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A Class of Non-Bazilevic Functions Subordinate to Gegenbauer Polynomials

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Abstract. In this paper, we introduce and investigate a class non-Bazilevic functions that associated by Gegenbauer Polynomials. The coefficient estimates of functions belonging to this class are derived. Moreover, we obtain the classical Fekete-Szegö inequality of functions belonging to this class.

1. Introduction

Let $\mathcal{A}$ be the family of all analytic functions f that are defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = 1 - f'(0)$. Any function $f \in \mathcal{A}$ has the following Taylor-Maclarin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where } z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. Let the functions f and g be analytic in $\mathbb{D}$, we say the function f is subordinate by the function g in $\mathbb{D}$, denoted by $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if there exists a Schwarz function w, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent over $\mathbb{D}$ then $f(z) \prec g(z)$ equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For more information about the Subordination Principle we refer the readers to to the monographs [11], [19] and [20].

As known univalent functions are injective (one-to-one) functions. Hence, they are invertible and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, the Koebe one-quarter Theorem tells us that the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ contains the disk $D(0, 1/4)$

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of center 0 and radius 1/4. Accordingly, every function $f \in \mathcal{S}$ has an inverse $f^{-1} = g$ which is defined as

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \ \ r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{1.2}$$

For this reason, we define the class Σ as follows. A function $f \in \mathcal{A}$ is said to be bi-univalent if both f and f^{-1} are univalent in $\mathbb{D}$. Therefore, let Σ denote the class of all bi-univalent functions in $\mathcal{A}$ which are given by equation (1.1). For example, the following functions belong to the class Σ:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \log\sqrt{\frac{1+z}{1-z}}.$$

However, Koebe function, $\dfrac{2z-z^2}{2}$ and $\dfrac{z}{1-z^2}$ do not belong to the class Σ. For more information about univalent and bi-univalent functions we refer the readers to the articles [17], [18], [21], the monograph [10], [12] and the references therein.

The subject of the geometric function theory in complex analysis has been investigated by many researchers in recent years, the typical problem in this field is studying a functional made up of combinations of the initial coefficients of the functions $f \in \mathcal{A}$. For a function in the class $\mathcal{S}$, it is well-known that $|a_n|$ is bounded by n. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of the class $\mathcal{S}$ gives the growth and distortion bounds for the class. In addition, the Fekete-Szegö functional arises naturally in the investigation of univalency of analytic functions. In the year 1933, Fekete and Szegö [15] found the maximum value of $|a_3 - \lambda a_2^2|$, as a function of the real parameter $0 \leq \lambda \leq 1$ for a univalent function f. Since then, the problem of dealing with the Fekete-Szegö functional for $f \in \mathcal{A}$ with any complex λ is known as the classical Fekete-Szegö problem. There are many researchers investigated the Fekete-Szegö functional and the other coefficient estimates problems, for example see the articles [3], [5], [7], [8], [13], [14], [15], [18], [28] and the references therein.

2. Preliminaries

In this section we present some information that are curial for the main results of this paper. For any real numbers $\lambda, t \in \mathbb{R}$,with $\lambda \geq 0$ and $-1 \leq t \leq 1$, and $z \in \mathbb{D}$ the generating function of Gegenbauer polynomials is given by

$$H_\lambda(z,t) = (z^2 - 2tz + 1)^{-\lambda}.$$

Moreover, for any fixed t the function $H_\lambda(z,t)$ is analytic on the unit disk $\mathbb{D}$ and its Taylor-Maclaurin series is given by

$$H_\lambda(z,t) = \sum_{n=0}^{\infty} C_n^\lambda(t) z^n.$$

In addition, Gegenbauer polynomials can be defined in terms of the following recurrence relation:

$$C_n^\lambda(t) = \frac{2t(n+\lambda-1)C_{n-1}^\lambda(t) - (n+2\lambda-2)C_{n-1}^\lambda(t)}{n}, \tag{2.1}$$

with initial values,

$$C_0^\lambda(t) = 1, \;\; C_1^\lambda(t) = 2\lambda t, \;\; \text{and} \;\; C_2^\lambda(t) = 2\lambda(\lambda+1)x^2 - \lambda. \tag{2.2}$$

It is well-known that the Gegenbauer polynomials and their special cases, are orthogonal polynomials, such as Legendre polynomials $L_n(t)$ and the Chebyshev polynomials of the second kind $T_n(x)$ where the values of λ are $\lambda = 1/2$ and $\lambda = 1$ respectively, more precisely

$$L_n(t) = C_n^{1/2}(t), \;\; \text{and} \;\; T_n(t) = C_n^1(t).$$

For more information about the Gegenbauer polynomials and their special cases, we refer the readers to the articles [2], [4], [6], [13], [16], [18], [23], [28], the monograph [10], [12], [27], and the references therein.

In the year 1988, Obradovic [22] introduced and studied the class of non-Bazilevic functions which, for $z \in \mathbb{D}$ and $0 < \beta < 1$, satisfies the following condition:

$$\mathcal{R}\left\{ f'(z)\left(\frac{z}{f(z)}\right)^{1+\beta} \right\} > 0.$$

In the year 2007, Owa, Sekine and Yamakawa [24] introduced and studied a Sakaguchi-type class $\mathcal{S}^*(\gamma, y)$ which, for $z \in \mathbb{D}$, $y \neq 1$, $|y| \leq 1$ and $0 \leq \gamma \leq 1$, satisfies the following condition:

$$\mathcal{R}\left\{ \frac{(1-y)zf'(z)}{f(z) - f(yz)} \right\} > \gamma.$$

Motivated by the aforementioned research papers, we define the following class $P_\Sigma(\alpha, \beta, \lambda, x, y)$ of non-Bazilevic functions. We say $f \in P_\Sigma(\alpha, \beta, \lambda, x, y)$ if the following subordination holds:

$$(1-\alpha)f'(z) + \alpha f'(z)\left(\frac{(x-y)z}{f(xz) - f(yz)}\right)^\beta \prec H_\lambda(z,t), \tag{2.3}$$

where $z \in \mathbb{D}$, $x \neq y$ in $\overline{\mathbb{D}}$, $\beta \geq 0$, $0 \leq \alpha \leq 1$ and $\lambda \geq 0$, and $t \in (1/2, 1]$.

Suppose $f \in \mathcal{A}$, then for $z \in \mathbb{D}$ we have

$$\frac{f(xz) - f(yz)}{x-y} = z + \sum_{n=2}^{\infty} \delta_n a_n z^n,$$

where

$$\delta_n = \frac{x^n - y^n}{x - y}, \ \ n \in \mathbb{N}.$$

Moreover, for $\beta \geq 0$ we get

$$\left(\frac{(x-y)z}{f(xz)-f(yz)}\right)^{\beta} = -\beta\delta_2 a_n z + \left(\frac{\beta(\beta+1)\delta_2^2 a_2^2}{2} - \beta\delta_3 a_3\right)z^2 + \cdots$$

Therefore, we have the following equation:

$$(1-\alpha)f'(z) + \alpha f'(z)\left(\frac{(x-y)z}{f(xz)-f(yz)}\right)^{\beta}$$

$$= 1 + (2a_2 - \alpha\beta\delta_2 a_2)\, z + \left((3-\alpha\beta\delta_3)a_3 - \alpha\beta\left(2\delta_2 - \frac{(\beta+1)\delta_2^2}{2}\right)a_2\right)z^2 + \cdots \tag{2.4}$$

For more information about this class and quasi-subordination, we refer the reader to the paper [26]. There are many special cases of this class that have been studied by many researchers. For example, if $x = 1$ and $\alpha = 1$ we get the class that is studied by Sharma and Raina [25]. If $\alpha = 1$ and $\lambda = 1$, we get the class that is studied by Dansu and Olatunji [9]. If $\alpha = 1$ and $0 < \beta < 1$, we get the class that is studied by Al-Khafaji et al. [1]. The following lemma (see, for details [14]) is a well-known fact, but it is crucial for our presented work.

Lemma 2.1. *Let the Schwarz function $w(z)$ be given by:*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \ \ \text{where} \ \ z \in \mathbb{D},$$

then $|w_1| \leq 1$ and for $t \in \mathbb{C}$

$$|w_2 - t w_1^2| \leq 1 + (|t| - 1)|w_1|^2 \leq \max\{1, |t|\}.$$

The results is sharp for the functions $w(z) = z$ and $w(z) = z^2$.

The primary goal of this article is determining the estimates for the initial Taylor-Maclarin coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class $P_{\Sigma}(\alpha, \beta, \lambda, x, y)$. Furthermore, we examine the corresponding Fekete-Szegö functional problem for functions in the presented class.

3. Coefficient Bounds for the Non-Bazilevic Functions

In this section, we provide estimates for the initial Taylor-Maclaurin coefficients for the functions belong to the class $P_{\Sigma}(\alpha, \beta, \lambda, x, y)$ which are given by equation (1.1).

Theorem 3.1. *Let the function f given by (1.1) be in the class $P_{\Sigma}(\alpha, \beta, \lambda, x, y)$. Then*

$$|a_2| \leq \frac{2\lambda t}{|2 - \alpha\beta(x+y)|} \tag{3.1}$$

and

$$|a_3| \leq \frac{2\lambda t}{|3 - \alpha\beta(x^2 + xy + y^2)|} \max\left\{1, \frac{\left|\Delta(\alpha,\beta,\lambda)t^2 - 1\right|}{2t}\right\}, \tag{3.2}$$

where

$$\Delta(\alpha,\beta,\lambda) = \frac{2(\lambda+1)(2-\alpha\beta\delta_2)^2 + 2\lambda\alpha\beta\left(4\delta_2 - (\beta+1)\delta_2^2\right)}{(2-\alpha\beta\delta_2)^2}.$$

Proof. Let f belong to the class $P_\Sigma(\alpha,\beta,\lambda,x,y)$. Then using (2.3) we can find an analytic functions p on the unit disk $\mathbb{D}$ such that

$$(1-\alpha)f'(z) + \alpha f'(z)\left(\frac{(x-y)z}{f(xz)-f(yz)}\right)^\beta \prec H_\lambda(t,p(z)), \tag{3.3}$$

where the analytic function $p(z)$ is given by

$$p(z) = p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad \text{where } \; z \in \mathbb{D},$$

and $p(0) = 0$, and for all $z \in \mathbb{D}$ $|p(z)| < 1$. Moreover, it is well-known that (see, for details [10]) for all $j \in \mathbb{N}$ we have $|p_j| \le 1$.

Now, upon comparing the coefficients in both sides of (2.4) and (3.3) we get the following equations:

$$(2-\alpha\beta\delta_2)a_2 = C_1^\lambda(t)p_1 \tag{3.4}$$

$$(3-\alpha\beta\delta_3)a_3 - \alpha\beta\left(2 - \frac{(\beta+1)\delta_2}{2}\right)\delta_2 a_2^2 = C_1^\lambda(t)p_2 + C_2^\lambda(t)p_1^2 \tag{3.5}$$

Using equation (3.4) we get the following equation

$$a_2 = \frac{C_1^\lambda(t)p_1}{2-\alpha\beta\delta_2} \tag{3.6}$$

Using equation (2.2) and $|p_1| \le 1$, we get the desired estimate of a_2:

$$|a_2| \le \frac{2\lambda t}{|2-\alpha\beta\delta_2|}.$$

Next, we look for the estimate of the coefficient a_3. Using equations (3.4) and (3.5) we get the following equation:

$$(3-\alpha\beta\delta_3)a_3 = C_1^\lambda(t)p_2 + \frac{2(2-\alpha\beta\delta_2)^2[C_2^\lambda(t)] + \alpha\beta[4-(\beta+1)\delta_2]\delta_2[C_1^\lambda(t)]^2}{2(2-\alpha\beta\delta_2)^2}p_1^2 \tag{3.7}$$

Using the initial values (2.2) and simple calculations, we obtain

$$(3-\alpha\beta\delta_3)a_3 = 2\lambda t\left(p_2 + \frac{\Delta(\alpha,\beta,\lambda)t^2 - 1}{2t}p_1^2\right), \tag{3.8}$$

where

$$\Delta(\alpha,\beta,\lambda) = \frac{2(\lambda+1)2(2-\alpha\beta\delta_2)^2 + 2\lambda\alpha\beta[4-(\beta+1)\delta_2]\delta_2}{(2-\alpha\beta\delta_2)^2}.$$

Using Lemma 2.1, we get the desired coefficient estimate of a_3. This completes the proof of Theorem 3.1. □

The following corollaries are just consequences of Theorem 3.1. If $\alpha = 1$, $x = 1$ and $y = 0$, we get the class $P^*_\Sigma(\beta, \lambda)$ where any function f belong to this class satisfies the following condition:

$$f'(z)\left(\frac{z}{f(z)}\right)^\beta \prec H_\lambda(z,t).$$

Corollary 3.1. *Let the function f given by (1.1) be in the class $P^*_\Sigma(\beta, \lambda)$. Then*

$$|a_2| \le \frac{2\lambda t}{|2-\beta|}$$

and

$$|a_3| \le \frac{2\lambda t}{|3-\beta|} \max\left\{1, \frac{\left|\Delta(\beta,\lambda)t^2 - 1\right|}{2t}\right\},$$

where

$$\Delta(\beta,\lambda) = \frac{2(\lambda+1)(2-\beta)^2 + 2\lambda\beta(3-\beta)}{(2-\beta)^2}.$$

If $\alpha = 0$ and $\beta = 0$, then we get the following corollary.

Corollary 3.2. *Let the function f given by (1.1) be in the class $P_\Sigma(\alpha, \beta, \lambda, x, y)$ with $\alpha = \beta = 0$. Then*

$$|a_2| \le \lambda t,$$

and

$$|a_3| \le \frac{2\lambda t}{3} \max\left\{1, \frac{\left|2(\lambda+1)t^2 - 1\right|}{2t}\right\}.$$

4. Fekete-Szegö Problem for the Non-Bazilevic Functions

In this section, we consider the classical Fekete-Szegö problem for functions belong to our class $P_\Sigma(\alpha, \beta, \lambda, x, y)$.

Theorem 4.1. *Let the function f given by (1.1) be in the class $P_\Sigma(\alpha, \beta, \lambda, x, y)$. Then for some $\zeta \in \mathbb{R}$,*

$$|a_3 - \zeta a_2^2| \le \begin{cases} \frac{2\lambda t}{|A|}, & \text{if } \zeta \in [\zeta_1, \zeta_2] \\ \frac{\lambda|2B(\lambda+1)t^2 - B - 2\lambda R t^2 - 4\lambda\zeta A t^2|}{|AB|}, & \text{if } \zeta \notin [\zeta_1, \zeta_2], \end{cases} \tag{4.1}$$

where

$$\zeta_1 = \frac{[2\lambda(B-R) + 2B]t^2 - (2t+1)B}{4\lambda A t^2},$$

$$\zeta_2 = \frac{[2\lambda(B-R) + 2B]t^2 + (2t-1)B}{4\lambda A t^2},$$

$A = 3 - \alpha\beta\delta_3$, $B = (2 - \alpha\beta\delta_2)^2$, and $R = \alpha\beta(4 - (\beta+1)\delta_2)\delta_2$.

Proof. In view of (3.4) and (3.5) we have

$$a_3 = \frac{C_1^\lambda(t)}{A}p_2 + \frac{C_2^\lambda(t)}{A}p_1^2 + \frac{R[C_1^\lambda(t)]^2}{2AB}p_1^2 \tag{4.2}$$

Now, for any real number ζ, using equation (3.4) we can write the last equation as

$$a_3 - \zeta a_2^2 = \frac{C_1^\lambda(t)}{A}\left\{p_2 + \left(\frac{C_2^\lambda(t)}{C_1^\lambda(t)} + \frac{RC_1^\lambda(t)}{2B} - \frac{\zeta A C_1^\lambda(t)}{B}\right)p_1^2\right\}.$$

Using the initial values (2.2) and Lemma 2.1, we obtain

$$|a_3 - \zeta a_2^2| \leq \frac{2\lambda t}{|A|}\max\left\{1, \left|\frac{2(\lambda+1)t^2-1}{2t} - \frac{\lambda t R}{B} - \frac{2\lambda\zeta A t}{B}\right|\right\} \tag{4.3}$$

Since $t > 0$, we have

$$\left|\frac{2(\lambda+1)t^2-1}{2t} - \frac{\lambda t R}{B} - \frac{2\lambda\zeta A t}{B}\right| \leq 1.$$

Hence, solving for ζ we get:

$$\frac{[2\lambda(B-R)+2B]t^2-(2t+1)B}{4\lambda A t^2} \leq \zeta \leq \frac{[2\lambda(B-R)+2B]t^2+(2t-1)B}{4\lambda A t^2}$$

Therefore, equation (4.3) becomes

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{2\lambda t}{|A|}, & \text{if } \zeta \in [\zeta_1, \zeta_2] \\ \frac{2\lambda t}{|A|}\left|\frac{2(\lambda+1)t^2-1}{2t} - \frac{\lambda t R}{B} - \frac{2\lambda\zeta A t}{B}\right|, & \text{if } \zeta \notin [\zeta_1, \zeta_2], \end{cases} \tag{4.4}$$

This completes the Theorem's proof. □

The following corollaries are just consequences of Theorem 4.1.

Corollary 4.1. *Let the function f given by (1.1) be in the class $P^*_\Sigma(\beta,\lambda)$. Then for some $\zeta \in \mathbb{R}$,*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{2\lambda t}{|3-\beta|}, & \text{if } \zeta \in [\zeta_1, \zeta_2] \\ \lambda\left|\frac{2(\lambda+1)t^2-1}{3-\beta} - \frac{2\lambda(\beta+2\zeta)t^2}{(2-\beta)^2}\right|, & \text{if } \zeta \notin [\zeta_1, \zeta_2], \end{cases} \tag{4.5}$$

where

$$\zeta_1 = \frac{2\lambda(\beta^2-7\beta+4)t^2 + (2t^2-2t-1)(2-\beta)^2}{4\lambda(3-\beta)t^2},$$

$$\zeta_2 = \frac{2\lambda(\beta^2-7\beta+4)t^2 + (2t^2+2t-1)(2-\beta)^2}{4\lambda(3-\beta)t^2}.$$

If $\alpha = 0$ and $\beta = 0$, then we get the following corollary.

Corollary 4.2. *Let the function f given by (1.1) be in the class $P_\Sigma(\alpha,\beta,\lambda,x,y)$ with $\alpha=\beta=0$. Then for some $\zeta \in \mathbb{R}$,*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{2\lambda t}{3}, & \text{if } \zeta \in [\zeta_1, \zeta_2] \\ \frac{\lambda}{3}\left|((2\lambda-3\zeta+2)t^2-1)\right|, & \text{if } \zeta \notin [\zeta_1, \zeta_2], \end{cases} \tag{4.6}$$

where

$$\zeta_1 = \frac{(2\lambda+2)t^2 - 2t - 1}{3\lambda t^2},$$

$$\zeta_2 = \frac{(2\lambda+2)t^2 + 2t - 1}{3\lambda t^2}.$$

5. Conclusion

This research paper has investigated a family of non-Bazilevic functions associated with the Gegenbauer polynomials. For functions belong to this function class, the author found estimates for the Taylor-Maclaurin initial coefficients and the calssical Fekete-Szegö functional problem.

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