsuit_4$).

(T_3) We get the conclusion from the first part of our proof and from Theorem 2.1 ($\heartsuit_6$). □

Remark 3.1. *The conclusions below are special cases of Theorem 3.1 and Theorem 2.1 ($\heartsuit_2$), ($\heartsuit_3$).*

- *The sequence $(\mho^n(\upsilon,\rho,\varrho),\mho^n(\rho,\varrho,\upsilon),\mho^n(\varrho,\upsilon,\rho))_{n\in\mathbb{N}}$ converges in Ω^3 to $(\upsilon^*,\rho^*,\varrho^*)$ as $n\to\infty$, where*

$$\left(\mho^{n+1}(\upsilon,\rho,\varrho),\mho^{n+1}(\rho,\varrho,\upsilon),\mho^{n+1}(\varrho,\upsilon,\rho)\right)=\mho^n(\mho(\upsilon,\rho,\varrho),\mho(\rho,\varrho,\upsilon),\mho(\varrho,\upsilon,\rho)),\ \forall n\in\mathbb{N}^*.$$

- *We have the following estimations*

$$\begin{pmatrix}\omega(\mho^n(\upsilon_0,\rho_0,\varrho_0),\upsilon^*)\\ \omega(\mho^n(\upsilon_0,\rho_0,\varrho_0),\rho^*)\\ \omega(\mho^n(\upsilon_0,\rho_0,\varrho_0),\varrho^*)\end{pmatrix}\leq Q^n(I-Q)^{-1}\begin{pmatrix}\omega(\upsilon_0,\mho(\upsilon_0,\rho_0,\varrho_0))\\ \omega(\rho_0,\mho(\upsilon_0,\rho_0,\varrho_0))\\ \omega(\varrho_0,\mho(\upsilon_0,\rho_0,\varrho_0))\end{pmatrix}.$$

4. Conclusion

In this article, we provide a new modification of the Perov-fixed point theorem for single-valued mappings in a complete generalized metric space. Also, we have discussed some TFP results for extended contractive mappings in the mentioned space. Moreover, the existence, uniqueness, and stability of these points have been investigated for single-value mappings. Finally, the theoretical results are involved to discuss the existence and availability of a solution to some periodic boundary value problems as a kind of application and to support our study.

5. Open problems

In this part, we present some problems that we are unable to complete, which in turn improve our paper, and we list them as follows:

- What would be the results of Theorem 2.1 and Theorem 3.1 if the mapping $\mho$ were replaced by a multivalued mapping under the same constraints?
- Will the results be the same if the generalized metric-like or generalized $b-$metric spaces are used?
- Can we include illustrative examples, especially numerical examples, to know the formula for the numerical solution to integral equations and periodic boundary value problems under study?

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