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# Fekete-Szegö and Second Hankel Determinant for a Subclass of Holomorphic $p$-Valent Functions Related to Modified Sigmoid

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**Abstract.** This research paper's primary focus is on applications of modified sigmoid functions to the class of holomorphic multivalent functions. Because of its multiple applications in computer sciences, engineering, and physics, we investigate the initial coefficient bounds for a new generalized subclass of holomorphic functions related to Sigmoid functions. Also, the relevant connections with the famous classical Fekete-Szegö inequality for these classes are discussed. The second Hankel determinant for the newly defined function class is obtained.

## 1. Introduction and Motivation

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p}\, z^{k+p}, \tag{1.1}$$

which are $p$-valently analytic in the open unit disk:

$$\mathcal{U} = \{z \in \mathbb{C} : 0 \leq |z| < 1\}.$$

The $p$-valently analytic functions have been investigated earlier from different aspects. $p$-valently analytic functions still inspire studies with interesting properties.

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The theory of special functions has been developed by Gauss, Jacobi, Klein and many others in 19th century. However, in the twentieth century, the theory of special functions has been overshadowed by other fields such as real and functional analysis, topology, algebra and differential equations. Special functions play an important role in geometric function theory. An example of special function is an activation function. An activation function acts as a squashing function which is the output of a neuron in a neural network taking certain values (usually 0 and 1, -1 or 1). There are three types of activation functions, namely threshold function, piecewise-linear function, and Sigmoid function.

The most popular activation function is the Sigmoid function. There are different methods to evaluate this function, such as truncating series expansion, looking-up tables, or piecewise approximation.

The Sigmoid function of the form

$$g(z) = \frac{1}{1+e^{-z}} \tag{1.2}$$

is differentiable and has the following properties.

- It outputs real numbers between 0 and 1.
- It maps from a very large input domain to a small range of outputs.
- never loses information because it is a one-to-one function.
- increases monotonically.

These properties enable us to use Sigmoid function in univalent function theory.

We briefly recall the following definitions which we needed in our investigation.

**Definition 1.1.** *( [13]) Let $f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}$, and $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$. The modified Hadamard product of two functions $f$ and $g$ which belong to $\mathcal{A}_p$ is defined by*

$$F(z) = (f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} \tag{1.3}$$

**Definition 1.2.** *( [14]) Let $f \in \mathcal{A}$. Then the $q^{th}$ Hankel determinant of $f$ is defined for $q \geq 1$ and $n \geq 1$ by*

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{1.4}$$

*Thus, the second Hankel determinant*

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2 \tag{1.5}$$

For two analytic functions $f$ and $g$, the function $f$ is subordinate to $g$, written as follows:

$$f(z) \prec g(z)$$

if there exists an analytic function $w$, with $w(0) = 0$ and $\left|w(z)\right| < 1$ such that $f(z) = g(w(z))$. In particular, if the function $g$ is univalent in $\mathcal{U}$, then$f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

**Definition 1.3.** *( [9]) Let $\eta \in \mathbb{C}/\{0\}$ and the class $M_\lambda(\eta, \varphi_{n,m})$ denote the subclass of $\mathcal{A}_p$ consisting of functions $f$ of the form (1.1), and satisfying the following subordination condition*

$$1 + \frac{1}{\eta}\left[\frac{zf^{'}(z) + \lambda z^2 f^{''}(z)}{(1-\lambda)\, f(z) + \lambda z f'(z)} - 1\right] \prec \varphi_{n,m} \tag{1.6}$$

*for $0 \le \lambda \le 1$ and $\varphi_{n,m}$ is a simple logistic Sigmoid activation function.*

In this study, we solve the Fekete-Szegö problem for functions in the class $M_{\lambda(*)}(\eta, \varphi_{n,m})$ and in the special instances, as well as provide bound estimates for the coefficients and an upper bound estimate for the second Hankel determinant.

**Definition 1.4.** *Let $\eta \in \mathbb{C}/\{0\}$ and the class $M_{\lambda(*)}(\eta, \varphi_{n,m})$ denote the subclass of $\mathcal{A}_p$ consisting of functions $f$ of the form (1.1), and satisfying the following subordination condition*

$$1 + \frac{1}{\eta}\left[\frac{z(f*g)^{'}(z) + \lambda z^2 (f*g)^{''}(z)}{(1-\lambda)\, (f*g)(z) + \lambda z(f*g)^{'}(z)} - 1\right] \prec \varphi_{n,m} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^m}{n!} z^n\right)^m \tag{1.7}$$

*for $0 \le \lambda \le 1$ and $\varphi_{n,m}$ is a simple logistic Sigmoid activation function.*

## 2. A Set of Lemmas

The following preliminary results needed for our investigation
Let $P$ be the family of all functions $p$ analytic in $\mathcal{U}$ for which $\Re\{\alpha(z)\} > 0$ and

$$p(z) = 1 + P_1 z + P_2 z^2 + \cdots, (for z \in \mathcal{U})$$

**Lemma 2.1.** *( [8]) If $p \in P$, then $|P_k| \le 2$ $(2, 3, 4, \cdots)$*

**Lemma 2.2.** *( [6]) Let $g$ be a Sigmoid function defined in (1.2) and*

$$\varphi(z) = 2g(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^m}{n!} z^n\right)^m \tag{2.1}$$

*then $\varphi(z) \in P$, $|z| < 1$ where $\varphi(z)$ is a modified Sigmoid function.*

**Lemma 2.3.** *( [6]) Let $g$ be a Sigmoid function defined in (1.1) and*

$$\varphi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^m}{n!} z^n\right)^m \tag{2.2}$$

*then $|\varphi_{n,m}(z)| < 2$ .*

**Lemma 2.4.** *( [6]) Let $\varphi(z) \in P$ and be starlike, then $f$ is a normalized univalent function of the form (1.1). Setting $m = 1$, Fadipe et al. [6] remarked that*

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2.3}$$

*where $c_n = \frac{(-1)^{n+1}}{2n!}$, then $|c_n| \leq 2$ for $n = 2, 3, 4, \cdots$ and the result is sharp for each $n$.*

## 3. Some coefficient estimates for the class of $M_{\lambda,(*)}\left(\eta, \varphi_{n,m}\right)$

In this section, we will find the estimates on the coefficients $a_{p+1}b_{p+1}, a_{p+2}b_{p+2}$ and $a_{p+3}b_{p+3}$ for functions in the class $M_{\lambda,(*)}\left(\eta, \varphi_{n,m}\right)$.

**Theorem 3.1.** *Let*

$$\varphi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^m}{n!} z^n\right)^m$$

*where $\varphi_{n,m}(z) \in A$ is a modified logistic Sigmoid activation function and $\varphi'_{n,m}(0) > 0$. If $F(z) = (f * g)(z)$ given by (1.1) belongs to the class $M_{\lambda,(*)}\left(\eta, \varphi_{n,m}\right)$ then,*

$$a_{p+1}b_{p+1} = \frac{(1-\lambda+\lambda p)\eta}{2p(1+\lambda p)} \tag{3.1}$$

$$a_{p+2}b_{p+2} = \frac{(1-\lambda+\lambda p)\eta^2}{4p(p+1)(1+\lambda(p+1))} \tag{3.2}$$

$$a_{p+3}b_{p+3} = \frac{\eta(1-\lambda+\lambda p)(3\eta^2 - p(p+1))}{24p(p+1)(p+2)(1+\lambda(p+2))} \tag{3.3}$$

*Proof.* Let $f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}$, and $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$. Then we can write the following qualities:

$$F(z) = (f*g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p}b_{k+p} z^{k+p} \Rightarrow (f*g)'(z) = pz^{p-1} + \sum_{k=1}^{\infty} (k+p)a_{k+p}b_{k+p} z^{k+p-1}$$

$$\Rightarrow (f*g)''(z) = p(p-1)z^{p-2} + \sum_{k=1}^{\infty} (k+p)(k+p-1)a_{k+p}b_{k+p} z^{k+p-2}$$

Thus, we obtain

$$z(f*g)'(z) + \lambda z^2 (f*g)''(z) = p(1-\lambda+\lambda p)z^p + \sum_{k=1}^{\infty} (k+p)(1+(k+p-1)\lambda)a_{k+p}b_{k+p} z^{k+p}$$

and

$$(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z) = (1-\lambda+\lambda p)z^p + \sum_{k=1}^{\infty} (1+(k+p-1)\lambda)a_{k+p}b_{k+p} z^{k+p}$$

If $F \in M_{\lambda(*)}(\eta, \varphi_{n,m})$, then we have

$$\frac{1}{\eta}\left[\frac{z(f*g)'(z) + \lambda z^2 (f*g)''(z)}{(1-\lambda)\,(f*g)(z) + \lambda z(f*g)'(z)} - 1\right] = \varphi_{n,m} - 1 \tag{3.4}$$

where $\varphi_{n,m}$ is a modified Sigmoid function given by

$$\varphi_{n,m} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + \cdots \tag{3.5}$$

In view of (3.4) and (3.5), expanding in series forms we have

$$\begin{gathered}\frac{1}{\eta}\left[(p-1)(1-\lambda+\lambda p)z^p + \sum_{k=1}^{\infty}(k+p-1)(1+(k+p-1)\lambda)a_{k+p}b_{k+p}\,z^{k+p}\right] = \\ \left[(1-\lambda+\lambda p)z^p + \sum_{k=1}^{\infty}(1+(k+p-1)\lambda)a_{k+p}b_{k+p}\,z^{k+p}\right]\left[\frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + \cdots\right]\end{gathered} \tag{3.6}$$

Comparing the coefficients of $z^{p+1}$, $z^{p+2}$ and $z^{p+3}$ in(3.6), we obtain

$$a_{p+1}b_{p+1} = \frac{(1-\lambda+\lambda p)\eta}{2p(1+\lambda p)} \tag{3.7}$$

$$a_{p+2}b_{p+2} = \frac{(1-\lambda+\lambda p)\eta^2}{4p(p+1)(1+\lambda(p+1))} \tag{3.8}$$

$$a_{p+3}b_{p+3} = \frac{\eta(1-\lambda+\lambda p)(3\eta^2 - p(p+1))}{24p(p+1)(p+2)(1+\lambda(p+2))} \tag{3.9}$$

□

**Corollary 3.1.** *For coefficient $a_{p+1}b_{p+1}$,*

$$\left|a_{p+1}b_{p+1}\right| = \frac{(1-\lambda+\lambda p)|\eta|}{2p(1+\lambda p)}$$

*is written and since $\varphi(\lambda) = \frac{(1-\lambda+\lambda p)}{(1+\lambda p)}$, $\varphi'(\lambda) < 0$ in the interval $0 \leq \lambda \leq 1$ and $\varphi(\lambda)$ is decreasing, it will be*

$$\frac{|\eta|}{2(p+1)} \leq |a_{p+1}b_{p+1}| \leq \frac{|\eta|}{2p} \tag{3.10}$$

*for $\frac{1}{2} \leq \frac{(1-\lambda+\lambda p)}{(1+\lambda p)} \leq 1$.*

Similarly, since the coefficients $a_{p+1}b_{p+1}$, $a_{p+2}b_{p+2}$ and $a_{p+3}b_{p+3}$ depend on $\lambda$ and are decreasing with respect to $\lambda$, the following inequalities can be written easily:

$$\frac{|\eta^2|}{4(p+1)(p+2)} \leq |a_{p+2}b_{p+2}| \leq \frac{|\eta|^2}{4p(p+1)} \tag{3.11}$$

$$\frac{|(\eta^3 - p(p+1)\eta)|}{24(p+1)(p+2)(p+3)} \leq |a_{p+3}b_{p+3}| \leq \frac{|(\eta^3 - p(p+1)\eta)|}{24p(p+1)(p+2)} \tag{3.12}$$

## 4. Some Results Connected with the Fekete-Szegö Inequality and Hankel Coefficient for the Class of $M_{\lambda,(*)}(\eta,\varphi_{n,m})$

The Fekete-Szegö problem may be considered one of the most important results about univalent functions, which is related to coefficients an of a function's Taylor series and was introduced by Fekete-Szegö [1]. The problem of maximizing the absolute value of functional $a_3 - \mu a_2^2$ is called the Fekete-Szegö problem. This result is sharp and is studied thoroughly by many researchers. The equality holds true for the Koebe function. In 1969, Keogh and Merkes [2] obtained the sharp upper bound of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for some subclasses of univalent function.
Recently, Murugusundarmoorthy and Janani [3], Olantunji et al. [5], Olantunji [4] and Orhan *at al.* [7] have studied Sigmoid function for various classes of analytic and univalent functions.
In this section, we first prove the following Fekete-Szegö result for the function in the classes $M_{\lambda,(*)}(\eta,\varphi_{n,m})$ with the values of $a_{p+1}b_{p+1}$ and $a_{p+2}b_{p+2}$.

**Theorem 4.1.** *If $F(z)\in\mathcal{A}_{\mathcal{P}}$ given by (1.1) belongs to the class $M_{\lambda,(*)}(\eta,\varphi_{n,m})$ then,*

$$|a_{p+2}b_{p+2} - \mu\left(a_{p+1}b_{p+1}\right)^2| = \frac{|\eta|^2}{4p(p+1)}\left(1+|\mu|\frac{(p+1)}{p}\right) \tag{4.1}$$

*Proof.* If the values of $a_{p+1}b_{p+1}$ and $a_{p+2}b_{p+2}$ determined by (3.7) and (3.8) are written instead of $a_{p+2}b_{p+2} - \mu\left(a_{p+1}b_{p+1}\right)^2$, we get

$$\begin{aligned} a_{p+2}b_{p+2} - \mu\left(a_{p+1}b_{p+1}\right)^2 &= \frac{(1-\lambda+\lambda p)\eta^2}{4p(p+1)(1+\lambda(p+1))} - \mu\left(\frac{(1-\lambda+\lambda p)\eta}{2p(1+\lambda p)}\right)^2 \\ &= \frac{(1-\lambda+\lambda p)\eta^2}{4p(p+1)(1+\lambda(p+1))} - \mu\frac{(1-\lambda+\lambda p)^2\eta^2}{4p^2(1+\lambda p)^2}. \end{aligned}$$

Taking absolute value on both sides of the above equation and applying triangle inequality, we get

$$|a_{p+2}b_{p+2} - \mu\left(a_{p+1}b_{p+1}\right)^2| \le \frac{(1-\lambda+\lambda p)|\eta|^2}{4p(p+1)(1+\lambda(p+1))} + |\mu|\frac{(1-\lambda+\lambda p)^2|\eta|^2}{4p^2(1+\lambda p)^2}.$$

Here $\zeta_1 = \frac{1-\lambda+\lambda p}{1+\lambda(p+1)}$ and $\zeta_2 = \frac{(1-\lambda+\lambda p)^2}{(1+\lambda p)^2}$ are taken and these functions depending on $\lambda$ are considered to be decreasing in the interval $0\le\lambda\le 1$, since

$$\max_{0\le\lambda\le1}\frac{1-\lambda+\lambda p}{1+\lambda(p+1)} = 1$$

and

$$\max_{0\le\lambda\le1}\frac{(1-\lambda+\lambda p)^2}{(1+\lambda p)^2} = 1$$

we get

$$|a_{p+2}b_{p+2} - \mu\left(a_{p+1}b_{p+1}\right)^2| \le \frac{|\eta|^2}{4p(p+1)} + |\mu|\frac{|\eta|^2}{4p^2}.$$

thus we obtain

$$|a_{p+2}b_{p+2} - \mu\left(a_{p+1}b_{p+1}\right)^2| \le \frac{|\eta|^2}{4p(p+1)}\left(1+|\mu|\frac{(p+1)}{p}\right)$$

Hence, we have reached the desired assertion of the Theorem(4.1),

$$\left|a_{p+2}b_{p+2}-\mu\left(a_{p+1}b_{p+1}\right)^2\right| \leq \begin{cases} \frac{|\eta|^2}{4p(p+1)}\left(1+|\mu|\frac{(p+1)}{p}\right), & \mu \geq 0 \\ \frac{|\eta|^2}{4p(p+1)}\left(1-|\mu|\frac{(p+1)}{p}\right), & \mu \leq 0 \end{cases}$$

This completes the proof of the Theorem. □

In the theory of singularities [10] and the investigation of power series with integral coefficients, the Hankel determinant is very important. The reader is encouraged to read [14] for more information. For several subfamilies of univalent functions, the growth of $H_q(n)$ has been explored. We know that the function $H_2(1) = a_3 - a_2^2$ for $q = 2$ and $n = 1$ is a well recognized Fekete-Szegö functional. For the bi-convex and bi-starlike classes, the second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2a_4 - a_3^2$ [12]. For some more recent papers regarding Hankel determinant, we may refer the readers to see [15–26].
The following theorem will give some results related to Hankel determinant for the functions belonging to classes $M_{\lambda,(*)}\left(\eta, \varphi_{n,m}\right)$.

**Theorem 4.2.** *If $F(z) \in \mathcal{A}_{\mathcal{P}}$ given by (1.1) belongs to the class $M_{\lambda,(*)}\left(\eta, \varphi_{n,m}\right)$ then,*

$$|(a_{p+1}b_{p+1})(a_{p+3}b_{p+3}) - \left(a_{p+2}b_{p+2}\right)^2| \leq \frac{|\eta|^2}{48p^2(p+1)^2(p+2)}\left((p+1)|3\eta^2 - p(p+1)| + 3(p+2)|\eta|^2\right) \tag{4.2}$$

*Proof.* From (3.7), (3.8)and (3.9) , we get

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2 \tag{4.3}$$

$$\begin{aligned}(a_{p+1}b_{p+1})(a_{p+3}b_{p+3}) - \left(a_{p+2}b_{p+2}\right)^2 = &\left(\frac{(1-\lambda+\lambda p)\eta}{2p(1+\lambda p)}\right)\left(\frac{\eta(1-\lambda+\lambda p)(3\eta^2 - p(p+1))}{24p(p+1)(p+2)(1+\lambda(p+2))}\right) \\ &-\left(\frac{(1-\lambda+\lambda p)\eta^2}{4p(p+1)(1+\lambda(p+1))}\right)^2\end{aligned} \tag{4.4}$$

$$= \frac{(1-\lambda+\lambda p)^2(3\eta^4 - p(p+1)\eta^2)}{48p^2(p+1)(p+2)(1+\lambda p)(1+\lambda(p+2))} - \frac{(1-\lambda+\lambda p)^2\eta^4}{16p^2(p+1)^2(1+\lambda(p+1))^2} \tag{4.5}$$

and thus

$$\begin{aligned}\left|(a_{p+1}b_{p+1})(a_{p+3}b_{p+3}) - \left(a_{p+2}b_{p+2}\right)^2\right| \leq &\frac{(1-\lambda+\lambda p)^2\left|(3\eta^4 - p(p+1)\eta^2)\right|}{48p^2(p+1)(p+2)(1+\lambda p)(1+\lambda(p+2))} \\ &+\frac{(1-\lambda+\lambda p)^2|\eta|^4}{16p^2(p+1)^2(1+\lambda(p+1))^2}\end{aligned} \tag{4.6}$$

Here $\zeta_3 = \frac{(1-\lambda+\lambda p)^2}{(1+\lambda p)(1+\lambda(p+2))}$ and $\zeta_4 = \frac{(1-\lambda+\lambda p)^2}{(1+\lambda(p+1))^2}$ are taken and these functions depending on $\lambda$ are considered to be decreasing in the interval $0 \leq \lambda \leq 1$, since

$$\max_{0\leq\lambda\leq 1} \frac{(1-\lambda+\lambda p)^2}{(1+\lambda p)(1+\lambda(p+2))} = 1$$

and

$$\max_{0\leq\lambda\leq 1} \frac{(1-\lambda+\lambda p)^2}{(1+\lambda(p+1))^2} = 1$$

thus we obtain

$$|(a_{p+1}b_{p+1})(a_{p+3}b_{p+3}) - \left(a_{p+2}b_{p+2}\right)^2| \leq \frac{|\eta|^2}{48p^2(p+1)^2(p+2)}\left((p+1)|3\eta^2 - p(p+1)| + 3(p+2)|\eta|^2\right) \tag{4.7}$$

This completes the proof of the Theorem. □

## 5. Conclusion

In this paper, we have introduced and investigated the class $M_{\lambda,(*)}\left(\eta, \varphi_{n,m}\right)$ of $p$-valent function related to the to modified Sigmoid functions. Thus, we obtained second, third and fourth Taylor and Maclaurin coefficients of functions in this class. We also found the second Hankel Determent for our defined function class. These results were an improvement on the estimates obtained in the recent studies.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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