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Some Results of n-EP Operators on Hilbert Spaces

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Abstract. Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H and $n \in \mathbb{N}$. An operator $T \in B(H)$ with closed range, is called n-EP operator if T^n commutes with T^+. In this paper, we present some new characterizations of n-EP operators, using Moore-penrose and Drazin inverse. Also, the problem of determining when the product of two operators is n-EP will be considered. As a consequence, we generalize a famous result on products of normal operators, due to I. Kaplansky to n-EP operators.

1. Introduction and preliminaries

Let $B(H)$ be the Banach algebra of all bounded linear operators on a complex Hilbert space H. For $T \in B(H)$, we use symbols $R(T)$, $N(T)$ and T^*, the range, the null subspace and the adjoint operator of T, respectively. It is known that every operator $T \in B(H)$ can be decomposed as $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ is the modulus of T and U is an appropriate partial isometry (i.e. $UU^*U = U$), such that $N(U) = N(T) = N(|T|)$. Now, we recall the definitions of some generalized inverses. For $T \in B(H)$, The Moore-Penrose inverse of T is the unique operator $T^+ \in B(H)$, which satisfies the four operator equations :

$$TT^+T = T, \ \ T^+TT^+ = T^+, \ \ (TT^+)^* = TT^+, \ \ (T^+T)^* = T^+T. \tag{1.1}$$

As we know, T has a Moore-Penrose inverse if and only if $\mathcal{R}(T)$ is closed. From (1.1), it can be proved that $\mathcal{R}(T^+) = \mathcal{R}(T^*)$ and that TT^+ and T^+T are orthogonal projections into $\mathcal{R}(T)$ and $\mathcal{R}(T^*)$, respectively. Notice that if $R(T)$ is closed, then T^* and $|T|$ and have closed ranges, in this case $(T^*)^+ = (T^+)^*$ and $T^+ = |T|^+U^*$, where $T = U|T|$ is the polar decomposition of T.
The Drazin inverse of $T \in B(H)$, is the unique operator A^D that satisfies :

$$TT^D = T^DT \ \ ,T^DTT^D = T^D \ \ ,T^{k+1}T^D = T^k, \quad \text{for some } k \geq 1.$$

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The smallest such k is called the Drazin index of T and denoted by $ind(A)$. In particular, when $ind(A) = 1$, the Drazin inverse T^D is called the group inverse of T and it is denoted by $T^{\#}$. Recall that $T \in B(H)$ is Drazin invertible if and only if $a(T)$ and $d(T)$ are finite, where $a(T)$, the ascent of T is the the minimal integer n such that $N(T^n) = N(T^{n+1})$ and $d(T)$, the descent of T the minimal integer n such that $R(T^n) = R(T^{n+1})$. If $a(T)$ and $d(T)$ are finite, they are equal and their common value is the index of T [9, Proposition 38.3].

An operator $T \in B(H)$ is said to be normal if $TT^* = T^*T$, n-normal if $T^nT^* = T^*T^n$, for some $n \in \mathbb{N}$ and EP operator if its range, $R(T)$, is closed and $R(T) = R(T^*)$. One of the interests of the EP operator lies in the fact that it commutes with its Moore-Penrose inverse. Clearly, every normal operator with closed range is EP but the converse is not true even in a finite dimensional space. Various known characterizations of EP matrices and EP operators were collected in [2–5]. Recently, Malik et al. [14] introduced the notion of n-EP matrices, generalizing the notion of EP matrices and later it was extended to closed range operators on an arbitrary Hilbert space by Wang and Deng [15]. An operators $T \in B(H)$ with closed range, is called n-EP if it satisfies $T^nT^+ = T^+T^n$, for some $n \in \mathbb{N}$. In [15], the authors obtained interesting characteristics of n-EP operators, also they studied the properties of the particular case of n-EP operators that are Drazin invertible in terms of the operator matrix decomposition. The goal of this paper is to explore futher characterizations of n-EP operators using the Moore-Penrose inverse and Drazin inverse.

The paper is organized as follow: In section 2, at first we review certain known fundamental properties of the class of n-EP operators and then establish new ones. Afterwards, we study the relationship between this class and the class of n-normal operators. Finally, we provide new characterizations of n-EP operators by certain conditions that relate to their powers and their Moore-Penrose and Drazin inverse. In particular, we show that if $ind(T) = n$ and T^DT is self-adjoint then T is n-EP. In section 3, it will be considered the problem of determining when the product of two operators is n-EP. As a consequence, we generalize the famous result on products of normal operators, due to I. Kaplansky [12] to n-EP operators. Also, some recent results obtained for EP complex matrices by P. Sam Johnson et al. [11] are extended to n-EP operators on an arbitrary Hilbert space.

To prove the main results, we shall begin with some lemmas.

Lemma 1.1. *[8] Let $T \in B(H)$ with closed range. If $S \in B(H)$ such that $ST = TS$ and $ST^* = T^*S$, then $ST^+ = T^+S$.*

Lemma 1.2. *[3] Let $T \in B(H)$ be an EP operator and $S \in B(H)$ such that $ST = TS$, then $ST^+ = T^+S$.*

Lemma 1.3. *[10] Let $T, S \in B(H)$ have closed ranges. Then TS has a closed range if and only if $R(T^+TSS^+)$ is closed.*

Lemma 1.4. *[6] Let $T, S \in \mathcal{B}(\mathcal{H})$ such that T, S and TS have closed ranges. Then the following statements are equivalent:*

(i) $(TS)^+ = S^+T^+$.

(ii) $\mathcal{R}(T^*TS) \subset \mathcal{R}(S)$ *and* $\mathcal{R}(SS^*T^*) \subset \mathcal{R}(T^*)$.

2. On the class of n-EP operators

In this section, first we give some fundamental properties of n-EP operators.

Proposition 2.1. *let $T \in B(H)$ be an n-EP operator. Then the following statements hold*

(1) *λT is n-EP operator, for all scalar λ.*
(2) *T^* is n-EP operator.*
(3) *If $S \in B(H)$ is unitary equivalente to T, then S is n-EP operator.*

Proof. (1) It is trivial in case $\lambda = 0$.
Now we suppose $\lambda \neq 0$. Then

$$(\lambda T)^n(\lambda T)^+ = (\lambda)^n \frac{1}{\lambda} T^n T^+ = \lambda^n \frac{1}{\lambda} T^+ T^n = (\lambda T)^+(\lambda T)^n$$

Hence, λT is n-EP operator.
(2) Since T is n-EP, then $T^nT^+ = T^+T^n$. By taking the adjoint, we find $(T^*)^n(T^*)^+ = (T^*)^+(T^*)^n$.So T^* is n-EP operator.
(3) Since S is unitary equivalente to T , then there exists an unitary operator $U \in B(H)$ such that $S = UTU^*$. So $R(UTU^*)$ is closed and $S^+ = UT^+U^*$. It follows that

$$S^nS^+ = UT^nT^+U^* = UT^+T^nU^* = S^+S^n \ .$$

Thus, S is is n-EP operator. □

Proposition 2.2. *If T is k-EP operator for a positive integer k, then T is $k+1$-EP. Hence T is n-EP for all $n \geq k$.*

Proof. Since $TT^+T = T$, then $T^kT^+T = T^k$ and $TT^+T^k = T^k$. According to T is k-EP, we find $T^{k+1}T^+ = T^k$ and $T^+T^{k+1} = T^k$. Thus, $T^{k+1}T^+ = T^+T^{k+1}$. Therefore, T is $k+1$-EP. □

Remark 2.1. *Following [1], an operator $T \in B(H)$ is said to be n-normal operator if $T^nT^* = T^*T^n$. By lemma 1.1 every n-normal operator with closed range is n-EP, but the converse is not true even in a finite dimensional space. Indeed, consider $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbb{C}^2$. Then $T^+ = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. By a direct calculation, we have*

$$T^2T^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = T^2T^+.$$

Hence, T is 2-Ep , while T is not 2-normal because

$$T^2T^* = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = T^*T^2$$

Next, we provide a necessary and sufficient condition under which an n-EP operator becomes n-normal. In the case $n = 1$, we obtain one well-known characterization of normal operators [3, Theorem 3.5].

Theorem 2.1. *Let $T \in B(H)$ with closed range. Then T is n-normal if and only if T is n-Ep and $(T^n)^*T^+ = T^+(T^n)^*$.*

Proof. If T is n-normal with closed range, then it's n-Ep. Applying involution to $T^nT^* = T^*T^n$, we have $(T^n)^*T = T(T^n)^*$ and since $(T^n)^*T^* = T^*(T^n)^*$, by Lemma 1.1, we get $(T^n)^*T^+ = T^+(T^n)^+$. Conversely, we Assume That T is n Ep operator. Then $T^nT^+ = T^+T^n$. From the the hypothesis $(T^n)^*T^+ = T^+(T^n)^*$, by taking the adjoint we get $(T^*)^+T^n = T^n(T^*)^+$. By using again Lemma 1.1, we obtain $T^nT^* = T^*T^n$ and so T is n-normal operator. □

Proposition 2.3. *Let T be a partial isometry. Then T is n-normal if and only if T is n-EP.*

Proof. Since T partial isometry, $T^+ = T^*$. Then the equivalence clearly holds. □

The following proposition generalizes Lemma 2.28 obtained for n-normal operators in [1] to n-EP operators.

Proposition 2.4. *let $T \in B(H)$ be an n-EP operator. If either T or T^* is injectif, then T is EP.*

Proof. Since T is n-EP, Then $T^nTT^+ = T^nT^+T$. That is $T^n(TT^+ - T^+T) = 0$. Since T is injective, $TT^+ - T^+T = 0$. Thus, T is EP. In case T^* is injective, since T^* is n-EP, then T^* is EP. Hence T is EP. □

Proposition 2.5. *Let $T \in B(H)$ with closed range. Then T is 2-EP if and only if T^+ is 2-EP.*

Proof. Suppose that T is 2-EP. It follows from [15, Theorem 3.5] that T^2 is EP and $(T^2)^+ = (T^+)^2$. Since $T^2T = TT^2$, by Lemma 1.2, $(T^2)^+T = T(T^2)^+$ and so $(T^+)^2T = T(T^+)^2$. Hence, T^+ is 2-EP. Conversely, If T^+ is 2-EP, then $(T^+)^+ = T$ is also 2-EP. □

Next, we show that Proposition 2.5 is not valid when the power 2 is replaced by 3.

Example 2.1. *Consider* $T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \in \mathbb{C}^3$. *Then $T^3 = 0$. which implies that T is 3-EP. It is easy to see that*

$$T^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } (T^+)^3 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence

$$(T^+)^3T = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = T(T^+)^3.$$

Thus, T^+ is not 3-EP.

Now, we prove a version of [1, Proposition 2.33] for n- EP operators.

Proposition 2.6. *Let $T \in B(H$ with closed range, $F = T^n + T^+$ and $G = T^n - T^+$. Then T is n- EP operator if and only if G commute with F.*

Proof. We have

$$\begin{aligned} FG = GF &\Leftrightarrow (T^n + T^+)(T^n - T^+) = (T^n - T^+)(T^n + T^+) \\ &\Leftrightarrow T^{2n} - T^nT^+ + T^+T^n - (T^+)^2 = T^{2n} + T^nT^+ - T^+T^n - (T^+)^2 \\ &\Leftrightarrow T^nT^+ = T^+T^n. \end{aligned}$$

Hence, T is n- EP operator if and only $FG = GF$. □

In [15], Wang and Deng showed that if T is an n-EP operator, then T is Drazin invertible with $Ind(T) \leq n$ and $T^D = T^n(T^+)^{n+1}$. In what follows, the ascent and the descent of a Hilbert space operators will be used to characterize n-EP operators.

Theorem 2.2. *Let $T \in B(H)$ with closed range, $B = T^nT^+$, $C = T^+T^n$, $F = T^n + T^+$, and $G = T^n - T^+$. Then the following statements are equivalent.*

(1) *T is n-Ep operator.*
(2) *$a(T) \leq n$ and B commute with F and G.*
(3) *$d(T) \leq n$ and C commute with F and G.*

Proof. (1) $\Rightarrow$ (2). Suppose that T is n-EP operator. Then $T^+T^{n+1} = T^nT^+T = T^n$. which means that $N(T^{n+1}) \subset N(T^n)$. As the converse inclusion is obvious it follows that $N(T^{n+1}) = N(T^n)$. Hence, $a(T) \leq n$. Again, since T is n-EP, we have

$$\begin{aligned} BF &= T^nT^+(T^n + T^+) \\ &= T^nT^+T^n + T^nT^+T^+ \\ &= T^nT^nT^+ + T^+T^nT^+ \\ &= (T^n + T^+)T^nT^+ = FB. \end{aligned}$$

By similar way we can prove that $BG = GB$.
(2) $\Rightarrow$ (1). If $BF = FB$ and $BG = GB$, then we get

$$T^nT^+T^n + T^nT^+T^+ = T^nT^nT^+ + T^+T^nT^+$$

and

$$T^nT^+T^n - T^nT^+T^+ = T^nT^nT^+ - T^+T^nT^+$$

It follows that

$$T^nT^+T^+ = T^+T^nT^+ \text{ and } T^nT^+T^n = T^nT^nT^+.$$

Hence, from the first equality we get $T^nT^+ = T^+T^n$ on $R(T^+) = R(T^*)$ and from the second we deduce that $T^{2n}T^+(x) = 0$, for all $x \in N(T)$. In addition, since $a(T) \leq n$, then $N(T^n) = N(T^{2n})$. Which implies

$$T^{2n}T^+(x) = T^nT^+(x) = 0, \text{ for all } x \in N(T).$$

So, $T^nT^+ = T^+T^n = 0$, on $N(T)$. Consequently,

$$T^nT^+ = T^+T^n \text{ on } H = R(T^*) \oplus N(T).$$

Therefore, T is n-EP.
(1) $\Rightarrow$ (3). The hypothesis T is n-EP implies

$$T^{n+1} = TT^nT^+ = TT^+T^n = T^n.$$

Then $R(T^n) = R(T^{n+1}T^+) \subset R(T^{n+1})$. Sine the inclusion $R(T^{n+1}) \subset R(T^n)$ is obvious, $R(T^{n+1}) = R(T^n)$. Thus $d(T) \leq n$. It is easy to check that the second part of (3)
(3) $\Rightarrow$ (1). First, since $d(T) \leq n$, then $R(T^n) = R(T^{n+1})$. This implies that $N((T^*)^{n+1}) = N((T^*)^n)$ and so $a(T^*) \leq n$. Next, by Applying adjoints of operators to $CF = FC$ and $CG = GC$ and using the relation $(T^*)^+ = (T^+)^*$, we obtain

$$(T^*)^n(T^*)^+((T^*)^n + (T^*)^+) = ((T^*)^n + (T^*)^+)(T^*)^n(T^*)^+$$

and

$$(T^*)^n(T^*)^+((T^*)^n - (T^*)^+) = ((T^*)^n - (T^*)^+)(T^*)^n(T^*)^+.$$

Therefore, by the implication (2) $\Rightarrow$ (1) and Proposition 2.1 (2), we deduce that T is n-EP. □

Remark 2.2. *Notice that the conditions $a(T) \leq n$ and $d(T) \leq n$ in Theorem 2.2 are indispensable, even if $n = 1$. This can be seen from the next example.*

Example 2.2. *Consider the left shift operator S, defined on the Hilbert space $\ell^2(\mathbb{N})$ by*

$$S(x_1, x_2, ...) = (x_2, x_3, ...).$$

Then

$$S^*(x_1, x_2, ...) = (0, x_1, x_2, ...).$$

For $e_{n+1} = (0, ..., 0, 1, 0, ...)$, where 1 is the n+1-th term, $e_{n+1} \in N(S^{n+1})$ while $e_{n+1} \notin N(S^n)$. So $a(S) = \infty$. On the other hand, since $SS^* = I$, then S is a partial isometry. This means that $S^+ = S^*$ and SS^+ commute with $S + S^+$ and $S - S^+$, but S is not EP because $S^+S \neq SS^+$. With similar arguments, we have $d(S^*) = \infty$ and S^* satisfies the second part of (3), while S^* is not EP.

In [13, Theorem 7.3] it was proved that if a is an element of a ring $\mathcal{R}$ with involution, then a is EP if and only if a is group invertible and $a^\# a$ is symmetric. The following proposition generalizes one implication of this result to n-EP Hilbert space operators.

Theorem 2.3. *Let $T \in B(H)$ with closed range such that $ind(T) = n$. If T^DT is self-adjoint then T is n-EP.*

Proof. If $ind(T) = n$ then T is Drazin invertible and $T^DTT = TT^DT$. Since T^DT is self-adjoint, it follows that $T^DTT^* = T^*T^DT$. According to lemma 1.1, we deduce that $T^DTT^+ = T^+T^DT$, which implies the following two equations:

$$T^nT^+ = T^{n+1}T^DT^+ = T^nT^DTT^+ = T^nT^+TT^D = T^nT^D$$

and

$$T^+T^n = T^+T^{n+1}T^D = T^+T^DTT^n = T^DTT^+T^n = T^DT^n.$$

Therefore, $T^nT^+ = T^+T^n$, because $T^nT^D = T^DT^n$. Hence T is n-EP.

□

In Theorem 2.3, the reverse implication does not hold as it is shown by the following example

Example 2.3. *Consider the matrix*

$$T = \begin{pmatrix} -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix wa studied in [14], where it was shown, that

$$T^+ = \begin{pmatrix} 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 & -1 & 2 \end{pmatrix}$$

and T is 3-EP. On the other hand, since $T^D = T^3(T^+)^4$, By a direct calculation, we get

$$T^D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 & -1 & 2 \end{pmatrix}$$

It follows that

$$T^DT = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} = (T^DT)^*.$$

Hence, T is not self-adjoint.

Now, we provide a condition under which the reciprocal implication of Theorem 2.3 holds.

Proposition 2.7. *Let $T \in B(H)$ with closed range. If T is n-EP such that $(T^{n+1})^+ = (T^+)^{n+1}$, then $T^D T$ is self-adjoint.*

Proof. If T is n-EP, then T^D exists and $T^D = (T^+)^{n+1}T^n$. Since $(T^{n+1})^+ = (T^+)^{n+1}$, we get that $T^D T = (T^+)^{n+1}T^{n+1} = (T^{n+1})^+ T^{n+1}$ is self-adjoint. □

As a consequence of Theorem 2.3, we obtain the following corollary.

Corollary 2.1. *Let $T \in B(H)$ with closed range. If T^n is EP, then T is n-EP.*

Proof. If T^n is EP, then $ind(T^n) \leq 1$ which implies $ind(T) \leq n$. So T^D exists and $(T^n)^+ = (T^n)^D$. Hence, we have

$$T^D T = (T^D T)^n = (T^D)^n T^n = (T^n)^D T^n = (T^n)^+ T^n.$$

Since $(T^n)^+ T^n$ is self-adjoint, by Theorem 2.3 we conclude that T is n-EP. □

3. Conditions that the product of operators is an n-EP operator

In [12], Kaplansky showed that if $S, T \in B(H)$ such that S and ST are normal, then TS is normal if and only if T commutes with $|S|$. Inspired by his work, we generalize this famous result to n-EP operators.

Theorem 3.1. *Let $T, S \in B(H)$ such that $R(T)$, $R(S)$ and $R(ST)$ are closed. If S is normal and ST is n-EP, then*

$$S^*ST = TS^*S \Longrightarrow TS \text{ is } n\text{-EP}.$$

Proof. Since S is normal, then by [7, Theorem 3] there exists a unitary operator U such that

$$S = U|S| = |S|U.$$

From the assumption $S^*ST = TS^*S$, we get $|S|T = T|S|$. Consequently, we have

$$TS = TU|S| = |S|TU = U^*U|S|TU = U^*STU.$$

Therefore, TS is unitary equivalent to ST. Since ST is n-EP, according to Proposition 2.1, we conclude that TS is also n-EP. □

Remark 3.1. *In Theorem 3.1, the reverse implication is false. Indeed, Consider the two matrices*

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then S is normal and by a simple computation, we have $(ST)^2 = (TS)^2 = 0$. Hence ST and TS are 2-EP, but

$$S^*ST = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = TS^*S.$$

We now prove the following Lemma which is needed in the proof of Theorem 3.2.

Lemma 3.1. *Let $S \in B(H)$ with closed ranges and $T \in B(H)$. If $(ST)^n$ is EP operator, then $(ST)^n S$ has closed range.*

Proof. By lemma 1.3, it is enough to prove that $R[((ST)^n)^+(ST)^n SS^+]$ is closed . Since $(ST)^n$ is EP, then we have

$$\begin{aligned} R[((ST)^n)^+(ST)^n SS^+] &= R[(SS^+((ST)^n)^+(ST)^n)^*] \\ &= R[(SS^+(ST)^n((ST)^n)^+)^*] \\ &= R[(ST)^n((ST)^n)^+)^*] \\ &= R[(ST)^n((ST)^n)^+] \end{aligned}$$

Hence, $R[((ST)^n)^+(ST)^n SS^+]$ is closed . □

The next result reformulates and extends Theorem 4.6 of [11] to n-EP operators.

Theorem 3.2. *Let $T, S \in B(H)$ and $n \in \mathbb{N}$ such that T, S, ST and TS have closed ranges. Then the following statements hold*

(i) *If $(ST)^n$ and $(TS)^n$ are EP operators, then we have $S^+(ST)^n = (TS)^n S^+$ and $(ST)^n T^+ = T^+(TS)^n$.*
(ii) *If $(ST)^+ = T^+S^+$, $S^+(ST)^n = (TS)^n S^+$ and $(ST)^n T^+ = T^+(TS)^n$, then ST is n-EP.*
(iii) *If $(TS)^+ = S^+T^+$, $S^+(ST)^n = (TS)^n S^+$ and $(ST)^n T^+ = T^+(TS)^n$, then TS is n-EP.*

Proof. (i) By Lemma 3.1, $R((ST)^n S)$ and $R((TS)^n T)$ are closed. Now, we poof that the reverse order law $S^+[(ST)^n]^+ = [(ST)^n S]^+$ holds. According to Lemma 1.4, this equality is equivalent to

$$R([(ST)^n]^*(ST)^n S) \subset R(S) \text{ and } R(SS^*[(ST)^n]^*) \subset R([(ST)^n]^*).$$

Since $(ST)^n$ and $(TS)^n$ are EP, the first inclusion follows as

$$\begin{aligned} R([(ST)^n]^*(ST)^n S) &\subset R([(ST)^n]^*) \\ &= R((ST)^n) \\ &\subset R(ST) \\ &\subset R(S), \end{aligned}$$

and the second inclusion follows as

$$\begin{aligned} R(SS^*[(ST)^n]^*) &= SR(S^*[(ST)^n]^*) \\ &= SR([(ST)^n S]^*) \\ &= SR([S(TS)^n]^*) \\ &= SR([(TS)^n]^* S^*) \\ &\subset SR([(TS)^n]^*) \\ &= SR((TS)^n) \\ &= R(S(TS)^n) \\ &= R((ST)^n S) \\ &\subset R((ST)^n) \\ &= R([(ST)^n]^*). \end{aligned}$$

Thus, $S^+[(ST)^n]^+ = [(ST)^nS]^+$. In a similar way we have

$$[(TS)^n]^+S^+ = [S(TS)^n]^+ ,\ [(ST)^n]^+T^+ = [T(ST)^n]^+ \text{ and } [(TS)^nT]^+ = T^+[(TS)^n]^+,$$

and consequently, we get that

$$S^+[(ST)^n]^+ = [(TS)^n]^+S^+ \text{ and } [(ST)^n]^+T^+ = T^+[(TS)^n]^+.$$

Since $(ST)^n$ and $(TS)^n$ are EP, By Lemma1.2, we obtain

$$S^+(ST)^n = (TS)^nS^+ \text{ and } (ST)^nT^+ = T^+(TS)^n.$$

(ii) if $S^+(ST)^n = (TS)^nS^+$ and $(ST)^nT^+ = T^+(TS)^n$, then multiplying the first equation from the left by T^+ and the second one by S^+ from the right, we get

$$T^+S^+(ST)^n = T^+(TS)^nS^+ \text{ and } (ST)^nT^+S^+ = T^+(TS)^nS^+.$$

Using the fact $(ST)^+ = T^+S^+$, we have $(ST)^n(ST)^+ = (ST)^+(ST)^n$. Therefore ST is n-EP.
(iii) It can be proved in a similar way to (ii). □

Also, the last result generalizes [11, Corollary 4.6] for EP matrices to n-EP operators on arbitrary Hilbert spaces.

Proposition 3.1. *Let $S \in B(H)$ wih closed range and $S = U|S|$ be its polar decomposition, where U is unitary and let $T \in B(H)$ such that T, ST, and TS have closed ranges. If $(ST)^+ = T^+S^+$ and $(TS)^+ = S^+T^+$, then the following statements hold*

(i) *Let $n \in \mathbb{N}$, if TU is EP and $(ST)^nU = U(TS)^n$, then ST and TS are n-EP.*
(ii) *Let $n \in \mathbb{N}$, if $(ST)^n$ and $(TS)^n$ are EP, then $(ST)^nU = U(TS)^n$.*

Proof. (i) First, by the hypothesis $(ST)^nU = U(TS)^n$ and since U is unitary, we observe

$$\begin{aligned}(ST)^nU = U(TS)^n &\iff ST(ST)^{n-1}U = UT(ST)^{n-1}S\\ &\iff U|S|T(ST)^{n-1}U = UT(ST)^{n-1}U|S|\\ &\iff U^*U|S|T(ST)^{n-1}U = U^*UT(ST)^{n-1}U|S|\\ &\iff |S|T(ST)^{n-1}U = T(ST)^{n-1}U|S| \end{aligned} \tag{3.1}$$

Since $|S|$ is EP, by using (3.1) and Lemma 1.2, it follows

$$(ST)^nU = U(TS)^n \iff |S|^+T(ST)^{n-1}U = T(ST)^{n-1}U|S|^+. \tag{3.2}$$

Obviously, by (3.1) and (3.2), we obtain

$$(ST)^nU = U(TS)^n \iff |S||S|^+T(ST)^{n-1}U = T(ST)^{n-1}U|S||S|^+. \tag{3.3}$$

Now, the equivalence (3.3) gives

$$\begin{aligned}(TS)^n S^+ &= T(ST)^{n-1} SS^+ \\ &= T(ST)^{n-1} U|S||S|^+ U^* \\ &= |S||S|^+ T(ST)^{n-1} UU^* \\ &= |S||S|^+ T(ST)^{n-1} \\ &= |S|^+ |S| T(ST)^{n-1} \\ &= |S|^+ U^* U|S| T(ST)^{n-1} \\ &= S^+ (ST)^n.\end{aligned}$$

On the other hand by (2.1), we have

$$\begin{aligned}|S|(TS)^{n-1} TU &= |S| T(ST)^{n-1} U \\ &= T(ST)^{n-1} U|S| \\ &= (TS)^n \\ &= TU|S|(TS)^{n-1}.\end{aligned}$$

Applying Lemma 1.2, the fact that TU is EP, we obtain

$$(TU)^+ |S|(TS)^{n-1} = |S|(TS)^{n-1} (TU)^+.$$

Since $(TU)^+ = U^* T^+$, by using the previous equality, we get

$$\begin{aligned}(ST)^n T^+ &= S(TS)^{n-1} TT^+ \\ &= U|S|(TS)^{n-1} TUU^* T^+ \\ &= U|S|(TS)^{n-1} TU(TU)^+ \\ &= U|S|(TS)^{n-1} (TU)^+ TU \quad (TU \text{ is } EP) \\ &= U(TU)^+ |S|(TS)^{n-1} TU \\ &= UU^* T^+ |S|(TS)^{n-1} TU \\ &= T^+ TU|S|(TS)^{n-1} \\ &= T^+ (TS)^n.\end{aligned}$$

Finally, from the given facts $(ST)^+ = T^+ S^+$ and $(TS)^+ = S^+ T^+$, it follows by Theorem 3.2, ST and TS are n-EP.

(ii) By (3.3) it is sufficient to prove that

$$|S||S|^+ T(ST)^{n-1} U = T(ST)^{n-1} U|S||S|^+.$$

Since $(ST)^n$ and $(TS)^n$ are EP, according to Theorem 3.2, we have

$$\begin{aligned}|S||S|^+T(ST)^{n-1}U &= |S|^+|S|T(ST)^{n-1}U \\ &= |S|^+U^*U|S|T(ST)^{n-1}U \\ &= S^+ST(ST)^nU \\ &= S^+(ST)^nU \\ &= (TS)^nS^+U \\ &= T(ST)^{n-1}SS^+U \\ &= T(ST)^{n-1}U|S||S|^+.\end{aligned}$$

□

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References

[1] S.A. Alzuraiqi, A.B. Patel, On n-Normal Operators, Gen. Math. Notes, 1 (2010), 61–73.

[2] O. Maria Baksalary, G. Trenkler, Characterizations of EP, Normal, and Hermitian Matrices, Linear Multilinear Algebra. 56 (2008), 299–304. https://doi.org/10.1080/03081080600872616.

[3] E. Boasso, V. Rakoc̆evic̀, Characterizations of EP and Normal Banach Algebra Elements and Banach Space Operators, Linear Algebra Appl. 435 (2011), 342–353. https://doi.org/10.1016/j.laa.2011.01.031.

[4] S.L. Campbell, C.D. Meyer, EP Operators and Generalized Inverses, Canad. Math. Bull. 18 (1975), 327–333. https://doi.org/10.4153/cmb-1975-061-4.

[5] D.S. Djorrdjevič, Products of EP operators on Hilbert spaces, Proc. Amer. Math. Soc. 129 (2000), 1727–1731. https://doi.org/10.1090/s0002-9939-00-05701-4.

[6] D.S. Djordjevic, Further Results on the Reverse Order Law for Generalized Inverses, SIAM J. Matrix Anal. Appl. 29 (2008), 1242–1246. https://doi.org/10.1137/050638114.

[7] T. Furuta, On the Polar Decomposition of an Operator, Acta Sci. Math. 46 (1983), 261–268.

[8] R.E. Harte, M. Mbekhta, On Generalized Inverses in C*-Algebras, Stud. Math. 103 (1992) 71–77.

[9] H.G. Heuser, Functional Analysis, John Wiley & Sons, 1982.

[10] S. Izumino, The Product of Operators With Closed Range and an Extension of the Reverse Order Law, Tohoku Math. J. (2) 34 (1982), 43–52. https://doi.org/10.2748/tmj/1178229307.

[11] P.S. Johnson, V. A., K. Kamaraj, Fuglede-Putnam type commutativity theorems for *EP* operators, Malaya J. Mat. 9 (2021) 709–714. https://doi.org/10.26637/mjm0901/0124.

[12] I. Kaplansky, Products of Normal Operators, Duke Math. J. 20 (1953), 257–260. https://doi.org/10.1215/s0012-7094-53-02025-0.

[13] J.J. Koliha, P. Patricio, Elements of Rings With Equal Spectral Idempotents, J. Aust. Math. Soc. 72 (2002), 137–152. https://doi.org/10.1017/s1446788700003657.

[14] S.B. Malik, L. Rueda, N. Thome, The Class of m-EP and m-Normal Matrices, Linear Multilinear Algebra. 64 (2016), 2119–2132. https://doi.org/10.1080/03081087.2016.1139037.

[15] X. Wang, C. Deng, Properties of m-EP Operators, Linear Multilinear Algebra. 65 (2016), 1349–1361. https://doi.org/10.1080/03081087.2016.1235131.