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Convergence of Modified S-Iteration to Common Fixed Points of Asymptotically Nonexpansive Mappings in Hyperbolic Spaces

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Abstract. In this paper, we provide some sufficient conditions for the strong convergence of an improved form of modified S-iteration for approximating common fixed points of two asymptotically nonexpansive mappings defined on a closed convex subset of a uniformly convex hyperbolic spaces.

1. Introduction

Let X be a Banach space and K be a closed convex subset of X. A mapping $T : K \to K$ is said to be an asymptotically nonexpansive mapping if there is a real sequence $\{k_n\}$ such that $\lim_{n\to\infty} k_n = 1$ and $\|T^n x - T^n y\| \leq k_n \|x - y\|$ $\forall x, y \in X$. Asymptotically nonexpansive mapping is a natural generalization of non- expansive mapping $(\|T x - T y\| \leq \|x - y\|)$.

In 1972, Gobel and Kirk [1] proved that if T is a asymptotically non- expansive self mapping on a closed bounded convex subset of a uniformly convex Banach space, then T has a fixed point. This result paved a way for a new development in the field of metric fixed point theory. J. Schu [2] initiated the study of convergence of iteration process to a fixed point of asymptotically nonexpansive mapping by considering the following modified Mann iteration.

$$x_1 \in K,\ x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, T^n x_n,\ \ n \geq 1$$

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Since then, many researchers have studied iteration process to approximate fixed point of asymptotically nonexpansive mappings (see [3–8]). K. K. Tan and Xu [9] considered the convergence of modified Ishikwa iteration process

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying some suitable conditions such as being bounded away from 0 and 1. Modified Ishikwa iteration process reduces to modified Mann iteration process when $\beta_n = 0$. Recently, Agarwal et.al. [1], in an attempt to obtain a faster rate of convergence, introduced the following iteration called modified S-iteration process

$$\begin{aligned} x_1 &\in K \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n T^n y_n \\ y_n &= (1-\alpha_n)\, x_n + \alpha_n T^n x_n. \end{aligned} \tag{1.1}$$

In the past decade or so, the study of iterative approximation in the setting of hyperbolic metric spaces has gained considerable momentum. The reason being hyberbolic metric spaces allows us to retain the essential aspects of metric spaces while endowed with a rich linear affine structure. There are several variations of hyberbolic spaces defined in the literature. But the following version defined by Kohlenbach appears to be the most standard definition.

Definition 1.1. *[10] A hyberbolic space (X, d, W) is a triple where (X, d) is a metric space and the mapping W is defined as, $W : X \times X \times [0, 1] \to X$ satisfying the following conditions:*

(1) $d(t, W(x, y, \alpha)) \leq (1-\alpha)d(t, x) + \alpha d(t, y)$;
(2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;
(3) $W(x, y, \alpha) = W(y, x, 1-\alpha)$;
(4) $d(W(x, z, \alpha), W(y, s, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, s)$ *for all* $x, y, z, s, t \in W$ *and* $\alpha, \beta \in [0, 1]$.

The iteration process Equation (1.1) is adopted to Hyberbolic spaces setting by Sahin and Basinir [11]

$$\begin{aligned} x_1 &\in K \\ x_{n+1} &= W(T^n x_n, T^n y_n,\, \alpha_n) \\ y_n &= \ (x_n,\, T^n x_n,\, \beta_n). \end{aligned} \tag{1.2}$$

G. S. Saluja [12] modified Equation (1.2) to approximate common fixed point of two mappings.

$$\begin{aligned} x_1 &\in K \\ x_{n+1} &= W(T^n x_n, S^n y_n,\, \alpha_n) \\ y_n &= \ W(x_n,\, T^n x_n,\, \beta_n), \end{aligned} \tag{1.3}$$

where S, $T : K \to K$ are asymptotically quasi nonexpansive mappings.

Motivated by the above results, we define the following iteration and discuss its convergence properties:

$$\begin{aligned} x_1 &\in K \\ x_{n+1} &= W(S^n x_n, T^n y_n, \alpha_n) \\ y_n &= W(S^n x_n, T^n x_n, \beta_n) \end{aligned} \tag{1.4}$$

2. Preliminaries

We recall some basic concepts related to our results.

Let $T : K \to X$ where K is a nonempty subset of a metric space X. Then $F(T) = \{x \in K : T(x) = x\}$. The map T is said to be demicompact if for every bounded sequence $\{x_n\}$ in K such that $d(x_n, Tx_n)$ is convergent, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ convergent in K.
A subset A of a hyperbolic space (X, d, W) is said to be convex if for any $\lambda \in (0, 1)$

$$W(x, y, \lambda) \in A, \text{ for all } x, y \in A.$$

Definition 2.1. *[13] Let (X, d, W) be a hyperpolic space, if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $u, x, y \in X$,*

$$d(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r,$$

whenever $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \epsilon r$, then the given hyperbolic space is said to be an uniformly convex hyperbolic space.

The following lemma is often used in our results.

Lemma 2.1. *[14] Let (X, d, W) be an uniformly convex hyperbolic space. Let $a \in X$ and t_n be a sequence in $[\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, \frac{1}{2}]$. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in (X, d, W) such that*

$$\limsup_{n\to\infty} d(x_n, a) \leq r \ , \ \limsup_{n\to\infty} d(y_n, a) \leq r \text{ and } \lim_{n\to\infty} d(W(x_n, y_n, t_n), a) = r,$$

for some $r > 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 2.2. *[15] Let $\{t_n\}, \{s_n\}$ and $\{r_n\}$ be any three non-negative real sequences such that the condition*

$$t_{n+1} \leq (1 + s_n)t_n + r_n$$

is satisfied with $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$. Then $\lim_{n\to\infty} t_n$ exists.

3. Main Results

Lemma 3.1. *Let K be a non-empty closed convex subset of a uniformly convex hyperbolic space* (X, d, W) *and let* $S, T : K \to K$ *be asymptotically nonexpansive mappings with a sequence* $\{k_n\}$ *satisfying* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$*, and assume that* $F = F(S) \cap F(T) \neq \phi$*. Suppose* $\{x_n\}$ *is defined by (1.4), then* $\lim_{n\to\infty} d(x_n, p)$ *and* $\lim_{n\to\infty} d(y_n, p)$ *exist for each* $p \in F$*.*

Proof. By definition of x_n and y_n, we have

$$\begin{aligned} d(y_n, p) &= d(W(S^n x_n, T^n x_n, \beta_n), p) \\ &\leq (1-\beta_n)\, d(S^n x_n, p) + \beta_n\, d(T^n x_n, p) \\ &\leq (1-\beta_n)\, k_n\, d(x_n, p) + \beta_n\, k_n\, d(x_n, p) \\ &= k_n\, d(x_n, p) \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(W(S^n x_n, T^n y_n, \alpha_n), p) \\ &\leq (1-\alpha_n)\, d(S^n x_n, p) + \alpha_n\, d(T^n y_n, p) \\ &\leq (1-\alpha_n)\, k_n\, d(x_n, p) + \alpha_n\, k_n\, d(y_n, p) \\ &\leq (1-\alpha_n)\, k_n\, d(x_n, p) + \alpha_n\, k_n^2\, d(x_n, p) \\ &= (k_n - \alpha_n k_n + \alpha_n k_n^2)\, d(x_n, p) \\ &= (1 + (k_n - 1) + \alpha_n k_n (k_n - 1))\, d(x_n, p). \end{aligned}$$

Therefore

$$d(x_{n+1}, p) \leq (1 + s_n) d(x_n, p)$$

where $s_n = (k_n - 1) + \alpha_n k_n (k_n - 1)$. Using the fact that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, we conclude from Lemma (2.1) that $\lim_{n\to\infty} d(x_n, p)$ exists. Let $\lim_{n\to\infty} d(x_n, p) = c$, where $c \geq 0$. Also from the inequality,

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\alpha_n) k_n\, d(x_n, p) + \alpha_n\, k_n\, d(y_n, p) \\ &= d(x_n, p) + \alpha_n k_n (d(y_n, p) - d(x_n, p)) \end{aligned}$$

and taking lim sup on both sides, we have

$$\begin{aligned} c &= c + \limsup \alpha_n\, (d(y_n, p) - c) \\ &\leq c, \end{aligned}$$

since $\limsup d(y_n, p) \leq c$. Therefore,

$$\limsup \alpha_n (d(y_n, p) - c) = 0.$$

Since $\alpha_n \in [\epsilon, 1-\epsilon]$, we conclude that $\lim_{n\to\infty} d(y_n, p) = c$. □

Remark 3.1. *Without loss of generality we have considered the same sequence* $\{k_n\}$ *for both S and T.*

Theorem 3.1. *Let K be a non-empty closed convex subset of a uniformly convex hyperbolic space (X, d, W) and let $S, T : K \to K$ be asymptotically nonexpansive mappings with a sequence $\{k_n\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and assume that $F = F(S) \cap F(T) \neq \phi$. Let $\{x_n\}$ be defined as in (1.4). Then $\{x_n\}$ converges strongly to an element of F iff $\lim_{n\to\infty} \inf d(x_n, F) = 0$.*

Proof. Suppose $\{x_n\}$ converges to a common fixed point of S and T. then $d(x_n, F) \to 0$. Assume $\lim_{n\to\infty} \inf d(x_n, F) = 0$. From the proof of Lemma (3.1)

$$d(x_{n+1}, p) \leq (1 + s_n)\, d(x_n, p)$$

where $s_n = (k_n - 1) + \alpha_n k_n (k_n - 1)$. Taking infimum over $p \in F$

$$d(x_{n+1}, F) \leq (1 + s_n)\, d(x_n, F). \tag{3.1}$$

Hence $\lim_{n\to\infty} d(x_n, F)$ converges by Lemma (3.1). So $\lim_{n\to\infty} d(x_n, F) = 0$ and there exists a positive integer N such that

$$d(x_n, F) < \frac{\epsilon}{4}, \ \forall\ n \geq N.$$

That is, $\inf\{d(x_n, p) : p \in F\} < \frac{\epsilon}{4}$ and we can find a $p \in F$ such that $d(x_n, p) < \frac{\epsilon}{2}$ for every $n \geq N$.

Let $n \geq N$ and $m > 0$, from (3.1) we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p) + d(p, x_n) \\ &\leq (1 + s_{n+m-1})(1 + s_{n+m-2}) \cdots (1 + s_N)\, d(x_N, p) \\ &+ (1 + s_{n-1})(1 + s_{n-2}) \cdots (1 + s_N)\, d(x_N, p). \end{aligned}$$

Since $\sum_{n=1}^{\infty} s_n < \infty$, the infinite product $\prod_{n=1}^{\infty}(1 + s_n)$ converges to a positive real number k. Hence

$$d(x_{n+m}, x_n) \leq 2k\, d(x_N, p) < k\epsilon.$$

This shows that $\{x_n\}$ is a Cauchy sequence and hence converges to $x \in F$. □

Theorem 3.2. *Let K be a non-empty closed convex subset of a uniformly convex hyperbolic space. Let $S, T : K \to K$ be asymptotically non-expansive mappings with $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and $F = F(S) \cap F(T) \neq \phi$. Let $\{\alpha_n\}, \{\beta_n\} \in [\epsilon, 1 - \epsilon]$ for some $0 < \epsilon \leq \frac{1}{2}$. Suppose $\{x_n\}$ is defined by (3) and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(y_n, Ty_n) = \lim_{n\to\infty} d(y_n, Sy_n) = 0.$$

Further, if one of the maps is demi compact, then $\{x_n\}$ and $\{y_n\}$ strongly converge to an element x^* of $F(T) \cap F(S)$.

Proof. Let $p \in F$. It follows from Lemma (3.1) that

$$\lim_{n\to\infty} d(x_n, q) = \lim_{n\to\infty} d(y_n, q) = c.$$

Since $\lim_{n\to\infty} d(x_{n+1}, p) = c$, from the proof of Lemma (3.1) we have,

$$\lim_{n\to\infty} d\left(W(S^n x_n, T^n y_n, \alpha_n), p\right) = c. \tag{3.2}$$

Applying lim sup to the inequalities $d(S^n x_n, p) \leq k_n\, d(x_n, p)$ and $d(T^n y_n, p) \leq k_n d(y_n, p)$, we obtain,

$$\limsup_{n\to\infty} d(S^n x_n, p) \leq c \text{ and } \limsup_{n\to\infty} d(T^n y_n, p) \leq c.$$

From (3.2) and applying Lemma (2.1), we have, $d(S^n x_n, T^n y_n) \to 0$. Also from

$$\lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d\left(W(S^n x_n, T^n x_n, \beta_n), p\right) = c$$

and

$$\lim_{n\to\infty} \sup\ d(S^n x_n, p) \leq c, \lim_{n\to\infty} \sup d(T^n x_n, p) \leq c$$

again using Lemma (2.1), we conclude that

$$\lim_{n\to\infty} d(S^n x_n, T^n x_n) = 0 \tag{3.3}$$

Now

$$\begin{aligned} d(x_{n+1}, S^n x_n) &= d\left(W(S^n x_n, T^n y_n, \alpha_n), S^n x_n\right) \\ &\leq (1-\alpha_n) d(S^n x_n, S^n x_n) + \alpha_n d(T^n y_n, S^n x_n) \\ &= \alpha_n d(T^n y_n, S^n x_n). \end{aligned}$$

Since $d(S^n x_n, T^n y_n) \to 0$,

$$\lim_{n\to\infty} d(x_{n+1}, S^n x_n) = 0, \tag{3.4}$$

and from

$$\begin{aligned} d(x_{n+1}, T^n y_n) &= d\left(W(S^n x_n, T^n y_n, \alpha_n), T^n y_n\right) \\ &\leq (1-\alpha_n) d(S^n x_n, T^n y_n) \end{aligned}$$

we have

$$\lim_{n\to\infty} d(x_{n+1}, T^n y_n) = 0. \tag{3.5}$$

Consider

$$\begin{aligned} d(y_n, S^n x_n) &= d\left(W(S^n x_n, T^n y_n, \beta_n), S^n x_n\right) \\ &\leq \alpha_n d(T^n x_n, S^n x_n) \end{aligned}$$

Using (3.3), we have

$$\lim_{n\to\infty} d(y_n, S^n x_n) = 0. \tag{3.6}$$

Similarly we can show that

$$\lim_{n\to\infty} d(y_n, T^n x_n) = 0. \tag{3.7}$$

Also

$$d(x_{n+1}, T^n x_n) \leq d(x_{n+1}, S^n x_n) + d(S^n x_n, T^n x_n).$$

From (3.3) and (3.4), we have $d(x_{n+1}, T^n x_n) \to 0$ and from (3.7),

$$d(x_{n+1}, y_n) \to 0. \tag{3.8}$$

The following can be deduced from the above assertions

$$\lim_{n\to\infty} d(T^n x_n, T^n y_n) = \lim_{n\to\infty} d(x_{n+1}, T^n y_n) = 0. \tag{3.9}$$

Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and

$$d(x_n, T^n x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^n x_n),$$

it follows that $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$. Further,

$$d(x_n, y_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) \to 0 \tag{3.10}$$

$$d(x_n, S^n x_n) \leq d(x_n, y_n) + d(y_n, S^n x_n) \to 0 \tag{3.11}$$

and

$$d(y_n, T^n y_n) \leq d(y_n, S^n x_n) + d(S^n x_n, T^n y_n).$$

Using (7) and (8), we have $d(y_n, T^n y_n) \to 0$. Also,

$$d(y_n, y_{n+1}) \leq d(y_n, x_{n+1}) + d(x_{n+1}, y_{n+1}).$$

From (3.8) and (3.10), $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Now

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + k_{n+1}d(x_n, x_{n+1}) + k_1 d(T^n x_n, x_n). \end{aligned}$$

As $n \to \infty$, each of the sequences in R.H.S tends to zero.

Therefore $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. In a similar fashion, we can show that $\lim_{n\to\infty} d(y_n, Ty_n) = 0$. Also, using (3.11) and $d(x_n, x_{n+1}) \to 0$, we see that $d(x_n, Sx_n) \to 0$. Since $d(Sy_n, Sx_n) \leq k_1 d(y_n, x_n)$ we have $d(Sx_n, Sy_n) \to 0$.

$$\begin{aligned} d(y_n, Sx_n) &\leq d(y_n, x_{n+1}) + d(x_{n+1}, S^{n+1}x_{n+1}) + d(S^{n+1}x_{n+1}, S^{n+1}x_n) + d(S^{n+1}x_n, Sx_n) \\ &\leq d(y_n, x_{n+1}) + d(x_{n+1}, S^{n+1}x_{n+1}) + k_{n+1}d(x_{n+1}, x_n) + k_1 d(Sx_n, x_n), \end{aligned}$$

which implies $d(y_n, Sx_n) \to 0$. Combining with $d(Sx_n, Sy_n) \to 0$, we have $d(y_n, Sy_n) \to 0$.

Assume T is demi-compact. Since $d(x_n, Tx_n) \to 0$ there exists $\{x_{n_i}\}$ such that $Tx_{n_i} \to x^*$

$$d(x_{n_i}, x^*) \leq d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, x^*) \to 0 \text{ as } i \to \infty,$$

since $\lim_{n\to\infty} d(x_n, Tx_n) \to 0$, we have $x^* \in F(T)$. Also, $\lim_{n\to\infty} d(x_n, x^*)$ exists. Therefore

$$x_n \to x^* \text{ and } d(x_n, y_n) \to 0 \text{ implies that } y_n \to x^*.$$

Further, $d(x_n, Sx_n) \to 0$ implies that $x^* \in F(S)$. Hence x_n and y_n strongly converge to $x^* \in F(T) \cap F(S)$. □

The following example illustrates our results.

Example 3.1. *Consider $K = B(0:0.9)$, a subset of $\left(\mathbb{R}^2, \|\|_2\right)$, the ball with center 0 and radius 0.9. Define $T : K \to K$ and $S : K \to K$ by*

$$T(x,y) = (x^2, 0) \text{ and } S(x,y) = (0, y^2).$$

Clearly both T and S are not nonexpansive. Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in K$. Then

$$T^n(x_1,x_2) = (x_1^{2^n}, 0) \text{ and } S^n(x_1,x_2) = (0, x_2^{2^n}).$$

If $y_1 \leq x_1$, then

$$\begin{aligned}\|T^n x - T^n y\|_2 &= \|(x_1^{2^n}, 0) - (y_1^{2^n}, 0)\|_2 \\ &= |x_1^{2^n} - y_1^{2^n}| \\ &= |x_1 - y_1|\,[x_1^{2^n-1} + y_1 x_1^{2^n-2} + \cdots + y_1^{2^n-1}] \\ &\leq \|x - y\|_2\,[2^n x_1^{2^n-1}].\end{aligned}$$

Let $k_n = \max\{1, 2^n x_1^{2^n-1}\} \leq \max\{1, 2^n (0.9)^{2^n-1}\}$. So $k_n \to 1$ and hence T is a asymptotically nonexpansive mapping. Similarly, S is also asmptotically nonexpansive. Now, for any $x_1 \in K$, define the sequence $\{x_n\}$ by

$$x_{n+1} = W(S^n x_n, T^n y_n, \alpha_n) \text{ and } y_n = W(S^n x_n, T^n x_n, \beta_n).$$

Let $x_n = (x_{n_1}, x_{n_2})$, $x_{n+1} = (x_{(n+1)_1}, x_{(n+1)_2})$ and $y_n = (y_{n_1}, y_{n_2})$ and $\alpha_n = \beta_n = \frac{1}{2}$ for all $n \in N$. Then

$$(x_{(n+1)_1}, x_{(n+1)_2}) = \frac{1}{2}(0, x_{n_2}^{2^n}) + \frac{1}{2}(y_{n_1}^{2^n}, 0) = \frac{1}{2}(y_{n_1}^{2^n}, x_{n_2}^{2^n})$$

and $(y_{n_1}, y_{n_2}) = \frac{1}{2}(x_{n_1}^{2^n}, x_{n_2}^{2^n})$.

Thus $y_{n_1} = \frac{1}{2} x_{n_1}^{2^n}$ and $(x_{(n+1)_1}, x_{(n+1)_2}) = \frac{1}{2}\left(\frac{1}{2^{2^n}} x_{n_1}^{2^{2n}}, x_{n_2}^{2^n}\right)$. Clearly $x_n = (x_{n_1}, x_{n_2})$ converges to the common fixed point $(0,0)$ of S and T.

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References

[1] K. Goebel, W.A. Kirk, A Fixed Point Theorem for Asymptotically Nonexpansive Mappings, Proc. Amer. Math. Soc. 35 (1972), 171–174.

[2] J. Schu, Weak and Strong Convergence to Fixed Points of Asymptotically Nonexpansive Mappings, Bull. Austral. Math. Soc. 43 (1991), 153–159. https://doi.org/10.1017/s0004972700028884.

[3] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative Construction of Fixed Points of Nearly Asymptotically Nonexpansive Mappings, J. Nonlinear Convex Anal. 8 (2007), 61–79.

[4] C.E. Chidume, E.U. Ofoedu, H. Zegeye, Strong and Weak Convergence Theorems for Asymptotically Nonexpansive Mappings, J. Math. Anal. Appl. 280 (2003), 364–374. https://doi.org/10.1016/s0022-247x(03)00061-1.

[5] H. Fukhar-ud-din, S.H. Khan, Convergence of Iterates With Errors of Asymptotically Quasi-Nonexpansive Mappings and Applications, J. Math. Anal. Appl. 328 (2007), 821–829. https://doi.org/10.1016/j.jmaa.2006.05.068.

[6] L. Qihou, Iterative Sequences for Asymptotically Quasi-nonexpansive Mappings, J. Math. Anal. Appl. 259 (2001), 1–7. https://doi.org/10.1006/jmaa.2000.6980.

[7] W. Kaczor, Weak Convergence of Almost Orbits of Asymptotically Nonexpansive Commutative Semigroups, J. Math. Anal. Appl. 272 (2002), 565–574. https://doi.org/10.1016/s0022-247x(02)00175-0.

[8] D.R. Sahu, J.S. Jung, Fixed-Point Iteration Processes for Non-Lipschitzian Mappings of Asymptotically Quasi-Nonexpansive Type, Int. J. Math. Math. Sci. 2003 (2003), 2075–2081. https://doi.org/10.1155/s0161171203302182.

[9] K.K. Tan, H.K. Xu, Fixed Point Iteration Processes for Asymptotically Nonexpansive Mappings, Proc. Amer. Math. Soc. 122 (1994), 733–739.

[10] U. Kohlenbach, Some Logical Metatheorems With Applications in Functional Analysis, Trans. Amer. Math. Soc. 357 (2005), 89–128

[11] A. Şahin, M. Başarır, On the Strong Convergence of a Modified S-Iteration Process for Asymptotically Quasi-Nonexpansive Mappings in a CAT (0) Space, Fixed Point Theory Appl. 2013 (2013), 12. https://doi.org/10.1186/1687-1812-2013-12.

[12] G.S. Saluja, Convergence of Modified S-Iteration Process for Two Asymptotically Quasi-Nonexpansive Type Mappings in CAT(0) Spaces, Demonstr. Math. 49 (2016), 107–118. https://doi.org/10.1515/dema-2016-0010.

[13] L. Leuştean, Nonexpansive Iterations in Uniformly Convex W-Hyperbolic Spaces. In: A. Leizarowitz, B.S. Mordukhovich, I. Shafrir, A. Zaslavski, (eds.) Nonlinear Analysis and Optimization I: Nonlinear Analysis, American Mathematical Society, pp. 193–209, (2010).

[14] A.R. Khan, H. Fukhar-ud-din, M.A. Ahmad Khan, An Implicit Algorithm for Two Finite Families of Nonexpansive Maps in Hyperbolic Spaces, Fixed Point Theory Appl. 2012 (2012), 54. https://doi.org/10.1186/1687-1812-2012-54.

[15] K.K. Tan, H.K. Xu, Approximating Fixed Points of Nonexpansive Mappings by the Ishikawa Iteration Process, J. Math. Anal. Appl. 178 (1993), 301–308. https://doi.org/10.1006/jmaa.1993.1309.