Int. J. Anal. Appl. (2024), 22:21

International Journal of Analysis and Applications

On the Spectral Theory of Regularized Quasi-Semigroups

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Abstract. We have shown a spectral inclusion between a different spectrum of a C_0-quasi-semigroups in [9]. Precisely for Saphar, essentially Saphar, quasi-Fredholm, Kato and essentially Kato spectra. In this paper, we extend these results for a C-quasi-semigroups (regularized quasi-semigroups) where C is a bounded injective operator.

1. Introduction

We consider a time-dependent abstract Cauchy problem described as follows:

$$x'(s) = A(s+t)x(s), \quad t,s \geq 0, \quad x(0) = Cx_0 \tag{1.1}$$

In this equation, the function $x(s)$ is an unknown function defined on the real interval $[0,T]$ into a Banach space denoted as X. The operator C is a bounded linear operator that is injective, acting on the Banach space X, and $A(s)$ represents a closed linear operator in X with the domain $\mathcal{D}(A(t)) = \mathcal{D}$. The solution to equation (1) can be formally expressed as $x(t) = U(t,s)x_0$, where $\{U(t,s)\}_{t,s\geq 0}$ forms a two-parameter family of operators acting on X, known as a C-quasi-emigroup or egularized quasi-semigroup of a bounded linear operators on a Banach spaces,. This notion was introduced by M.Janfada in [2] as a generalization of C_0-semigroups of operators . For more information about this last notion, see [4].

The main objective is to establish the existence of a solution to Cauchy problem without any qualitative information about it. To gain insight into the solution $x(t)$, a conventional approach involves examining the spectrum of the quasi-semigroup $U(t,s)$ directly. However, in many practical applications, we only have explicit access to the generator $A(t)$, and thus, there arises a need to establish a relationship between the spectrum of the quasi-semigroup $U(t,s)$ and the spectrum of its generator $A(t)$.

Received: Dec. 1, 2023.

2020 *Mathematics Subject Classification.* 47A10, 47D06, 47D60.

Key words and phrases. C-quasi-semigroup; Saphar; essentially Saphar; Kato; essentially Kato; quasi-Fredholm.

https://doi.org/10.28924/2291-8639-22-2024-21

ISSN: 2291-8639

The works done on C_0-semigroups [7], on C-semigroups [10], [11], on C_0-quasi-semigroups [8], and recently on regularized quasi-semigroups [9], has led us to seek other results concerning the latter concept.

We begin with the definition of regularized quasi-semigroups of bounded operators introduced by Janfada in [2].

Definition 1.1. *Suppose that C is an injective bounded linear operator on a Banach space X. A commutative two parameter family $\{U(t,s)\}_{t,s\geq 0} \subseteq \mathcal{B}(X)$ is called a regularized quasi-semigroups (or C-quasi-semigroups) if for every $t, s_1, s_2 \geq 0$ and $x \in X$, we have*

(1) $U(t,0) = C$;
(2) $CU(t, s_1 + s_2) = U(t + s_1, s_2)U(t, s_1)$;
(3) $\{U(t,s)\}_{t,s\geq 0}$ *is strongly continuous, that is,*

$$\lim_{(t,s)\longrightarrow(t_0,s_0)} \left\|U(t,s)x - U(t_0,s_0)x\right\| = 0, \ x \in X;$$

(4) *there exists a continuous and increasing mapping $M : [0,+\infty[\longrightarrow [0,+\infty[$ such that, for any $t, s > 0$, $\left\|U(t,s)\right\| \leq M(t+s)$.*

For a C-quasi-semigroup $\{U(t,s)\}_{t,s\geq 0}$ on a Banach space X, let $\mathcal{D}$ be the set of all $x \in X$ for which the following limits exist in the range of C:

$$\lim_{s\to 0^+} \frac{U(0,s)x - Cx}{s} \text{ and } \lim_{s\to 0^+} \frac{U(t,s)x - Cx}{s} = \lim_{s\to 0^+} \frac{U(t-s,s)x - Cx}{s}, \ t > 0.$$

In this case, for $t \geq 0$, we define an operator $A(t)$ on $\mathcal{D}$ as

$$A(t)x = C^{-1} \lim_{s\to 0^+} \frac{U(t,s)x - Cx}{s}.$$

The family $\{A(t)\}_{t\geq 0}$ is called the infinitesimal generator of the regularized quasi-semigroup $\{U(t,s)\}_{t,s\geq 0}$.

Remark 1.1.

(1) *If $C = I$ (the identity operator), then $\{U(t,s)\}_{t,s\geq 0}$ is a C_0-quasi-semigroup [4].*
(2) *Let $\{U(t,s)\}_{t,s\geq 0}$ be a C-quasi-semigroup. Letting $r \mapsto 0$ in (2) of definition , we obtain:*
   (a) $\forall t \geq 0 : U(t,s)C = CU(t,s)$;
   (b) $\forall t \geq 0, \forall x \in R(C) : U(t,s)x \in R(C)$;
   (c) $\forall x \in R(C), t \geq 0 : C^{-1}U(t,s)x = U(t,s)C^{-1}x$

For more information, examples and properties on the regularized quasi-semigroups, see [2].

Throughout this paper, X a complex Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. Let T be a closed linear operator on X with domain $D(T)$. We denote by $R(T)$, $R^\infty(T) := \cap_{n\geq 1} R(T^n)$, $Ker(T)$ and $\rho(T)$ respectively the the range, the hyper range, the kernel, the resolvent and the spectrum of T.

2. Main Results

2.1. Preliminaries and first result.

we start with the following results shown in the recent paper [9].

Lemma 2.1. *Let $A(t)$ be a closed and densely defined generator of a $C-$quasi-semigroup $(U(t,s))_{t,s\geq 0}$ on a Banach space X, and let $C \in B(X)$ be injective. Then for all $t \geq s \geq 0$ and all $\lambda \in \mathbb{C}$, we have*

(1) *For all $x \in \mathcal{D}$,*

$$D_\lambda(t,s)(\lambda I - A(t))x = [e^{\lambda s}C - U(t-s,s)]x.$$

(2) *For all $x \in X$, we have $D_\lambda(t,s)x \in \mathcal{D}$ and*

$$(\lambda I - A(t))D_\lambda(t,s)x = [e^{\lambda s}C - U(t-s,s)]x.$$

where $D_\lambda(t,s)x = \displaystyle\int_0^s e^{\lambda(s-h)}U(t-h,h)xdh$ is a bounded and linear operator.

For $t \geq 0$, we fix $\mathcal{D}^0 = \mathcal{D}(A(t)^0) = X$, $A(t)^0 = I$, and for $n \in \mathbb{N}$ we define by recurrence:

$$\mathcal{D}^n = \mathcal{D}(A(t)^n) := \{x \in \mathcal{D}(A(t)^{n-1}) : A(t)^{n-1}x \in \mathcal{D}(A(t))\},$$

$$A(t)^n x = A(t)A(t)^{n-1}x \text{ pour } x \in \mathcal{D}(A(t)^n),$$

We obtain :

$$X = D(A(t)^0) \supseteq D(A(t)) \supseteq D(A(t)^2) \supseteq ... \supseteq D(A(t)^n).$$

Corollary 2.1. *Let $A(t)$ be a closed and densely defined generator of a $C-$quasi-semigroup $(U(t,s))_{t,s\geq 0}$ on a Banach space X, and let $C \in B(X)$ be injective. Then for all $t \geq s \geq 0$, $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}\setminus\{0\}$, we obtain*

(1) *For all $x \in X$,*

$$(\lambda I - A(t))^n[D_\lambda(t,s)]^n x = [e^{\lambda s}C - U(t-s,s)]^n x.$$

(2) *For all $x \in \mathcal{D}^n$,*

$$[D_\lambda(t,s)]^n(\lambda I - [A(t)]^n)x = [e^{\lambda s}C - U(t-s,s)]^n x.$$

(3) $Ker[\lambda I - A(t)] \subseteq Ker[e^{\lambda s}C - U(t-s,s)]$.
(4) $R[e^{\lambda s}C - U(t-s,s)] \subseteq R[\lambda I - A(t)]$.
(5) $Ker[\lambda I - A(t)]^n \subseteq Ker[e^{\lambda s}C - U(t-s,s)]^n$.
(6) $R[e^{\lambda s}C - U(t-s,s)]^n \subseteq R[\lambda I - A(t)]^n$.
(7) $R^\infty[e^{\lambda s}C - U(t-s,s)] \subseteq R^\infty[\lambda I - A(t)]$.

Lemma 2.2. *Let $A(t)$ be a closed and densely defined generator of a $C-$quasi-semigroup $(U(t,s))_{t,s\geq 0}$ on a Banach space X, and let $C \in B(X)$ be injective. Then for all $t \geq s > 0$ and all $\lambda \in \mathbb{C}$, we have*

$$(\lambda I - A(t))L_\lambda(t,s) + \frac{1}{s}e^{-\lambda s}D_\lambda(t,s) = C.$$

With $L_\lambda(t,s) = \displaystyle\frac{1}{s}\int_0^s e^{-\lambda h}D_\lambda(t,h)dh$.

Furthermore, the operators $L_\lambda(t,s)$, $D_\lambda(t,s)$ and $(\lambda I - A(t))$ are mutually commuting. Also, C is commute with each one $D_\lambda(t,s)$ and $L_\lambda(t,s)$.

From de last theorem, we have the following corollary.

Corollary 2.2. *Let $A(t)$ be the generator of the C-quasi-semigroup $\{U(t,s)\}_{t,s\geq 0}$ such that $A(t)$ is closed and densely defined. Then for all $t \geq s > 0$, $x \in R^{\infty}(C)$ and $\lambda \in \mathbb{C}$, we have*

(1) *For all $n \in \mathbb{N}^*$, there exists an operator $\alpha_{\lambda,n}(t,s)$ such that,*

$$(\lambda I - A(t))^n [C^{-1} L_\lambda(t,s)]^n x + D_\lambda(t,s)\alpha_{\lambda,n}(t,s)x = x.$$

Moreover, the operator $\alpha_{\lambda,n}(t,s)$ is commute with each one of $D_\lambda(t,s)$ and $L_\lambda(t,s)$.

(2) *For all $n \in \mathbb{N}^*$, there exists an operator $\beta_{\lambda,n}(t,s)$ such that,*

$$(\lambda I - A(t))^n \beta_{\lambda,n}(t,s)x + [\alpha_{\lambda,n}(t,s)]^n [D_\lambda(t,s)]^n x = x.$$

Furthermore, , the operator $\beta_{\lambda,n}(t,s)$ is commute with each one of $D_\lambda(t,s)$ and $\alpha_{\lambda,n}(t,s)$.

Proof. (1) Let $n \in \mathbb{N}^*$, Then, from lemma 2.2, for all $\lambda \in \mathbb{C}^\star$, $x \in R^{\infty}(C)$ and $t,s > 0$, we have

$$(\lambda I - A(t))C^{-1}L_\lambda(t,s)x + \frac{1}{s}e^{-\lambda s}C^{-1}D_\lambda(t,s)x = x$$

and

$$\begin{aligned}
[(\lambda I - A(t))C^{-1}L_\lambda(t,s)]^n x &= [I - \frac{1}{s}e^{-\lambda s}C^{-1}D_\lambda(t,s)]^n x \\
&= \sum_{i=0}^{n} \binom{n}{i} [-\frac{1}{s}e^{-\lambda s}C^{-1}D_\lambda(t,s)]^i x \\
&= x + \sum_{i=1}^{n} \binom{n}{i} [-\frac{1}{s}e^{-\lambda s}C^{-1}D_\lambda(t,s)]^i x \\
&= x - D_\lambda(t,s)C^{-1} \sum_{i=1}^{n} \binom{n}{i} [\frac{1}{s}e^{-\lambda s}]^i [-C^{-1}D_\lambda(t,s)]^{i-1} x \\
&= x - D_\lambda(t,s)\alpha_{\lambda,n}(t,s)x
\end{aligned}$$

Finally, we have

$$(\lambda I - A(t))^n [C^{-1} L_\lambda(t,s)]^n x + D_\lambda(t,s)\alpha_{\lambda,n}(t,s)x = x,$$

With

$$\alpha_{\lambda,n}(t,s) = C^{-1} \sum_{i=1}^{n} \binom{n}{i} [\frac{1}{s}e^{-\lambda s}]^i [-C^{-1}D_\lambda(t,s)]^{i-1}.$$

On the other hand , for commutativity, it's clear from lemma 2.2.

(2) According to (1), we have for all $n \in \mathbb{N}^*$ and $x \in R^{\infty}(C)$,

$D_\lambda(t,s)\alpha_{\lambda,n}(t,s)x = x - (\lambda I - A(t))^n[C^{-1}L_\lambda(t,s)]^n x$, then

$$\begin{aligned}[D_\lambda(t,s)\alpha_{\lambda,n}(t,s)]^n x &= \Big[x - (\lambda I - A(t))^n[C^{-1}L_\lambda(t,s)]^n\Big]^n x\\ &= x - \sum_{i=1}^{n}\binom{n}{i}\Big[(\lambda I - A(t))^n[C^{-1}L_\lambda(t,s)]^n\Big]^i x\\ &= x - (\lambda I - A(t))^n\sum_{i=1}^{n}\binom{n}{i}\Big[(\lambda I - A(t))^{n(i-1)}[C^{-1}L_\lambda(t,s)]^{ni}x\\ &= x - (\lambda I - A(t))^n\beta_{\lambda,n}(t,s)x,\end{aligned}$$

Therefore, we obtain

$$[D_\lambda(t,s)]^n[\alpha_{\lambda,n}(t,s)]^n x + (\lambda I - A(t))^n\beta_{\lambda,n}(t,s)x = x,$$

where $\beta_{\lambda,n}(t,s) = \sum_{i=1}^{n}\binom{n}{i}(\lambda I - A(t))^{n(i-1)}[C^{-1}L_\lambda(t,s)]^{ni}$.

□

2.2. **Spectral Inclusion For Saphar Spectrum.**

Let X, Y be Banach spaces, let $T : X \to Y$ be an operator. T has a generalized inverse [10] if and only if there exists an operator $S : Y \to X$ such that $TST = T$. A closed operator T is called Saphar, in symbol $T \in \mathcal{S}(X)$, if T has a generalized inverse and $Ker(T) \subseteq R^\infty(T)$.

For the subspaces E and F of X, we say that E is essentially contained in F and we write $E \subseteq_e F$, if there exists a finite-dimensional subspace $G \subseteq X$ such that $E \subseteq F + G$.

A closed operator T is called essentially Saphar, in symbol $T \in e\mathcal{S}(X)$, if T has a generalized inverse and $Ker(T) \subseteq_e R^\infty(T)$.

The Saphar and essentially Saphar spectra are defined by

$$\sigma_S(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{S}(X)\}, \quad \sigma_{eS}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin e\mathcal{S}(X)\}.$$

During this article we define the spectra $\sigma_\star(C,T)$ by replacing the identity I by an injective bounded operator $C \in B(X)$.

Theorem 2.1. *Let $A(t)$ be the generator of a $C-$quasi-semigroup $(U(t,s))_{t,s\geq 0}$ on a Banach space X such that $A(t)$ is closed and densely defined. For all $t,s > 0$, we have*

$$e^{s\sigma_S(A(t))} \subseteq \sigma_S(C, U(t-s,s)).$$

Proof. Let $\lambda \in \mathbb{C}$ such that $e^{\lambda s}C - U(t-s,s)$ is a Saphar operator, so $e^{\lambda s}C - U(t-s,s)$ has a generalized inverse and $Ker(e^{\lambda s}C - U(t-s,s)) \subseteq R^\infty[e^{\lambda s}C - U(t-s,s)]$.

We show that $\lambda I - A(t)$ is a Saphar operator.

Since $e^{\lambda s}C - U(t-s,s)$ has a generalized inverse, then there exists an operator $S : X \to X$ such that,

$$(e^{\lambda s}C - U(t-s,s))S(e^{\lambda s}C - U(t-s,s)) = e^{\lambda s}C - U(t-s,s).$$

Then, from lemma 2.1 and corollary 2.2, for all $\lambda \in \mathbb{C}$, $x \in R(C)$ and for all $t, s > 0$, we have

$$(\lambda I - A(t))C^{-1}L_\lambda(t,s)x + \frac{1}{s}e^{-\lambda s}C^{-1}D_\lambda(t,s)x = x$$

and

$$\begin{aligned} & (\lambda I - A(t)) \\ = \; & (\lambda I - A(t))C^{-1}L_\lambda(t,s)(\lambda I - A(t)) + \frac{1}{s}e^{-\lambda s}C^{-1}D_\lambda(t,s)(\lambda I - A(t)) \\ = \; & (\lambda I - A(t))C^{-1}L_\lambda(t,s)(\lambda I - A(t)) + \frac{1}{s}e^{-\lambda s}C^{-1}(e^{\lambda s}C - U(t-s,s)) \\ = \; & (\lambda I - A(t))C^{-1}L_\lambda(t,s)(\lambda I - A(t)) + \frac{1}{s}e^{-\lambda s}C^{-1}(e^{\lambda s}C - U(t-s,s))S(e^{\lambda s}C - U(t-s,s)) \\ = \; & (\lambda I - A(t))C^{-1}L_\lambda(t,s)(\lambda I - A(t)) + \frac{1}{s}e^{-\lambda s}C^{-1}(\lambda I - A(t))D_\lambda(t,s)S(\lambda I - A(t))D_\lambda(t,s) \\ = \; & (\lambda I - A(t))[C^{-1}L_\lambda(t,s) + \frac{1}{s}e^{-\lambda s}C^{-1}D_\lambda(t,s)SD_\lambda(t,s)](\lambda I - A(t)) \end{aligned}$$

So, $\lambda I - A(t)$ has a generalized inverse.

On the other hand, from corollary 2.1, we have

$$Ker(\lambda I - A(t)) \subseteq Ker(e^{\lambda s}C - U(t-s,s)) \subseteq R^\infty[e^{\lambda s}C - U(t-s,s)] \subseteq R^\infty(\lambda I - A(t))$$

Consequently, $\lambda I - A(t)$ essentially Saphar.

□

Corollary 2.3. *For all $t, s > 0$, we have*

$$e^{s\sigma_{eS}(A(t))} \subseteq \sigma_{eS}(C, U(t-s,s)).$$

Proof. Let $\lambda \in \mathbb{C}$ such that $e^{\lambda s}C - U(t-s,s)$ is a essentially Saphar operator, so $e^{\lambda s}C - U(t-s,s)$ has a generalized inverse and $Ker(e^{\lambda s}C - U(t-s,s)) \subseteq_e R^\infty[e^{\lambda s}C - U(t-s,s)]$.

According to the theorem 2.1, $\lambda I - A(t)$ has a generalized inverse.

In addition, let G a finite dimensional subspace of X. We have,

$$Ker(\lambda I - A(t)) \subseteq Ker(e^{\lambda s}C - U(t-s,s)) \subseteq R^\infty[e^{\lambda s}C - U(t-s,s)] + G \subseteq R^\infty(\lambda I - A(t)) + G.$$

Hence $\lambda I - A(t)$ essentially Saphar.

□

2.3. **Spectral Inclusion For Kato Spectrum.**

A closed operator T is called Kato, in symbol $T \in \mathcal{D}(X)$, if $R(T)$ is closed and $Ker(T) \subseteq R^\infty(T)$.

A closed operator T is called essentially Kato, in symbol $T \in e\mathcal{D}(X)$, if $R(T)$ is closed and $Ker(T) \subseteq_e R^\infty(T)$.

The Kato and essentially Kato spectra are defined by

$$\sigma_K(T) = \{\lambda \in \mathbb{C} \,:\, \lambda I - T \notin \mathcal{D}(X)\}, \quad \sigma_{eK}(T) = \{\lambda \in \mathbb{C} \,:\, \lambda I - T \notin e\mathcal{D}(X)\}.$$

To obtain results concerning the Kato spectrum and essentially Kato spectrum , we start with the following proposition.

Proposition 2.1. *Let $A(t)$ be the generator of a $C-$quasi-semigroup $(U(t,s))_{t,s\geq 0}$ on a Banach space X such that $A(t)$ is closed and densely defined. For all $p \in \mathbb{N}^*$, if $R[e^{\lambda s}C - U(t-s,s)]^p$ is closed, then $R[\lambda I - A(t)]^p$ is also closed.*

Proof. Let $(y_n)_{n\in\mathbb{N}} \subseteq X$ such that $y_n \to y \in X$ and there exists $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ satisfying

$$(\lambda I - A(t))^p x_n = y_n.$$

By corollary 2.2, for all $p \in \mathbb{N}^*$, there exists $\alpha_{\lambda,p}(t,s)$ and $\beta_{\lambda,p}(t,s)$ such that

$$(\lambda I - A(t))^p \beta_{\lambda,p}(t,s) + [\alpha_{\lambda,p}(t,s)]^p [D_\lambda(t,s)]^p = I.$$

Hence, we conclude that

$$\begin{aligned} y_n &= (\lambda I - A(t))^p \beta_{\lambda,p}(t,s) y_n + [\alpha_{\lambda,p}(t,s)]^p [D_\lambda(t,s)]^p y_n \\ &= (\lambda I - A(t))^p \beta_{\lambda,p}(t,s) y_n + [\alpha_{\lambda,p}(t,s)]^p [D_\lambda(t,s)]^p (\lambda I - A(t))^p x_n \\ &= (\lambda I - A(t))^p \beta_{\lambda,p}(t,s) y_n + [\alpha_{\lambda,p}(t,s)]^p [e^{\lambda s}C - U(t-s,s)]^p x_n \end{aligned}$$

So,

$$[e^{\lambda s}C - U(t-s,s)]^p [\alpha_{\lambda,p}(t,s)]^p x_n = y_n - (\lambda I - A(t))^p \beta_{\lambda,p}(t,s) y_n$$

Thus,

$$y_n - (\lambda I - A(t))^p \alpha_{\lambda,p}(t,s) y_n \in R[e^{\lambda s}C - U(t-s,s)].$$

Moreover, since $R[e^{\lambda s}C - U(t-s,s)]^p$ is closed, and

$$y_n - (\lambda I - A(t))^p \alpha_{\lambda,p}(t,s) y_n \to y - (\lambda I - A(t))^p \alpha_{\lambda,p}(t,s) y,$$

we conclude that

$$y - (\lambda I - A(t))^p \alpha_{\lambda,p}(t,s) y \in R[e^{\lambda s}C - U(t-s,s)]^p.$$

Then there exists $z \in R(C)$ such that

$$[e^{\lambda s}C - U(t-s,s)]^p z = y - (\lambda I - A(t))^p \alpha_{\lambda,p}(t,s) y.$$

Hence, we have

$$\begin{aligned} y &= [e^{\lambda s}C - U(t-s,s)]^p z + (\lambda I - A(t))^p \alpha_{\lambda,p}(t,s) y; \\ &= (\lambda I - A(t))^p D_\lambda(t,s)^p z + (\lambda I - A(t))^p \alpha_{\lambda,p}(t,s) y; \\ &= (\lambda I - A(t))^p [D_\lambda(t,s)^p z + \alpha_{\lambda,p}(t,s) y. \end{aligned}$$

Finally, we obtain

$$y \in R(\lambda I - A(t))^p.$$

□

Corollary 2.4. *For all $t, s > 0$, we have*

$$e^{s\sigma_K(A(t))} \subseteq \sigma_K(C, U(t-s,s)) \ , \ e^{s\sigma_{eK}(A(t))} \subseteq \sigma_{eK}(C, U(t-s,s)).$$

Proof. It is automatic due to the theorem 2.1 and proposition 2.1. □

2.4. **Spectral Inclusion For Quasi-Fredholm Spectrum.**

The degree of stable iteration $dis(T)$ of an operator T is defined by

$$dis(T) = inf\{n \in \mathbb{N} \ : \ \forall m \geq n, R(T^n) \cap Ker(T) = R(T^m) \cap Ker(T)\},$$

with $inf(\emptyset) = \infty$.

A closed operator T is called quasi-Fredholm [3], in symbol $T \in q\Phi(X)$, if there is $d \in \mathbb{N}$ such that for all $n \geq d$, $R(T^n)$ and $R(T) + Ker(T^n)$ are closed and $dis(T) = d$.

The quasi-Fredholm spectrum is defined by

$$\sigma_{qe}(T) = \{\lambda \in \mathbb{C} \ : \ \lambda I - T \notin q\Phi(X)\}.$$

We start by the following proposition.

Proposition 2.2. *Let $A(t)$ be a closed and densely defined generator of a $C-$quasi-semigroup $(U(t,s))_{t,s\geq 0}$ on a Banach space X. Then $dis(\lambda I - A(t)) \leq dis(e^{\lambda s}C - U(t-s,s))$.*

Proof.

- If $dis(e^{\lambda s}C - U(t-s,s)) = +\infty$. So, the result is evident.
- If $d = 0$, for all $n \geq d$, we have

$$R(e^{\lambda s}C - U(t-s,s))^n \cap Ker(e^{\lambda s}C - U(t-s,s)) = Ker(e^{\lambda s}C - U(t-s,s)).$$

Consequently,

$$Ker(e^{\lambda s}C - U(t-s,s)) \subseteq R(e^{\lambda s}C - U(t-s,s))^n.$$

According to corollary 2.1, we obtain

$$Ker(\lambda I - A(t)) \subseteq Ker(e^{\lambda s}C - U(t-s,s)) \subseteq R(e^{\lambda s}C - U(t-s,s))^n \subseteq R(\lambda I - A(t))^n.$$

Therefore, $Ker(\lambda I - A(t)) \cap R(\lambda I - A(t))^n = Ker(\lambda I - A(t)) \cap R(\lambda I - A(t))^0$

Consequently, $dis(\lambda - A(t)) = 0$.

- If $dis(e^{\lambda s}C - U(t-s,s)) = d \in \mathbb{N}^\star$. Then for all $n \geq d$,

$$R\left(e^{\lambda s}C - U(t-s,s)\right)^n \cap Ker\left(e^{\lambda s}C - U(t-s,s)\right) = R\left(e^{\lambda s}C - U(t-s,s)\right)^d \cap Ker\left(e^{\lambda s}C - U(t-s,s)\right).$$

Therefore, it's enough to show that for all $n \geq d$,

$$R(\lambda I - A(t))^n \cap Ker(\lambda I - A(t)) = R(\lambda I - A(t))^d \cap Ker(\lambda I - A(t)).$$

Let $y \in R(\lambda I - A(t))^d \cap Ker(\lambda I - A(t))$, then there exists $x \in \mathcal{D}^d$ such that

$$y = (\lambda I - A(t))^d x \text{ and } (\lambda I - A(t))y = 0.$$

Hence, from corollary 2.2, we have

$$(\lambda I - A(t))^d \beta_{\lambda,d}(t,s) + [\alpha_{\lambda,d}(t,s)]^d [D_\lambda(t,s)]^d = I.$$

Therefore,

$$\begin{aligned} y &= (\lambda I - A(t))^d \beta_{\lambda,d}(t,s) y + [\alpha_{\lambda,d}(t,s)]^d [D_\lambda(t,s)]^d y \\ &= (\lambda I - A)^{d-1} \beta_{\lambda,d}(t,s)(\lambda I - A(t)) y + (e^{\lambda s}C - U(t-s,s))^d [\alpha_{\lambda,d}(t,s)]^d x \\ &= \left(e^{\lambda s}C - U(t-s,s)\right)^d [\alpha_{\lambda,d}(t,s)]^d x. \end{aligned}$$

Then, according to (3) of corollary 2.1 we have

$$y \in R\left(e^{\lambda s}C - U(t-s,s)\right)^d \cap Ker\left(e^{\lambda s}C - U(t-s,s)\right) \subseteq R\left(e^{\lambda s}C - U(t-s,s)\right)^n \subseteq R(\lambda I - A(t))^n.$$

So, $y \in R(\lambda I - A(t))^n \cap Ker(\lambda I - A(t))$.
Therefore, $R(\lambda I - A(t))^n \cap Ker(\lambda I - A(t)) = R(\lambda I - A(t))^d \cap Ker(\lambda I - A(t))$.
Consequently, $dis(\lambda I - A(t)) \leq d$

□

The following theorem provides the spectral inclusion between the quasi-Fredholm spectrum of C-quasi-semigroup and the quasi-Fredholm spectrum of its generator.

Theorem 2.2. *Let $A(t)$ be a closed and densely defined generator of a $C-$quasi-semigroup $(U(t,s))_{t,s\geq 0}$ on a Banach space X such that $\mathcal{D} \subset R^\infty(C)$. For all $t,s \geq 0$, we have*

$$e^{s\sigma_{qF}(A(t))} \subseteq \sigma_{qF}(C, U(t-s,s)).$$

Proof. according to the previous proposition, it remains to show that
if $R(e^{\lambda s}C - U(t-s,s)) + Ker(e^{\lambda s}C - U(t-s,s))^n$ is closed in X then $R(\lambda I - A(t)) + Ker(\lambda I - A(t))^n$ is closed .

Suppose that $R(e^{\lambda s}C - U(t-s,s)) + Ker(e^{\lambda s}C - U(t-s,s))^n$ is closed in X.
Let $y_n = (\lambda I - A(t))x_n + z_n$ be a sequence which converges to y, with $x_n \in \mathcal{D}$ and $z_n \in Ker(\lambda I - A(t))^n$.

So,

$$D_\lambda^n(t,s)y_n = D_\lambda^n(t,s)(\lambda I - A(t))x_n + D_\lambda^n(t,s)z_n \in R(e^{\lambda s}C - U(t-s,s)) + Ker(e^{\lambda s}C - U(t-s,s))^n$$

Since ,$R(e^{\lambda s}C - U(t-s,s)) + Ker(e^{\lambda s}C - U(t-s,s))^n$ is closed, then

$$D_\lambda^n(t,s)y \in R(e^{\lambda s}C - U(t-s,s)) + Ker(e^{\lambda s}C - U(t-s,s))^n$$

There exist $x \in X$ and $z \in Ker(e^{\lambda s}C - U(t-s,s))^n$ such that $D_\lambda^n(t,s)y = (e^{\lambda s}C - U(t-s,s))x + z$.
So, $D_\lambda^{2n}(t,s)y = D_\lambda^n(t,s)(e^{\lambda s}C - U(t-s,s))x + D_\lambda^n(t,s)z$.

According to 2.2, for all $n \in \mathbb{N}^*$ we have

$$\begin{aligned} y &= (\lambda I - A(t))^{2n}\beta_{\lambda,n}(t,s)y + [\alpha_{\lambda,n}(t,s)]^{2n}[D_\lambda(t,s)]^{2n}y \\ &= (\lambda I - A(t))^{2n}\beta_{\lambda,n}(t,s)y + [\alpha_{\lambda,n}(t,s)]^{n}D_\lambda^n(t,s)\left(e^{\lambda s}C - U(t-s,s)\right)x + [\alpha_{\lambda,n}(t,s)]^{n}D_\lambda^n(t,s)z \\ &= (\lambda I - A(t))\left[(\lambda I - A(t))^{2n-1}\beta_{\lambda,n}(t,s)y + [\alpha_{\lambda,n}(t,s)]^{n}D_\lambda^{n+1}(t,s)x\right] + [\alpha_{\lambda,n}(t,s)]^{n}D_\lambda^n(t,s)z. \end{aligned}$$

Hence, $y \in R(\lambda I - A(t)) + Ker(\lambda I - A(t))^n$.

Finally, $R(\lambda I - A(t)) + Ker(\lambda I - A(t))^n$ is closed □

Acknowledgments: A huge thank you to Professor Abdelaziz Tajmouati for exceptional guidance throughout my academic journey, from undergraduate to doctoral studies. His influence has been invaluable and will be etched in my academic success.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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