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A Study on Quotient Structures of Bipolar Fuzzy Finite State Machines

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Abstract. This article introduces different congruence relations on the bipolar fuzzy set associated with the bipolar fuzzy finite state machine. Each congruence relation associates a semigroup with the bipolar fuzzy finite automata. We also discuss characterizing a bipolar fuzzy finite state machine by defining an admissible relation.

1. INTRODUCTION

Zadeh [12] introduced the concept of fuzzy sets (FSs) first in 1965. An FS A of a universe ξ is a function $A : \xi \rightarrow [0,1]$. There are numerous extensions of fuzzy sets, such as intuitionistic fuzzy sets (IFSs), interval-valued fuzzy sets (IVFSs), vague sets (VSs), etc. Lee [7] introduces bipolar-valued fuzzy sets (BFSs), which are an extension of FS whose membership degree (MSD) range is enlarged from the interval $[0,1]$ to $[-1,1]$. BFSs have membership degrees (MSDs) that represent the degree of satisfaction with the property corresponding to an FS and its counter property. This has given me an intense interest in research in special sectors such as algebraic structures, graph theory, medical science, decision-making, machine learning, automata theory, pattern recognition, etc.

Malik et al. [8,9] developed the concept and approach of fuzzy finite state machines (FFSMs), fuzzy finite state submachines (FFSbMs) and their decomposition. In 2002, Kumbhojkar and Chaudhari [6] proposed and studied the product of FFSMs and other related aspects. Jun [2]

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introduced the notion of intuitionistic fuzzy finite state machines (IFFSMs) in 2005, Jun [3] suggested a commutative intuitionistic fuzzy finite switchboard state machine called the intuitionistic fuzzy finite switchboard state machine (InFFSSM) in 2006. In 2007, Jun [4] discussed the quotient structures of intuitionistic fuzzy finite state machines. Jun and Kavikumar [5] popularized the notion of bipolar fuzzy finite state machines (BiFFSMs) in 2011, proposed the algebraic features of BiFFSMs and evaluated particular results. Prabhu et al. [10] studied finite state machines via bipolar neutrosophic set theory. Reena [11] investigated bipolar vague finite switchboard state machines (BVFSSMs) in 2019. Iampan et al. [1] introduced the notion of bipolar neutrosophic finite switchboard state machines (BNFSSTMs), homomorphisms and strong homomorphisms of bipolar neutrosophic finite state machines (BNFSTMs).

2. Preliminaries

This segment reviews the definitions that are fundamental to this paper.

Throughout this article, we represent the notations, SG for a semigroup, $\mathrm{C_R}$ for a congruence relation, $\mathrm{E_R}$ for an equivalence relation, $\mathrm{B_{fadR}}$ for a bipolar fuzzy admissible relation, $\mathrm{H_M}$ for a homomorphism.

Definition 2.1. *[12] Let ξ be a non-empty set. A mapping $\mu_A : \xi \to [0,1]$ is called an FS over ξ.*

Definition 2.2. *[7] Let ξ be a universal set and H be a set over ξ that is defined by a positive membership function (+veMSF) and a negative membership function (-veMSF), where $\mu_H^+ : \xi \to [0,1]$ and $\mu_H^- : \xi \to [-1,0]$. Then H is called a BFS over ξ, and can be written in the form*

$$H = \{< x : \mu_H^+(j), \mu_H^-(j) >\mid j \in \xi\}.$$

Definition 2.3. *[3] A BFFSM is a triple $Z = (U, \xi, \iota)$, where U: the set of states and ξ: the set of input symbols are finite non-empty sets, and $\iota = (\iota_*^+, \iota_*^-)$ is a BFS in $U \times \xi \times U$. The set of all words of elements of ξ of finite length is symbolized by ξ^*, and the empty word in ξ^* is symbolized by λ, and the length of u for every $u \in \xi^*$ is symbolized by $\mid u \mid$.*

Definition 2.4. *[3] Let $Z = (U, \xi, \iota)$ be a BFFSM. Define a BFS $\iota_* = (\iota_*^+, \iota_*^-)$ in $U \times \xi^* \times U$ by for all $\tau, \varsigma \in U, u \in \xi^*$, and $a \in \xi$,*

$$\iota_*^+(\varsigma, \lambda, \tau) = \begin{cases} 1 & if \quad \varsigma = \tau, \\ 0 & if \quad \varsigma \neq \tau, \end{cases}$$

$$\iota_*^-(\varsigma, \lambda, \tau) = \begin{cases} -1 & if \quad \varsigma = \tau, \\ 0 & if \quad \varsigma \neq \tau, \end{cases}$$

$$\iota_*^+(\varsigma, \xi a, \tau) = \sup_{r \in U}[\iota_*^+(\varsigma, \xi, r) \wedge \iota_*^+(r, a, \tau)],$$

$$\iota_*^-(\varsigma, \xi a, \tau) = \inf_{r \in U}[\iota_*^-(\varsigma, \xi, r) \wedge \iota_*^-(r, a, \tau)].$$

Remark 2.1. *[3] Let $Z = (U, \xi, \iota)$ be a BFFSM. Then for all $\tau, \varsigma \in U$ and $p, k \in \xi^*$,*

$$\iota_*{}^{+}(\varsigma, pk, \tau) = \sup_{r \in U}[\iota_*{}^{+}(\varsigma, p, r) \wedge \iota_*{}^{+}(r, k, \tau)],$$
$$\iota_*{}^{-}(\varsigma, pk, \tau) = \inf_{r \in U}[\iota_*{}^{-}(\varsigma, p, r) \vee \iota_*{}^{-}(r, k, \tau)].$$

Notation 2.1. *Consider ξ^* is an SG with identity λ with respect to a binary operation concatenation of two words. Let $p, k \in \xi^*$. Determine a relation $\sim$ on ξ^* by $p \sim k$ if and only if $\iota_*{}^{+}(\varsigma, p, \tau) = \iota_*{}^{+}(\varsigma, k, \tau)$ and $\iota_*{}^{-}(\varsigma, p, \tau) = \iota_*{}^{-}(\varsigma, k, \tau)$ for all $\tau, \varsigma \in U$. Then $\sim$ is a C_R on ξ^*. For any $p \in \xi^*$, we denote $[p] = \{k \in \xi^* \mid p \sim k\}$ and $S(Z) = \{[p] \mid p \in \xi^*\}$.*

Theorem 2.1. *[3] Let $Z = (U, \xi, \iota)$ be a BFFSM. Define a binary operation $\odot$ on $S(Z)$ by $[\xi] \odot [\zeta] = [\xi\zeta]$ for all $[\xi], [\zeta] \in S(Z)$. Then $(S(Z), \odot)$ is a finite SG with identity.*

3. Quotient structures of BFFSMs

Let $Z = (U, \xi, \iota)$ be a BFFSM. We now define another C_R on ξ^*. Let $g, \zeta \in \xi^*$. Define $g \equiv \zeta$ if and only if ($\iota_*{}^{+}(\varsigma, g, \tau) > 0 \Leftrightarrow \iota_*{}^{+}(\varsigma, \zeta, \tau) > 0$ and $\iota_*{}^{+}(\varsigma, g, \tau) < 0 \Leftrightarrow \iota_*{}^{+}(\varsigma, \zeta, \tau) < 0$). It is clear that $\equiv$ is an E_R on ξ^*. Let $z \in \xi^*$ and $g \equiv \zeta$ be assumed. For each $\tau, \varsigma \in U$, we have

$$\begin{aligned}
\iota_*{}^{+}(\varsigma, zg, \tau) > 0 &\Leftrightarrow \sup_{r \in U}[\iota_*{}^{+}(\varsigma, z, r) \wedge \iota_*{}^{+}(r, g, \tau)] > 0 \\
&\Leftrightarrow \exists r \in U, [\iota_*{}^{+}(\varsigma, z, r) \wedge \iota_*{}^{+}(r, g, \tau)] > 0 \\
&\Leftrightarrow \exists r \in U, [\iota_*{}^{+}(\varsigma, z, r) \wedge \iota_*{}^{+}(r, \zeta, \tau)] > 0 \\
&\Leftrightarrow \sup_{r \in U}[\iota_*{}^{+}(\varsigma, z, r) \wedge \iota_*{}^{+}(r, \zeta, \tau)] > 0 \\
&\Leftrightarrow \iota_*{}^{+}(\varsigma, z\zeta\tau) > 0,
\end{aligned}$$

$$\begin{aligned}
\iota_*{}^{-}(\varsigma, zg, \tau) < 0 &\Leftrightarrow \inf_{r \in Q}[\iota_*{}^{-}(\varsigma, z, r) \vee \iota_*{}^{-}(r, g, \tau)] < 0 \\
&\Leftrightarrow \exists r \in U, [\iota_*{}^{-}(\varsigma, z, r) \vee \iota_*{}^{-}(r, g, \tau)] < 0 \\
&\Leftrightarrow \exists r \in U, [\iota_*{}^{-}(\varsigma, z, r) \vee \iota_*{}^{-}(r, \zeta, \tau)] < 0 \\
&\Leftrightarrow \inf_{r \in U}[\iota_*{}^{-}(\varsigma, z, r) \vee \iota_*{}^{-}(r, \zeta, \tau)] < 0 \\
&\Leftrightarrow \iota_*{}^{-}(\varsigma, z\zeta\tau) < 0.
\end{aligned}$$

Hence, $zg \equiv z\zeta$. Similarly, $gz \equiv \zeta z$. Therefore, $\equiv$ is a C_R on ξ^*. For any $g \in \xi^*$, we represent $\tilde{g} = \{\zeta \in \xi^* \mid g \equiv y\}$ and $\tilde{S}(Z) = \{\tilde{g} \mid g \in \xi^*\}$. Determine $\tilde{\odot}$ a binary operation on $\tilde{S}(Z)$ by $\tilde{g}\tilde{\odot}\tilde{\zeta} = \widetilde{g\zeta}$ for all $\tilde{g}, \tilde{\zeta} \in \tilde{S}(Z)$. Obviously, $\tilde{\odot}$ is well-defined and associative. For each $\tilde{g} \in \tilde{S}(Z)$, $\tilde{g}\tilde{\odot}\tilde{\lambda} = \widetilde{g\lambda} = \tilde{g} = \widetilde{\lambda g} = \tilde{\lambda}\tilde{\odot}\tilde{g}$. Thus, $\tilde{\lambda}$ is the identity of $(\tilde{S}(M), \tilde{\odot})$. Assume now $g \in \xi^*$ such that $g = g_1 g_2 \ldots g_n$, where $g_1, g_2, \ldots, g_n \in \xi$. Now, for all $\tau, \varsigma \in U$, we get

$$\iota_*{}^{+}(\varsigma, g, \tau) = \sup_{r_1, r_2, \ldots, r_{n-1} \in U}[\iota^{+}(\varsigma, g_1, r_1) \wedge \iota^{+}(r_1, g_2, r_2) \wedge \ldots \wedge \iota^{+}(r_{n-1}, g_n, \tau)],$$
$$\iota_*{}^{-}(\varsigma, g, \tau) = \inf_{r_1, r_2, \ldots, r_{n-1} \in U}[\iota^{-}(\varsigma, g_1, r_1) \vee \iota^{-}(r_1, g_2, r_2) \vee \ldots \vee \iota^{-}(r_{n-1}, g_n, \tau)].$$

As the image of ι is finite, the image of ι^* is also finite. Let $\Theta : S(Z) \to \tilde{S}(Z)$ be determined by $\Theta([g]) = \tilde{g}$ for all $[g] \in S(Z)$. Let $g, \zeta \in \xi^*$ be such that $[g] = [\zeta]$. Then $\iota_*{}^+(\varsigma, g, \tau) = \iota_*{}^+(\varsigma, \zeta, \tau)$ and $\iota_*{}^-(\varsigma, g, \tau) = \iota_*{}^-(\varsigma, \zeta, \tau)$ for all $\tau, \varsigma \in U$. Thus, for all $\tau, \varsigma \in U$,

$$\iota_*{}^+(\varsigma, g, \tau) > 0 \Leftrightarrow \iota_*{}^+(\varsigma, \zeta, \tau) > 0,$$

$$\iota_*{}^-(\varsigma, g, \tau) < 0 \Leftrightarrow \iota_*{}^-(\varsigma, \zeta, \tau) < 0.$$

Thus, $g \equiv \zeta$. Therefore, $\tilde{g} = \tilde{\zeta}$. Thus, Θ is well-defined and also it is onto. For each $[g], [\zeta] \in S(M)$, we have

$$\Theta([g] \odot \Theta[\zeta]) = \Theta([g\zeta]) = \widetilde{g\zeta} = \tilde{g}\tilde{\odot}\tilde{\zeta} = \Theta([g])\tilde{\odot}\Theta([\zeta]).$$

Since $S(Z)$ is finite, we have $\tilde{S}(Z)$ is finite.

Theorem 3.1. *Let $M = (\varsigma, \xi, \iota)$ be a BFFSM. Determine a binary operation $\tilde{\odot}$ on $\tilde{S}(M)$ by $\tilde{\xi}\tilde{\odot}\tilde{\zeta} = \widetilde{\xi\zeta}$ for all $\tilde{\xi}, \tilde{\zeta} \in \tilde{S}(M)$. Then $(\tilde{S}(M), \tilde{\odot})$ is a finite SG with identity and $[\xi] \mapsto \tilde{\xi}$ is a H_M of $S(M)$ onto $\tilde{S}(M)$.*

Proof. It is straightforward. □

Notation 3.1. *Let $M = (\varsigma, \xi, \iota)$ be a BFFSM. For all $u \in \xi^*$, define a BFS $u^M = (\iota_{u^M}{}^+, \iota^-_{u^M})$ in $\varsigma \times \varsigma$ by for all $\tau, \theta \in \varsigma$,*

$$\iota_{u^M}{}^+(\theta, \tau) = \iota_*{}^+(\theta, \xi, \tau),$$

$$\iota_{u^M}{}^-(\theta, \tau) = \iota_*{}^-(\theta, \xi, \tau).$$

Let L, R, T be non-empty sets. Let $C = (\iota^+_C, \iota^-_C)$ and $D = (\iota^+_D, \iota^-_D)$ be BFSSs in $L \times R$ and $R \times T$, respectively. Define the BFS $C \circ D = (\iota^+_{C \circ D}, \iota^-_{C \circ D})$ in $L \times T$ by for all $\tau \in L$ and $t \in T$,

$$\iota_{C \circ D}{}^+(\tau, t) = \sup_{r \in R}[\iota_C{}^+(\tau, r) \wedge \iota_D{}^+(r, t)],$$

$$\iota_{C \circ D}{}^-(\tau, t) = \inf_{r \in R}[\iota_C{}^-(\tau, r) \vee \iota_D{}^-(r, t)].$$

Theorem 3.2. *Assume $M = (\varsigma, \xi, \iota)$ as a BFFSM and $S_M = \{\Theta^M \mid \Theta \in \xi^*\}$. Then*

- *(i) for all $\Theta, \zeta \in \xi^*, \Theta^M \circ \zeta^M = (\Theta\zeta)^M$,*
- *(ii) $(S_M, \circ)$ is a finite SG with identity, isomorphic to $(S(M), \odot)$.*

Proof. (i) Assume $\tau, \mu \in \varsigma$. Then

$$\begin{aligned}
\iota_{\Theta\zeta^M}{}^+(\mu, \tau) &= \iota^+_*(\mu, \Theta\zeta, \tau) \\
&= \sup_{r \in \varsigma}[\iota^+_*(\mu, \Theta, r) \wedge \iota^+_*(r, \zeta, \tau)] \\
&= \sup_{r \in \varsigma}[\iota^+_{\Theta^M}(\mu, r) \wedge \iota^+_{\zeta^M}(r, \tau)] \\
&= \iota^+_{\Theta^M \circ \zeta^M}(\mu, \tau),
\end{aligned}$$

$$\begin{aligned}\iota_{\Theta\zeta^M}{}^{-}(\mu,\tau) &= \iota_*^-(\mu,\Theta\zeta,\tau)\\ &= \inf_{r\in\varsigma}[\iota_*^-(\mu,\Theta,r)\vee\iota_*^-(r,\zeta,\tau)]\\ &= \inf_{r\in\varsigma}[\iota_{\Theta^M}^-(\mu,r)\vee\iota_{\zeta^M}^-(r,\tau)]\\ &= \iota_{\Theta^M\circ\zeta^M}^-(\mu,\tau).\end{aligned}$$

Hence, for all $\Theta,\zeta\in\xi^*,\Theta^M\circ\zeta^M=(\Theta\zeta)^M$.

(ii) Obviously, $(S_M,\circ)$ is an SG with identity λ^M. Since ς and the image of ι are finite, we have S_M is finite. Now define a function $f:S_M\to S(M)$ by $f(\Theta^M)=[\Theta]$ for all $\Theta^M\in S_M$. Let $\Theta^M,\zeta^M\in S_M$. Then

$$\begin{aligned}\Theta^M=\zeta^M &\Leftrightarrow \{[\iota_{\Theta^M}^+(\mu,\tau)]=[\iota_{\zeta^M}^+(\mu,\tau)] \text{ and } [\iota_{\Theta^M}^-(\mu,\tau)]=[\iota_{\zeta^M}^-(\mu,\tau)]\}\\ &\Leftrightarrow \{\iota_*^+(\mu,\Theta,\tau)=\iota_*^+(\mu,\zeta,\tau) \text{ and } \iota_*^-(\mu,\Theta,\tau)=\iota_*^-(\mu,\zeta,\tau)\}\\ &\Leftrightarrow [\Theta]=[\zeta].\end{aligned}$$

Thus, f is well-defined and 1-1, and it is also onto. Now,

$$f(\Theta^M\circ\zeta^M)=f((\Theta\zeta)^M)=[\Theta\zeta]=[\Theta]\odot[\zeta]=f(\Theta^M)\odot f(\zeta^M).$$

Hence, f is a $\mathrm{H_M}$. □

Definition 3.1. *Let $M=(\varsigma,\xi,\iota)$ be a BFFSM. The index of an E_R is the number of distinct equivalence classes.*

Let $\approx$ be a $\mathrm{C_R}$ of finite index on X^*. For any $\Theta\in\xi^*$, we denote

$$<\Theta>=\{\zeta\in\xi^*\mid\Theta\approx\zeta\},$$

$$\tilde{\varsigma}=\{<\Theta>\mid\Theta\in\xi^*\}.$$

Define a BFS $\sigma=(\iota_\sigma^+,\iota_\sigma^-)$ in $\tilde{\varsigma}X\xi X\tilde{\varsigma}$ by

$$\iota_\sigma^+(<\Theta>,a,<w>)=\begin{cases}\iota_\sigma^+(<\Theta>,a,<\Theta a>) & \text{if} \quad w\approx\Theta a,\\ 0 & \text{if} \quad \text{otherwise,}\end{cases}$$

$$\iota_\sigma^-(<\Theta>,a,<w>)=\begin{cases}\iota_\sigma^-(<\Theta>,a,<\Theta a>) & \text{if} \quad w\approx\Theta a,\\ 0 & \text{if} \quad \text{otherwise,}\end{cases}$$

for all $<\Theta>,<w>\in\tilde{\varsigma}$ and $a\in\xi$, where $\iota_\sigma^+(<\Theta>,a,<\Theta a>)$ and $\iota_\sigma^-(<\Theta>,a,<\Theta a>)$ are arbitrary elements in $(0,1]$ and $[-1,0)$, respectively. Let $<\Theta>,<\zeta>,<\nu>,<\eta>\in\tilde{\varsigma}$ and $a,b\in\xi$ be such that $(<\Theta>,a,<\nu>)=(<\zeta>,b,<\eta>)$. Then $<\Theta>=<\zeta>,a=b$, and $<\nu>=<\eta>$. Now, $\nu\approx\Theta a$ if and only if $\eta\approx yb$. Thus, $\iota_\sigma^+(<\Theta>,a,<\nu>)=\iota_\sigma^+(<\zeta>,b,<\eta>)$ and $\iota_\sigma^-(<\Theta>,a,<\nu>)=\iota_\sigma^-(<\zeta>,b,<\eta>)$. Hence, ι_σ^+ and ι_σ^- are single valued. Therefore, $M=(\varsigma,\xi,\sigma)$ is a BFFSM.

Now consider an extension $\sigma^*=(\iota_{\sigma_*}^+,\iota_{\sigma_*}^-)$ of $\sigma=(\iota_\sigma^+,\iota_\sigma^-)$ as $\iota=(\iota^+,\iota^-)$ was extended to $\iota_*=(\iota_*^+,\iota_*^-)$ in Definition 2.4.

Lemma 3.1. *Let $\tilde{M} = (\tilde{\varsigma}, \xi, \sigma)$ be stated as above. Then the following points hold:*

(i) for all $a, \Theta \in \xi^*$, ${\iota_{\sigma^*}}^+(<\Theta>, a, <\Theta a>) > 0$ and ${\iota_{\sigma^*}}^-(<\Theta>, a, <\Theta a>) < 0$,

(ii) for all $w, \Theta, z \in \xi^*$, ${\iota_{\sigma^*}}^+(<z>, \Theta, <w>) > 0$ and ${\iota_{\sigma^*}}^-(<z>, \Theta, <w>) < 0 \Rightarrow <z\Theta> = <w>$.

Proof. (i) Let $a, \Theta \in \xi^*$ and $|a| = n$. If $n = 0$, then $a = \lambda$. Hence, ${\iota_{\sigma^*}}^+(<\Theta>, a, <\Theta a>) = {\iota_{\sigma^*}}^+(<\Theta>, \lambda, <\Theta>) = 1 > 0$. Also, ${\iota_{\sigma^*}}^-(<\Theta>, a, <\Theta a>) = {\iota_{\sigma^*}}^-(<\Theta>, \lambda, <\Theta>) = -1 < 0$. Let the result hold for all $\zeta \in \xi^*$ such that $|\zeta| = n-1, n > 0$. Let $a = yb$, where $b \in \xi$. Then

$$\begin{aligned} {\iota_{\sigma^*}}^+(<\Theta>, a, <\Theta a>) &= {\iota_{\sigma^*}}^+(<\Theta>, yb, <\Theta\zeta b>) \\ &= \sup_{<q>\in\tilde{\varsigma}} [{\iota_{\sigma^*}}^+(<\Theta>, \zeta, <q>) \wedge {\iota_\sigma}^+(<q>, b, <\Theta\zeta b>)] \\ &\geq {\iota_{\sigma^*}}^+(<\Theta>, \zeta, <\Theta\zeta>) \wedge {\iota_\sigma}^+(<\Theta\zeta>, b, <\Theta\zeta b>) \\ &> 0, \end{aligned}$$

$$\begin{aligned} {\iota_{\sigma^*}}^-(<\Theta>, a, <\Theta a>) &= {\iota_{\sigma^*}}^-(<\Theta>, yb, <\Theta\zeta b>) \\ &= \inf_{<q>\in\tilde{\varsigma}} [{\iota_{\sigma^*}}^-(<\Theta>, \zeta, <q>) \vee {\iota_\sigma}^-(<q>, b, <\Theta\zeta>)] \\ &\leq {\iota_{\sigma^*}}^+(<\Theta>, \zeta, <\Theta\zeta>) \vee {\iota_\sigma}^+(<\Theta\zeta>, b, <\Theta\zeta b>) \\ &< 0. \end{aligned}$$

(ii) Consider $w, \Theta, z \in \xi^*$, $|\Theta| = n$ as such that ${\iota_{\sigma^*}}^+(<z>, \Theta, <w>) > 0$ and ${\iota_{\sigma^*}}^-((<z>, \Theta, <w>)) < 0$. If $n = 0$, then $\Theta = \lambda$. Thus, ${\iota_{\sigma^*}}^+(<z>, \lambda, <w>) = {\iota_{\sigma^*}}^+(<z>, \Theta, <w>) > 0$ and ${\iota_{\sigma^*}}^-(<z>, \lambda, <w>) = {\iota_{\sigma^*}}^-(<z>, \Theta, <w>) < 0$. Thus, $<z> = <w>$ and so $<z\Theta> = <w>$ for all $\zeta \in \xi^*$ such that $|\zeta| = n-1, n > 0$. Let $\Theta = ya$, where $a \in \xi$. Let ${\iota_{\sigma^*}}^+(<z>, \zeta a, <w>) = {\iota_{\sigma^*}}^+(<z>, \Theta, <w>) > 0$ and ${\iota_{\sigma^*}}^-(<z>, \zeta a, <w>) = {\iota_{\sigma^*}}^-(<z>, \Theta, <w>) < 0$. Then

$$0 < {\iota_{\sigma^*}}^+(<z>, \zeta a, <w>) = \sup_{<q>\in\tilde{\varsigma}} [{\iota_{\sigma^*}}^+(<z>, \zeta, <q>) \wedge {\iota_\sigma}^+(<q>, a, <w>)],$$

$$0 > {\iota_{\sigma^*}}^-(<z>, \zeta a, <w>) = \inf_{<q>\in\tilde{\varsigma}} [{\iota_{\sigma^*}}^-(<z>, \zeta, <q>) \vee {\iota_\sigma}^-(<q>, a, <w>)],$$

which now imply that ${\iota_{\sigma^*}}^+(<z>, \zeta, <q>) > 0, {\iota_{\sigma^*}}^+(<q>, a, <w>) > 0, {\iota_{\sigma^*}}^-(<z>, \zeta, <q>) < 0$, and ${\iota_{\sigma^*}}^-(<q>, a, <w>) < 0$ for some $q \in \tilde{\varsigma}$. By induction hypothesis, we have $<z\zeta> = <q>$ and $<qa> = <w>$. Hence, $<z\Theta> = <z\zeta a> = <qa> = <w>$. □

Theorem 3.3. *A BFFSM $M = (\varsigma, \xi, \iota)$ determined by given $C_R \approx$ on ξ^* of finite index like in a way that $\approx$ is the same $C_R \equiv$ like on $M = (\varsigma, \xi, \iota)$.*

Proof. Let $\delta, \Theta \in \xi^*$ be such that $\delta \approx \Theta$. Let $<z>, <w> \in \tilde{\varsigma}$ be such that ${\iota_{\sigma^*}}^+(<z>, \delta, <w>) > 0$ and ${\iota_{\sigma^*}}^-(<z>, \delta, <w>) < 0$. As now $\delta \approx \Theta$ and $\approx$ is a $\mathrm{C_R}$, we have $<z\Theta> = <z\delta> = <w>$. Hence, ${\iota_{\sigma^*}}^+(<z>, \Theta, <w>) > 0$ and ${\iota_{\sigma^*}}^-(<z>, \Theta, <w>) < 0$. Likewise if ${\iota_{\sigma^*}}^+(<z>, \Theta, <w>) > 0$ and ${\iota_{\sigma^*}}^-(<z>, \Theta, <w>) < 0$, then ${\iota_{\sigma^*}}^+(<z>, \delta, <w>) > 0$ and ${\iota_{\sigma^*}}^-(<z>, \delta, <w>) < 0$. Hence, $\delta \equiv \Theta$.

Conversely, suppose that $\delta \equiv \Theta$. Let $< z > \in \tilde{\xi}$. Now, ${\iota_{\sigma^*}}^+(< z >, \delta, < z\delta >) > 0$ and ${\iota_{\sigma^*}}^-(< z >, \delta, < z\delta >) < 0$. Thus, ${\iota_{\sigma^*}}^+(< z >, \Theta, < z\delta >) > 0$ and ${\iota_{\sigma^*}}^-(< z >, \Theta, < z\delta >) < 0$, so $< z\Theta > = < z\delta >$. Choose $z = \lambda$, then $< \delta > = < \Theta >$ and so $\delta \approx \Theta$. Hence, $\delta \approx \Theta$ if and only if $\delta \equiv \Theta$. □

Definition 3.2. *Let $M = (\varsigma, \xi, \iota)$ be a BFFSM and let $\sim$ be an E_R on ς. Then $\sim$ is called a B_{fadR} if it satisfies for all $d, k, w \in \varsigma$ and $a \in \xi$, $d \sim k, \iota^+(d, a, w) > 0$, and $\iota^-(d, a, w) < 0$ imply $\exists t \in \varsigma$ such that $t \sim w, \iota^+(k, a, t) \geq \iota^+(d, a, r)$, and $\iota^-(k, a, t) \leq \iota^-(d, a, r)$.*

Now, we present a characterization of a $\mathrm{B_{fadR}}$.

Theorem 3.4. *Let $M = (\varsigma, \xi, \iota)$ be a BFFSM and let $\sim$ be an E_R on ς. Then $\sim$ is a B_{fadR} if and only if it satisfies for all $b, k, w \in \varsigma^*$ and for all $u \in \xi^*$, $b \sim k, {\iota_*}^+(b, u, w) > 0$, and ${\iota_*}^-(b, u, w) < 0$ imply $\exists t \in \varsigma$ such that $t \sim w, {\iota_*}^+(k, u, t) \geq {\iota_*}^+(b, u, w)$, and ${\iota_*}^-(k, u, t) \leq {\iota_*}^-(b, u, w)$.*

Proof. Suppose $\sim$ is a $\mathrm{B_{fadR}}$ on ς. Let $b, k, w \in \varsigma$ and $u \in \xi^*$ be such that $b \sim k, {\iota_*}^+(b, u, w) > 0$, and ${\iota_*}^-(b, u, w) < 0$. Suppose $|u| = n$. Suppose $n = 0$. Then $u = \lambda$. It implies ${\iota_*}^+(b, \lambda, w) = {\iota_*}^+(b, u, w) > 0$ and ${\iota_*}^-(b, \lambda, w) = {\iota_*}^-(b, u, w) < 0$. Thus, $b = w, {\iota_*}^+(b, u, b) = 1$, and ${\iota_*}^-(b, u, b) = -1$. If we take $t = k$, then $t \sim w$ and also

$$
{\iota_*}^+(k, u, t) = {\iota_*}^+(k, u, k) = 1 = {\iota_*}^+(b, u, b) = {\iota_*}^+(b, u, w),
$$

$$
{\iota_*}^-(k, u, t) = {\iota_*}^-(k, u, k) = -1 = {\iota_*}^-(b, u, b) = {\iota_*}^-(b, u, w).
$$

Thus, for $n = 0$, the result is true. Assume the result is true for every $\Theta \in \xi^*$, where $|\Theta| = n - 1, n > 0$. Assume $u = \Theta a$, where $a \in \xi$. Now, it implies

$$
{\iota_*}^+(b, u, w) = {\iota_*}^+(b, \Theta a, w) = \sup_{k_1 \in \varsigma}[{\iota_*}^+(b, \Theta, k_1) \wedge {\iota_*}^+(k_1, a, w)] > 0,
$$

$$
{\iota_*}^-(b, u, w) = {\iota_*}^-(b, \Theta a, w) = \inf_{k_1 \in \varsigma}[{\iota_*}^-(b, \Theta, k_1) \vee {\iota_*}^-(k_1, a, w)] < 0.
$$

Let $s \in \varsigma$ be such that

$$
{\iota_*}^+(b, \Theta, s) \wedge {\iota_*}^+(s, a, w) = \sup_{k_1 \in \varsigma}[{\iota_*}^+(b, \Theta, k_1) \wedge {\iota_*}^+(k_1, a, w)],
$$

$$
{\iota_*}^-(b, \Theta, s) \vee {\iota_*}^-(s, a, w) = \inf_{k_1 \in \varsigma}[{\iota_*}^-(b, \Theta, k_1) \vee {\iota_*}^-(k_1, a, w)].
$$

Thus, ${\iota_*}^+(b, \Theta, s) > 0, {\iota_*}^+(s, a, w) > 0, {\iota_*}^-(b, \Theta, s) < 0$, and ${\iota_*}^-(s, a, w) < 0$. As from the induction hypothesis, we get the existence of $t_s \in \varsigma$ such that $t_s \sim S, {\iota_*}^+(k, \zeta, t_s) \geq {\iota_*}^+(b, \zeta, s)$, and ${\iota_*}^-(k, \zeta, t_s) \leq {\iota_*}^-(b, \zeta, s)$. Now, ${\iota_*}^+(s, a, w) > 0, {\iota_*}^-(s, a, w) < 0$, and $t_s \sim S$. Since $\sim$ is a $\mathrm{B_{fadR}}$, $\exists t \in \varsigma$ such that $\iota^+(t_s, a, t) \geq \iota^+(s, a, w), \iota^-(t_s, a, t) \leq \iota^-(s, a, w)$, and $t \sim w$. Thus, ${\iota_*}^+(k, \zeta, t_s) \wedge \iota^+(t_s, a, t) \geq {\iota_*}^+(b, \zeta, s) \wedge {\iota_*}^+(s, a, w)$ and ${\iota_*}^-(k, \zeta, t_s) \vee \iota^-(t_s, a, t) \leq {\iota_*}^-(b, \zeta, s) \wedge {\iota_*}^-(s, a, w)$ for some $t \in \varsigma$. Hence,

$$
\begin{aligned}
{\iota_*}^+(b, u, w) &= {\iota_*}^+(b, \zeta, s) \wedge {\iota_*}^+(s, a, w) \\
&\leq {\iota_*}^+(k, \zeta, t_s) \wedge \iota^+(t_s, a, t) \\
&\leq \sup_{w_1 \in \varsigma}[{\iota_*}^+(k, \zeta, w_1) \wedge \iota^+(w_1, a, t)]
\end{aligned}
$$

$$
\begin{aligned}
&= {\iota_*}^{+}(k,\zeta a,t)\\
&= {\iota_*}^{+}(k,u,t),
\end{aligned}
$$

$$
\begin{aligned}
{\iota_*}^{-}(b,\xi,w) &= {\iota_*}^{-}(b,\zeta,s) \vee {\iota_*}^{-}(s,a,w)\\
&\geq {\iota_*}^{-}(k,\zeta,t_s) \vee \iota^{-}(t_s,a,t)\\
&\geq \inf_{w_1\in Q}[{\iota_*}^{-}(k,\zeta,w_1) \vee \iota^{-}(w_1,a,t)]\\
&= {\iota_*}^{-}(k,\zeta a,t)\\
&= {\iota_*}^{-}(k,u,t) \text{ and } t \sim w.
\end{aligned}
$$

Thus, the result is followed by induction.

The converse part is trivial. □

4. Conclusion

In this exploration, we presented the quotient structures of BFFSMSs, each association of an SG and a BFFSM by supposed a $\mathrm{C_R}$. We likewise characterized the idea of a $\mathrm{B_{fadR}}$.

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