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Application of an Ansatz Method on a Delay Model With a Proportional Delay Parameter

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Abstract. Delay differential equations are fundamental tools to modeling various real-world problems. A particular type of these models is considered in this paper in the form $y'(t) = ay(t) + ae^{at}y(at)$, where a is a proportional delay parameter. Solving delay equations is usually a difficult task. This is because there are no standard/well-known methods for solving such kind of equations. This paper proposes a simple procedure to solve the above delay equation. The solution is obtained in closed form which is optimal. The suggested analysis can be invested to analyze more complex models in physics and engineering sciences.

1. Introduction

The field of delay differential equations attracts the attention of many researchers in the past two decades. One can find a number of numerical methods to solve delay equations such as the Taylor method [1], the Chebyshev polynomial [2], the collocation method [3], the orthonormal Bernstein polynomials [4], the spectral methods [5], and the transferred Legendre pseudospectral method [6]. In addition, there are analytical methods such as the Adomian decomposition method (ADM) [7], the Homotopy Perturbation Method (HPM) [8], the direct ansatz method [9], and the Laplace Transform (LT) method [10]. Moreover, the authors [11,12] considered multi and generalized forms of delay models.

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In this paper, a particular type of delay equations with a proportional delay parameter a is considered in the form:

$$y'(t) = ay(t) + ae^{at}y(at), \quad y(0) = \lambda, \quad a \neq 0, \tag{1.1}$$

where a and λ are real constants. Although the just mentioned analytical methods, i.e., the ADM the HPM, and the LT showed accurate results when solving the standard pantograph delay model and other scientific models in physics/engineering/medical sciences [13-27], their applications to the present model may face some difficulties due to the existence of the exponential term. The use of the ADM encounters the difficulty of calculating the Adomian polynomials while the HPM forces us to implement an auxiliary parameter. Beside, the ADM and the HPM are based on putting the equation being solved in the canonical form which may not be effective to reach the desired solution. For example, the authors [7,8,13,14] showed that the canonical form of an equation can be expressed in many ways and usually lead to different approximate series solutions which may diverge in some cases. Also, the LT may face a particular difficulty to evaluate the inverse LT of complex expressions. Accordingly, searching for a simple but effective procedure is desirable for solving the delay problems.

2. Ansatz method

Suppose that the solution of Eq. (1.1) is in the form:

$$y(t) = e^{\mu t} \sum_{n=0}^{\infty} d_n e^{\omega a^n t}, \tag{2.1}$$

where μ and ω are auxiliary parameters and to be determined. From (2.1), we have

$$y'(t) = e^{\mu t} \sum_{n=0}^{\infty} \omega a^n d_n e^{\omega a^n t} + \mu e^{\mu t} \sum_{n=0}^{\infty} d_n e^{\omega a^n t}, \tag{2.2}$$

and

$$y(at) = e^{\mu a t} \sum_{n=0}^{\infty} d_n e^{\omega a^{n+1} t}. \tag{2.3}$$

Inserting Eqs. (2.1)-(2.3) into Eq. (1.1), then

$$\sum_{n=0}^{\infty} (\omega a^n + \mu) d_n e^{\omega a^n t} = a \sum_{n=0}^{\infty} d_n e^{\omega a^n t} + ae^{(a+\mu a-\mu)t} \sum_{n=0}^{\infty} d_n e^{\omega a^{n+1} t}, \tag{2.4}$$

or

$$(\omega+\mu-a)d_0+\sum_{n=0}^{\infty}\left(\omega a^{n+1}+\mu-a\right)d_{n+1}e^{\omega a^{n+1}t}=ae^{(a+\mu(a-1))t}\sum_{n=0}^{\infty}d_n e^{\omega a^{n+1}t}. \tag{2.5}$$

Setting

$$\omega+\mu-a=0, \qquad a+\mu(a-1)=0, \tag{2.6}$$

we get

$$\omega=a-\mu, \qquad \mu=\frac{a}{1-a}. \tag{2.7}$$

Accordingly, Eq. (2.5) becomes

$$\sum_{n=0}^{\infty}\left[\left(\omega a^{n+1}+\mu-a\right)d_{n+1}-ad_n\right]e^{\omega a^{n+1}t}=0, \tag{2.8}$$

which gives

$$d_{n+1}=\frac{ad_n}{\omega a^{n+1}+\mu-a}, \quad n\geq 0. \tag{2.9}$$

Substituting $\omega=a-\mu$ into (2.9) yields

$$d_{n+1}=\frac{\left(\frac{a}{a-\mu}\right)d_n}{a^{n+1}-1}, \quad n\geq 0. \tag{2.10}$$

3. Analytic solution

The above analysis is to be invested in this section to establish the analytic solution of Eq. (1.1). To do so, we have at first to find explicit form for the coefficients d_n of the series in the ansatz (2.1). From (2.10) one can easily find that

$$d_n=\frac{\left(\frac{a}{a-\mu}\right)^n d_0}{\prod_{k=1}^{n}\left(a^k-1\right)}, \quad n\geq 1. \tag{3.1}$$

Eq. (2.1) can be rewritten as

$$y(t)=e^{\mu t}\sum_{n=0}^{\infty}d_n e^{(a-\mu)a^n t}, \tag{3.2}$$

or

$$y(t)=d_0e^{at}+e^{\mu t}\sum_{n=1}^{\infty}d_n e^{(a-\mu)a^n t}. \tag{3.3}$$

Hence,

$$y(t)=d_0\left[e^{at}+e^{\mu t}\sum_{n=1}^{\infty}\frac{\left(\frac{a}{a-\mu}\right)^n}{\prod_{k=1}^{n}\left(a^k-1\right)}e^{(a-\mu)a^n t}\right], \tag{3.4}$$

or equivalently

$$y(t) = d_0 e^{\mu t} \sum_{n=0}^{\infty} \frac{\left(\frac{a}{a-\mu}\right)^n}{\prod_{k=1}^{n} (a^k - 1)} e^{(a-\mu)a^n t}. \tag{3.5}$$

Applying the given condition gives d_0 as

$$d_0 = \frac{\lambda}{\sum_{n=0}^{\infty} \frac{\left(\frac{a}{a-\mu}\right)^n}{\prod_{k=1}^{n} (a^k - 1)}}. \tag{3.6}$$

Therefore,

$$y(t) = \lambda e^{\mu t} \sum_{n=0}^{\infty} \frac{\left(\frac{a}{a-\mu}\right)^n}{\prod_{k=1}^{n} (a^k - 1)} e^{(a-\mu)a^n t} \Big/ \sum_{n=0}^{\infty} \frac{\left(\frac{a}{a-\mu}\right)^n}{\prod_{k=1}^{n} (a^k - 1)}. \tag{3.7}$$

4. Numerical validation

The objective of this section is to validate the obtained theoretical results in the previous sections. The closed form solution (3.7) is to be compared with the available exact solution of a specific delay model at a certain value of the parameter a. Before doing so, it may be useful to extract the m-term approximate solution by replacing the infinity in (3.7) by a finite number of terms m. Accordingly, the m-term approximate solution can be expressed as

$$\Phi_m(t) = \lambda e^{\mu t} \sum_{n=0}^{m-1} \frac{\left(\frac{a}{a-\mu}\right)^n}{\prod_{k=1}^{n} (a^k - 1)} e^{(a-\mu)a^n t} \Big/ \sum_{n=0}^{m-1} \frac{\left(\frac{a}{a-\mu}\right)^n}{\prod_{k=1}^{n} (a^k - 1)}, \quad m \geq 1. \tag{4.1}$$

In order to extract numerical results, the value of the initial condition is chosen as a fixed value $\lambda = 1$ in all figures. At $a = 1/2$, the exact solution of the model $y'(t) = 1/2y(t) + 1/2e^{t/2}y(t/2)$ is available and given by $y(t) = e^t$.

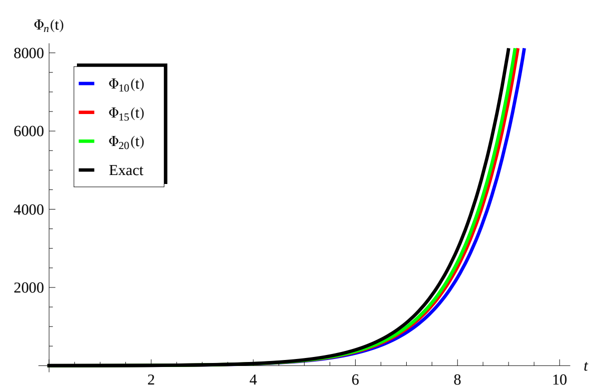

Figure 1. Comparison between the approximate solutions $\Phi_m(t)$, $m = 10, 15, 20$ and the exact solution at $a = 1/2$.

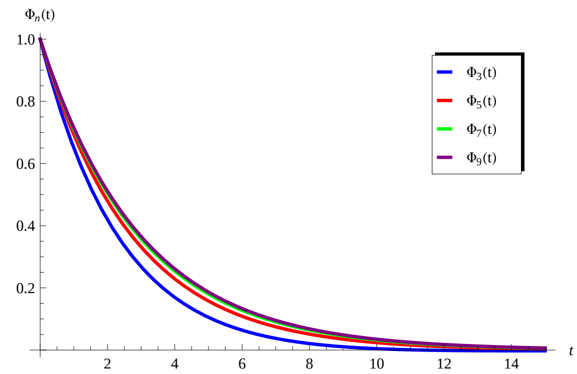

FIGURE 2. Convergence of the approximate solutions $\Phi_m(t)$, $m = 3, 5, 7, 9$ at $a = -1/2$.

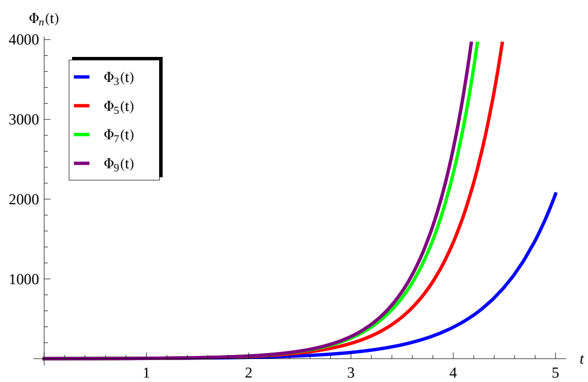

FIGURE 3. Convergence of the approximate solutions $\Phi_m(t)$, $m = 3, 5, 7, 9$ at $a = 3/4$.

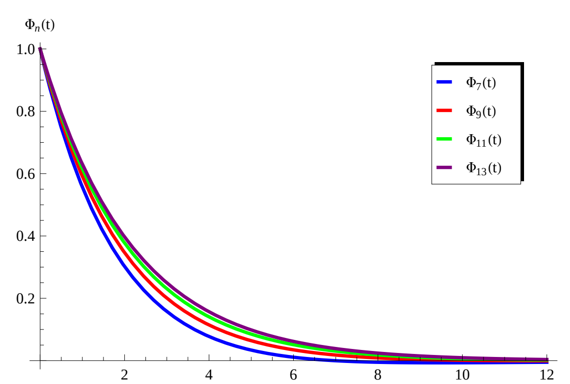

FIGURE 4. Convergence of the approximate solutions $\Phi_m(t)$, $m = 7, 9, 11, 13$ at $a = -3/4$.

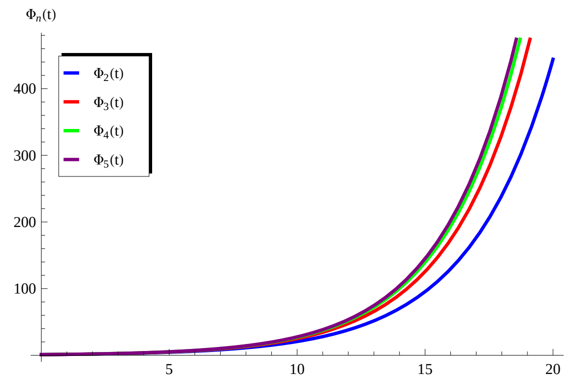

FIGURE 5. Convergence of the approximate solutions $\Phi_m(t)$, $m = 2, 3, 4, 5$ at $a = 1/4$.

Figure 1 displays the comparison between the approximate solutions $\Phi_m(t)$, $m = 10, 15, 20$ and the exact one at $a = 1/2$. This figure indicates that the curves of the approximations Φ_m become closer to the exact curve as the number of terms m increases, hence, the present analysis can be trusted.

When $a \neq 1/2$, the exact solution of the current model is not available in the literature. So, the convergence of the approximations $\Phi_m(t)$ should be examined. For illustration, the convergence of the approximate solutions $\Phi_m(t)$, $m = 3, 5, 7, 9$ at $a = -1/2$ is shown in Fig. 2. The results shows that the convergence occurs even for approximations with a low number of terms. In addition, the results introduced in Figs. (3)-(6) confirm this fact.

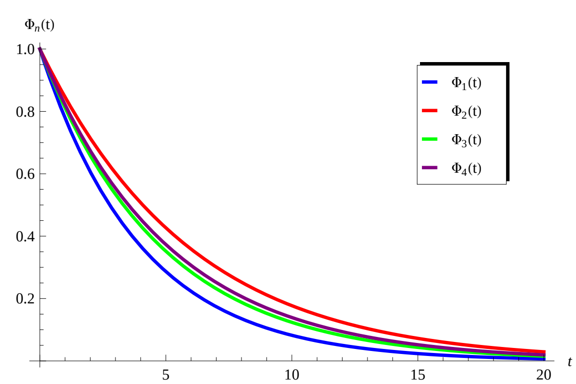

FIGURE 6. Convergence of the approximate solutions $\Phi_m(t)$, $m = 1, 2, 3, 4$ at $a = -1/4$.

5. Conclusions

In this paper, an ansatz method was developed to solve a delay model with a proportional delay parameter. The advantage of the developed approach is confirmed via performing a comparison between the current solution and the exact one at a specific value of the proportional delay parameter. Moreover, the convergence of the present approximate solution was demonstrated through several plots. The main advantage of the current approach is that it is straightforward and can further applied to solve more complex delay models of the same type.

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