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Bertrand Offsets of Spacelike Ruled Surfaces With Blaschke Approach

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Abstract. Dual parametrizaions of the Bertrand offset- spacelike ruled surfaces are assigned and sundry modern outcomes are acquired in view of their integral invariants. A modern characterization of the Bertrand offsets of spacelike developable surfaces is specified. Further, many connections among the striction curves of Bertrand offsets of spacelike ruled surfaces and their integral invariants are gained.

1. Introduction

The context of Bertrand offset ($\mathcal{BO}$) for ruled surface ($\mathcal{RS}$) is a paramount and influential instrument in model-depend industrialization of mechanical outputs, and geometrical exampling. The $\mathcal{BO}$ can be applied to produce geometrical models of shell-style forms and solid surfaces [1–4]. Thus, abundant engineers and geometers have searched and gained much geometric-kinematical ownerships of these kind surfaces in Euclidean and non-Euclidean spaces; for epitome Ravani and Ku used the Bertrand curves ($\mathcal{BC}$) for ruled surfaces depend on line geometry [5]. They manifested that a $\mathcal{RS}$ can have an infinity of $\mathcal{BO}$, in the same view of a planar curve can have an infinity of Bertrand mates. Via the E. Study map, Küçük and Gürsoy considered numerous descriptives of $\mathcal{BO}$ of trajectory $\mathcal{RS}$ in view of the interrelations via the projection areas for the spherical indicatrix of $\mathcal{BO}$ and their integral invariants [6]. In [7], Kasap and Kuruoglu acquired the connections through integral invariants of the common of the Bertrand $\mathcal{RS}$ in Euclidean 3-space $\mathcal{E}^3$. In [8] Kasap and Kuruoglu inaugurated the address of $\mathcal{BO}$ of $\mathcal{RS}$ in Minkowski 3-space. The involute-evolute offsets of $\mathcal{RS}$ is defined by Kasap et al. in [9]. Orbay et al. [10] examined the realization of Mannheim offsets of the $\mathcal{RS}$. Onder and Ugurlu acquired the connections via the invariants of Mannheim offsets of timelike ($\mathcal{TL}$) $\mathcal{RS}$ and they gave the cases for these surface offsets to be developable [11]. Aldossary and Abdel-Baky utilized the theory of $\mathcal{BC}$ for $\mathcal{RS}$, via the E. Study map [12]. Senturk and Yuce have appraised the integral invariants of the offsets by

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the geodesic Frenet frame [13]. Serious achievements to the $\mathcal{BO}$ of these ruled surfaces have been researched in [14–18].

In this work, an extension of the $\mathcal{BO}$ is provided for spacelike ($\mathcal{SL}$) ruled and developable surfaces in Minkowski 3-space $\mathcal{E}_1^3$. In view of the E. Study map, two $\mathcal{SL}$ ruled surfaces which are offsets in the sense of Bertrand are contemplated. It is offered that, generally, any $\mathcal{SL}\,\mathcal{RS}$ can have a binary infinity of $\mathcal{BO}$; however for a $\mathcal{SL}$ developable $\mathcal{RS}$ to have a $\mathcal{SL}$ developable $\mathcal{BO}$, a linear equation should be specified through the curvature and torsion of its edge of regression. Further, it is expounded that the $\mathcal{SL}$ developable offsets of a developable surface are parallel offsets. The ramifications, in addendum to being of theoretical regard, have achievements in geometricial modelling and the manufacturing of outputs.

2. Basic Concepts

In this section we list an abstract notations of dual numbers and dual Lorentzian vectors [1–3, 14–18]. A non-null directed line $\mathcal{L}$ in Minkowski 3-space $\mathcal{E}_1^3$ can be designated by a point $y \in \mathcal{L}$ and a unit vector λ of $\mathcal{L}$, that is, $\|\lambda\|^2 = \pm 1$. To have coordinates for $\mathcal{L}$, one demonstrate the moment vector $\lambda^* = y \times \lambda$. If y is substituted by any point $x = y + t\lambda$, $t \in \mathbb{R}$ on $\mathcal{L}$, this show that λ^* is not based on y. For the two non-null vectors λ and λ^* we find

$$<\lambda, \lambda>= \pm 1, \quad <\lambda^*, \lambda>=0. \tag{2.1}$$

The 6-components λ_i, $\lambda_i^* (i = 1, 2, 3)$ of λ and λ^* are the normalized Plűcker coordinates of $\mathcal{L}$. Hence, the two non-null vectors λ and λ^* specified the directed line $\mathcal{L}$.

A dual number ($\mathcal{DN}$) $\widehat{\lambda}$ is a number $\lambda + \varepsilon\lambda^*$, where $(\lambda, \lambda^*) \in \mathbb{R} \times \mathbb{R}$, ε is a dual unit with $\varepsilon \neq$, and $\varepsilon^2 = 0$. Thus, the set

$$\mathcal{D}^3 = \{\widehat{\lambda} := \lambda + \varepsilon\lambda^* = (\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3)\}, \tag{2.2}$$

with the Lorentzian scalar product

$$<\widehat{\lambda}, \widehat{\lambda}>= \widehat{\lambda}_1^2 - \widehat{\lambda}_2^2 + \widehat{\lambda}_3^2, \tag{2.3}$$

is dual Lorentzian 3-space $\mathcal{D}_1^3$. Then a point $\widehat{\lambda} = (\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3)^t$ has dual coordinates $\widehat{\lambda}_i = (\lambda_i + \varepsilon\lambda_i^*) \in \mathcal{D}$. If $\lambda \neq 0$ the norm $\left\|\widehat{\lambda}\right\|$ of $\widehat{\lambda} = \lambda + \varepsilon\lambda^*$ is

$$\left\|\widehat{\lambda}\right\| = \sqrt{\left|<\widehat{\lambda}, \widehat{\lambda}>\right|} = \|\lambda\| \, (1 + \varepsilon\frac{<\lambda, \lambda^*>}{\|\lambda\|^2}). \tag{2.4}$$

So, the vector $\widehat{\lambda}$ is a $\mathcal{TL}$ ($\mathcal{SL}$)) dual unit vector ($\mathcal{DUV}$) if $\left\|\widehat{\lambda}\right\|^2 = -1 (\left\|\widehat{\lambda}\right\|^2 = 1)$. Then,

$$\left\|\widehat{\lambda}\right\|^2 = \pm 1 \Longleftrightarrow \|\lambda\|^2 = \pm 1, \ <\lambda, \lambda^*>=0. \tag{2.5}$$

For any two dual vectors $\widehat{\lambda}=\left(\widehat{\lambda}_1,\widehat{\lambda}_2,\widehat{\lambda}_3\right)$ and $\widehat{\epsilon}=(\widehat{\epsilon}_1,\widehat{\epsilon}_2,\widehat{\epsilon}_3)$ of $\mathcal{D}_1^3$, the vector product is

$$\widehat{\lambda}\times\widehat{\epsilon}=\begin{vmatrix}\widehat{e}_1 & -\widehat{e}_2 & \widehat{e}_3\\ \widehat{\lambda}_1 & \widehat{\lambda}_2 & \widehat{\lambda}_3\\ \widehat{\epsilon}_1 & \widehat{\epsilon}_2 & \widehat{\epsilon}_3\end{vmatrix},$$

where $\widehat{e}_1,\widehat{e}_2,\widehat{e}_3$ is the canonical dual basis of $\mathcal{D}_1^3$. The hyperbolic and Lorentzian (de Sitter space) $\mathcal{DU}$ spheres with the joint center $\widehat{0}$, respectively, are:

$$\mathcal{H}_+^2=\left\{\widehat{\lambda}\in\mathcal{D}_1^3 \mid \widehat{\lambda}_1^2-\widehat{\lambda}_2^2+\widehat{\lambda}_3^2=-1\right\}, \tag{2.6}$$

and

$$\mathcal{S}_1^2=\left\{\widehat{\lambda}\in\mathcal{D}_1^3 \mid \widehat{\lambda}_1^2-\widehat{\lambda}_2^2+\widehat{\lambda}_3^2=1\right\}. \tag{2.7}$$

Hence, we have the following map (E. Study's map): The ring mod hyperboloid is in bijection with the set of $\mathcal{SL}$ lines, the mutual asymptotic cone is in bijection with the set of null-lines, and the oval shaped hyperboloid is in bijection with the set of $\mathcal{TL}$ lines (see Fig. 1). Then, a regular curve on $\mathcal{H}_+^2$ symbolizes a $\mathcal{TL}$ $\mathcal{RS}$ in $\mathcal{E}_1^3$. Also a regular curve on $\mathcal{S}_1^2$ symbolizes a $\mathcal{SL}$ or $\mathcal{TL}$ $\mathcal{RS}$ in $\mathcal{E}_1^3$ [14–19].

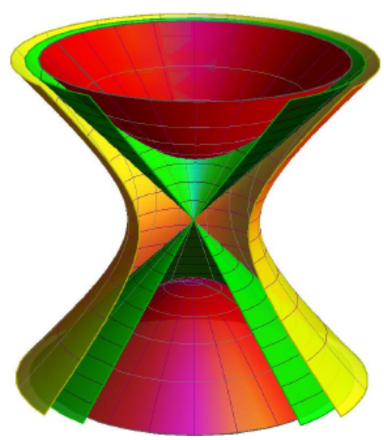

Figure 1. Hyperbolic and Lorentzian (de Sitter space) $\mathfrak{DU}$ spheres.

2.1. **The Blaschke approach.** A differentiable dual curve ($\mathcal{DC}$)

$$v\in\mathbb{R}\mapsto\widehat{g}(v)\in\mathcal{S}_1^2,\, v\in\mathbb{R},$$

is a $\mathcal{TL}$ or $\mathcal{SL}$ $\mathcal{RS}$ $(\widehat{g})$ in Minkowski 3-space $\mathcal{E}_1^3$. It will be supposed a $\mathcal{SL}$ $\mathcal{RS}$ in our study. The $\mathcal{TL}$ $\mathcal{DUV}$

$$\widehat{g}_2(v)=g_2+\varepsilon g_2^*=\frac{d\widehat{g}}{dv}\left\|\frac{d\widehat{g}}{dv}\right\|^{-1},$$

is the tangent vector on $\widehat{g}$. Inserting the $\mathcal{SL}$ $\mathcal{DU}$ vector $\widehat{g}_3(v) = g_3(v) + \varepsilon g_3^*(v) = \widehat{g} \times \widehat{g}_2$ we have the moving frame $\{\widehat{g} = \widehat{g}_1(v), \widehat{g}_2(v), \widehat{g}_3(v)\}$ on $\widehat{g}(v)$ named Blaschke frame. Then,

$$\left.\begin{array}{l} <\widehat{g}_1, \widehat{g}_1> = -<\widehat{g}_2, \widehat{g}_2> = <\widehat{g}_3, \widehat{g}_3> = 1, \\ \widehat{g}_3 = \widehat{g}_1 \times \widehat{g}_2, \widehat{g}_2 = \widehat{g}_1 \times \widehat{g}_3, \widehat{g}_1 = \widehat{g}_2 \times \widehat{g}_3. \end{array}\right\} \tag{2.8}$$

The $\mathcal{DU}$ vectors $\widehat{g}_1$, $\widehat{g}_2$, and $\widehat{g}_3$ represents three orthogonally intersected oriented lines at a point c named the central point. The places of the central points is the striction curve on $(\widehat{g})$. Via the spherical kinematics, the locomotion of the Blaschke frame is a turnover with the Darboux vector $\widehat{\omega}$ of this frame. Then,

$$\frac{d}{dv}\begin{pmatrix} \widehat{g}_1 \\ \widehat{g}_2 \\ \widehat{g}_3 \end{pmatrix} = \begin{pmatrix} 0 & \widehat{p} & 0 \\ \widehat{p} & 0 & \widehat{q} \\ 0 & \widehat{q} & 0 \end{pmatrix}\begin{pmatrix} \widehat{g}_1 \\ \widehat{g}_2 \\ \widehat{g}_3 \end{pmatrix} = \widehat{\omega} \times \begin{pmatrix} \widehat{g}_1 \\ \widehat{g}_2 \\ \widehat{g}_3 \end{pmatrix}, \tag{2.9}$$

where $\widehat{\omega}(v) = \widehat{q}(v)\widehat{g}_1(v) - \widehat{p}(v)\widehat{g}_3(v)$ and

$$\widehat{p}(v) = p(v) + \varepsilon p^*(v) = \left\|\frac{d\widehat{g}}{dv}\right\|, \ \widehat{q}(v) = q(v) + \varepsilon q^*(v) = -\det(\widehat{g}, \frac{d\widehat{g}}{dv}, \frac{d^2\widehat{g}}{dv^2}),$$

are the Blaschke invariants of $\widehat{g}(v) \in \mathcal{S}_1^2$. Also, we realize the $\mathcal{SL}$ $\mathcal{DUV}$

$$\widehat{e}(v) := e + \varepsilon e^* = \frac{\widehat{\omega}}{\|\widehat{\omega}\|} = \frac{\widehat{q}}{\sqrt{\widehat{p}^2 + \widehat{q}^2}}\widehat{g}_1 - \frac{\widehat{p}}{\sqrt{\widehat{p}^2 + \widehat{q}^2}}\widehat{g}_3. \tag{2.10}$$

It is visible that $\widehat{e}$ is the Disteli-axis (curvature-axis or striction-axis) of $(\widehat{g})$. The tangent of the striction curve $c(v)$ is specified by [12]:

$$\frac{dc(v)}{dv} = q^* g_1(v) - p^*(v) g_3(v). \tag{2.11}$$

Via the presumption that $p(v) \neq 0$, we appoint the functions

$$\chi(v) = \frac{q(t)}{p(t)}, \ F(v) = \frac{q^*(v)}{p(v)}, \ \varkappa(v) = \frac{p^*(v)}{p(v)}, \tag{2.12}$$

which are invariants of $\widehat{g}(v) \in \mathcal{S}_1^2$. Let $d\widehat{u} = du + \varepsilon du^*$ be the dual-arc length of $\widehat{g}(v)$, that is, $d\widehat{u} = \widehat{p}dv = p(1 + \varepsilon\varkappa)dv$. Then, from Equations 2.9 and 2.12 we acquire

$$\frac{dc(u)}{du} = F(u) g_1(u) - \varkappa(u) g_3(u), \tag{2.13}$$

and

$$\begin{pmatrix} \widehat{g}_1' \\ \widehat{g}_2' \\ \widehat{g}_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \widehat{\chi} \\ 0 & \widehat{\chi} & 0 \end{pmatrix}\begin{pmatrix} \widehat{g}_1 \\ \widehat{g}_2 \\ \widehat{g}_3 \end{pmatrix}; \ (' = \frac{d}{d\widehat{u}}), \tag{2.14}$$

where $\widehat{\chi}(\widehat{u}) := \frac{\widehat{q}}{\widehat{p}} = \chi + \varepsilon\chi^*$ is the dual geodesic curvature of $\widehat{g}(\widehat{u}) \in \mathcal{S}_1^2$. Thus, a non-developable $\mathcal{SL}$ $\mathcal{RS}$ $(\widehat{g})$ can be realized as follows:

$$(\widehat{g}) : \Gamma(u, t) = \int_0^u (F(u) g_1(u) + \varkappa(u) g_3) du + t g_1(u), \ u \in I, \ \in t \in \mathbb{R}. \tag{2.15}$$

The $\mathcal{TL}$ unit normal vector field at any point is

$$\xi(u,t) = \frac{\frac{\partial\Gamma(u,t)}{\partial u}\times\frac{\partial\Gamma(u,t)}{\partial t}}{\left\|\frac{\partial\Gamma(u,t)}{\partial u}\times\frac{\partial\Gamma(u,t)}{\partial t}\right\|} = \pm\frac{\varkappa g_2 - t g_3}{\sqrt{\varkappa^2 - t^2}},\ |\varkappa| > |t|, \tag{2.16}$$

which is the $\mathcal{TL}$ central normal at the striction point ($t = 0$). Let φ be a hyperbolic rotation angel through ξ and the central normal g_2, then

$$\xi(u,t) = \cosh\varphi g_2 - \sinh\varphi g_3, \text{ with } \tanh\varphi = \frac{t}{\varkappa}. \tag{2.17}$$

Equation 2.17 is a Minkowski version of the well known Chasles Theorem [1–3].

3. Bertrand Offsets of Spacelike Ruled Surfaces

In this section, we meditate the $\mathcal{B}$ offsets of $\mathcal{SL}$ ruled and developable surfaces, then a theory hassling to the theory of $\mathcal{B}$ curves can be advanced for such surfaces.

Definition 3.1. *Let $(\widehat{g})$ and $(\widehat{\overline{g}})$ be two non-developable $\mathcal{SL}$ ruled surfaces in $\mathcal{E}_1^3$. The surface $(\widehat{\overline{g}})$ is said to be $\mathcal{BO}$ of $(\widehat{g})$ if there exists a bijection through their generators such that both surfaces have a joint $\mathcal{TL}$ central normal at the conformable central points.*

Let $(\widehat{\overline{g}})$ be a $\mathcal{SL}$ $\mathcal{BO}$ of $(\widehat{g})$ with the Blaschke frame $\{\widehat{\overline{g}}_1(\widehat{\overline{u}}), \widehat{\overline{g}}_2(\widehat{\overline{u}}), \widehat{\overline{g}}_3(\widehat{\overline{u}})\}$, it can be stated as aforementioned in the above equations. Let $\widehat{\psi} = \psi + \varepsilon\psi^*$ be the $\mathcal{SL}$ dual angle through the generators of $(\widehat{g})$ and $(\widehat{\overline{g}})$ at the corresponding points, that is,

$$<\widehat{\overline{g}}, \widehat{g}> = \cos\widehat{\psi}. \tag{3.1}$$

By differentiating of Equation 3.1 with respect to $\widehat{u}$, we find

$$<\widehat{\overline{g}}_2, \widehat{g}> \widehat{\overline{u}}' + <\widehat{\overline{g}}, \widehat{g}_2> = -\widehat{\psi}' \sin\widehat{\psi}. \tag{3.2}$$

Since $(\widehat{g})$ and $(\widehat{\overline{g}})$ are $\mathcal{BO}$ ($\widehat{g}_2 = \widehat{\overline{g}}_2$), then we have $\widehat{\psi}' = 0$, so that $\widehat{\psi} = \psi + \varepsilon\psi^*$ is an stationary dual number.

Theorem 3.1. *The offset angle ψ and the offset distance ψ^* among the rgenerators of a non-developable $\mathcal{SL}$ $\mathcal{RS}$ and its $\mathcal{BO}$ are stationary.*

It is apparent via Theorem 3.1 that a $\mathcal{SL}$ $\mathcal{RS}$, mostly, has a couple infinity of $\mathcal{SL}$ $\mathcal{BO}$. Each $\mathcal{BO}$ can be traced by an stationary linear offset $\psi^* \in \mathbb{R}$ and an stationary angular offset $\psi \in [0, 2\pi]$. Any two $\mathcal{SL}$ ruled surfaces of this set of $\mathcal{SL}$ ruled surfaces are reciprocal of one another; if $(\widehat{\overline{g}})$ is a $\mathcal{SL}$ $\mathcal{BO}$ of $(\widehat{g})$, then $(\widehat{g})$ is as well a $\mathcal{SL}$ $\mathcal{BO}$ of $(\widehat{\overline{g}})$. Moreover, we can set

$$\widehat{\overline{g}}_1(\widehat{\overline{u}}) = \cos\widehat{\psi}\widehat{g}_1(\widehat{u}) + \sin\widehat{\psi}\widehat{g}_3(\widehat{u}). \tag{3.3}$$

Via Theorem 1 the $\mathcal{TL}$ central tangents $\widehat{g}_3$ and $\widehat{\overline{g}}_3$ are also have the same stationary dual angle at the matching central points. Then,

$$\begin{pmatrix} \widehat{\overline{g}}_1 \\ \widehat{\overline{g}}_2 \\ \widehat{\overline{g}}_3 \end{pmatrix} = \begin{pmatrix} \cos\widehat{\psi} & 0 & \sin\widehat{\psi} \\ 0 & 1 & 0 \\ -\sin\widehat{\psi} & 0 & \cos\widehat{\psi} \end{pmatrix} \begin{pmatrix} \widehat{g}_1 \\ \widehat{g}_2 \\ \widehat{g}_3 \end{pmatrix}. \tag{3.4}$$

The considerable point to note here is the technique we have utilized (compared with [5, 6]). Furthermore we also have

$$\frac{d}{d\widehat{\overline{u}}} \begin{pmatrix} \widehat{\overline{g}}_1 \\ \widehat{\overline{g}}_2 \\ \widehat{\overline{g}}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \widehat{\overline{\chi}} \\ 0 & \widehat{\overline{\chi}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\overline{g}}_1 \\ \widehat{\overline{g}}_2 \\ \widehat{\overline{g}}_3 \end{pmatrix}, \tag{3.5}$$

where

$$d\widehat{\overline{u}} = (\cos\widehat{\psi} + \widehat{\chi}\sin\widehat{\psi})d\widehat{u},\ \widehat{\overline{\chi}}d\widehat{\overline{u}} = (\widehat{\chi}\cos\widehat{\psi} - \sin\widehat{\psi})d\widehat{u}.$$

By eliminating $d\widehat{\overline{u}}/d\widehat{u}$, we gain

$$(\widehat{\overline{\chi}} - \widehat{\chi})\cos\widehat{\psi} + (1 + \widehat{\overline{\chi}}\widehat{\chi})\sin\widehat{\psi} = 0. \tag{3.6}$$

This is a dual Minkowski version for $\mathcal{SL}$ $\mathcal{RS}$ and its $\mathcal{SL}$ $\mathcal{BO}$ $\mathcal{RS}$ in terms of their dual geodesic curvatures.

Theorem 3.2. *Any two non-developable $\mathcal{SL}$ ruled surfaces are $\mathcal{BO}$ iff the Equation 3.6 is fulfilled.*

The equation of the striction curve of the offset surface $(\widehat{\overline{g}})$, in view of its base surface $(\widehat{g})$, can therefore be located as

$$\overline{c}(\overline{u}) = c(u) + \psi^{*}\widehat{g}_2(u). \tag{3.7}$$

So, the equation of $(\widehat{\overline{g}})$ in view of $(\widehat{g})$ can be acquired as

$$(\widehat{\overline{g}}) : \overline{\Gamma}(\overline{u}, t) = c(\overline{u}) + \psi^{*}\widehat{g}_2(u) + t\left(\cos\psi\widehat{g}_1(u) + \sin\psi\widehat{g}_3(u)\right),\ t \in \mathbb{R}. \tag{3.8}$$

Let $\overline{\xi}(\overline{u}, t)$ be the $\mathcal{TL}$ unit normal of an arbitrary point on $(\widehat{\overline{g}})$. Then, as in Eq. (16), we have:

$$\overline{\xi}(\overline{u}, t) = \pm\frac{\varkappa\overline{g}_2 - t\overline{g}_3}{\sqrt{\overline{\varkappa}^2 - t^2}},\ \left|\overline{\varkappa}\right| > |t|, \tag{3.9}$$

where $\overline{\varkappa}$ is the distribution parameter of $(\widehat{\overline{g}})$. It is lucid from Equations 2.16 and 3.9 that the $\mathcal{TL}$ normal to a $\mathcal{SL}$ $\mathcal{RS}$ and its $\mathcal{BO}$ are not the same. This importances that the $\mathcal{BO}$ of a $\mathcal{SL}$ $\mathcal{RS}$ are, mostly, not parallel offsets. Thus, the parallel situations through $(\widehat{\overline{g}})$ in view of $(\widehat{g})$ can be described by the next theorem:

Theorem 3.3. *Any two non-developable $\mathcal{SL}$ $\mathcal{BO}$ ruled surfaces are parallel offsets iff (a) $\varkappa = \overline{\varkappa}$, (b) each axis of the Blaschke frame of $(\widehat{g})$ is collinear with the congruent axis for $(\widehat{\overline{g}})$.*

Proof. Let $(\widehat{g})$ and $(\widehat{\overline{g}})$ are parallel offsets non-developable $\mathcal{SL}$ $\mathcal{RS}$, that is, $\overline{\xi}(\overline{u},t)\times\xi(u,t)=0$. Then, we have the following

$$t(\varkappa\cos\psi-\overline{\varkappa})g_1-t^2\sin\psi g_2+t\varkappa\sin\psi g_3=0.$$

The above equation should be hold true for any value $t\neq 0$, which leads to $\psi=0$ and $\varkappa=\overline{\varkappa}$.

Suppose that the two conditions (a) and (b) are hold true, that is, $\psi=0,\varkappa=\overline{\varkappa}$. Then substitute them into $\overline{\xi}(\overline{u},t)\times\xi(u,t)$, that is,

$$\overline{\xi}(\overline{u},t)\times\xi(u,t)=\frac{\varkappa\overline{g}_2-t\overline{g}_3}{\sqrt{\overline{\varkappa}^2-t^2}}\times\frac{\varkappa g_2-tg_3}{\sqrt{\varkappa^2-t^2}}$$

It is obvious that past equation is the zero vector, which implies that $(\widehat{g})$ and $(\widehat{\overline{g}})$ are parallel offsets. □

Once more in the same method, but now for developable $\mathcal{SL}$ $\mathcal{RS}$, that is, $\varkappa=\overline{\varkappa}=0$, we have:

Corollary 3.1. *Any two developable $\mathcal{SL}$ $\mathcal{BO}$ ruled surfaces are parallel offsets iff their Blaschke frames are colinear.*

3.1. **The striction curves.** In this subsection we investigate the possessions and connections of the striction curves. Furthermore, we assign novel geometric and kinematical descriptions of the invariants of the $\mathcal{B}$ offsets. In view of Equation 3.7, the tangent of the striction curve $\overline{c}(\overline{u})$ of $(\widehat{\overline{g}})$ is

$$\frac{d\overline{c}(\overline{u})}{d\overline{u}}=[(F+\psi^*)g_1+(\varkappa+\chi\psi^*)g_3]\frac{du}{d\overline{u}}, \tag{3.10}$$

whereas, as in Equations 2.11-2.13, is:

$$\frac{d\overline{c}(\overline{u})}{d\overline{u}}=\overline{F}(\overline{u})\overline{g}_1(\overline{u})+\overline{\varkappa}(\overline{u})\overline{g}_3(\overline{u}). \tag{3.11}$$

From Equations 3.10, and 3.11 we attain

$$\frac{d\overline{u}}{du}=\frac{F+\psi^*}{\overline{F}\cos\psi-\overline{\varkappa}\sin\psi}=\frac{\varkappa+\chi\psi^*}{\overline{F}\sin\psi+\overline{\varkappa}\cos\psi}. \tag{3.12}$$

A)- In the case of $(\widehat{g})$ is a $\mathcal{SL}$ tangential developable surface ($\mathcal{TDS}$), that is, $\varkappa(u)=0$. In this issue, we attian

$$\overline{\varkappa}=\overline{F}\frac{(F+\psi^*)\sin\psi-\chi\psi^*\cos\psi}{(F+\psi^*)\cos\psi-\chi\psi^*\sin\psi}. \tag{3.13}$$

Then, the $\mathcal{BO}$ of a $\mathcal{SL}$ $\mathcal{TDS}$ is not $\mathcal{TDS}$, that is, $\overline{\varkappa}(u)\neq 0$. Also, we have $dc/du=F(u)g_1(u)$. Let s be arc length parameter of $c(u)$ and $\{t_1(s), t_2(s), t_3(s)\}$ is the moving Serret-Frenet frame of $c(s)$. Then,

$$\frac{d}{ds}\begin{pmatrix}t_1\\t_2\\t_3\end{pmatrix}=\begin{pmatrix}0&\kappa(s)&0\\\kappa(s)&0&\tau(s)\\0&\tau(s)&0\end{pmatrix}\begin{pmatrix}t_1\\t_2\\t_3\end{pmatrix},$$

where $\kappa(s)$ and $\tau(s)$ are the natural curvature and torsion of the striction curve $c(s)$, respectively;

$$\kappa(s)=\frac{1}{F(s)},\ \tau(s)=\frac{\chi(s)}{F(s)}, \text{with } F(s)\neq 0.$$

So, the function $F(s)$ is the radii of curvature of the $\mathcal{SL}$ striction curve $c(s)$. If $(\widehat{\overline{g}})$ is also a $\mathcal{SL}$ tangential surface, that is, $\overline{\varkappa}(\overline{u})=0$. Then,

$$\overline{\kappa}(s)=\frac{1}{\overline{F}(s)},\ \overline{\tau}(\overline{s})=\frac{\overline{\chi}(s)}{F(s)}, \text{with } \overline{F}(\overline{s})\neq 0,$$

is obtained. In this case, the Equation 3.13 reduces to

$$(1+\psi^{*}\kappa(s))\sin\psi-\tau(s)\psi^{*}\cos\psi=0. \tag{3.14}$$

Corollary 3.2. *$(\widehat{g})$ and $(\widehat{\overline{g}})$ are $\mathcal{SL}$ $\mathcal{BO}$ tangential surfaces iff their striction curves are $\mathcal{B}$ curves.*

From Equation 3.14 we may also state the following:
1) If $\psi=0$, then $\psi^{*}=0$ or $\tau(s)=0$,
2) If $\psi^{*}=0$, then $\psi=0$, that is, the rulings are colinear,
3) If $\tau(s)=0$, and $\psi^{*}\neq 0$, then $\kappa(s)=-1/\psi^{*}$ is constant or $\psi=0$,
4) $\psi=\pi/2$, and $\psi^{*}\neq 0$, then $\kappa(s)=-1/\psi^{*}$ is constant.

(B) If $(\widehat{g})$ is a $\mathcal{SL}$ binormal ruled surface ($\mathcal{BRS}$), that is, $F=0$. In this issue, from Equation 3.13, we find

$$\overline{F}=-\overline{\varkappa}\frac{\psi^{*}\cos\psi+(\chi\psi^{*}+\varkappa)\sin\psi}{\psi^{*}\cos\psi-(\chi\psi^{*}+\varkappa)\cos\psi}. \tag{3.15}$$

Then, the $\mathcal{BO}$ of a $\mathcal{SL}$ $\mathcal{RS}$ is not $\mathcal{BRS}$, that is, $\overline{F}(\overline{u})\neq 0$. Also, we have $dc/du=-\varkappa(u)g_{3}(u)$. Correspondingly, let s be arc length parameter of $c(u)$ and $\{b_1(s), b_2(s), b_3(s)\}$ is the mobile Serret-Frenet frame of $c(s)$. Then

$$\frac{d}{ds}\begin{pmatrix} b_1\\ b_2\\ b_3\end{pmatrix}=\begin{pmatrix} 0 & \kappa(s) & 0\\ \kappa(s) & 0 & \tau(s)\\ 0 & \tau(s) & 0\end{pmatrix}\begin{pmatrix} b_1\\ b_2\\ b_3\end{pmatrix},$$

where $\kappa(s)$ and $\tau(s)$ are the natural curvature and torsion of the striction curve $c(u)$, respectively;

$$\kappa(s)=\frac{\chi(s)}{\varkappa(s)},\ \tau(s)=\frac{1}{\varkappa(s)}, \text{with } \varkappa(s)\neq 0.$$

Therefore, the curvature function $\varkappa(s)$ is the radii of torsion of the spacelike striction curve $c(s)$. Further, if the $\mathcal{SL}$ $\mathcal{BO}$ $(\widehat{\overline{g}})$ is also a $\mathcal{BRS}$, then we reach

$$(1+\psi^{*}\kappa)\sin\phi+\psi^{*}\tau\cos\psi=0.$$

Corollary 3.3. *$(\widehat{g})$ and $(\widehat{\overline{g}})$ are $\mathcal{SL}$ $\mathcal{BO}$ binormal surfaces iff their striction curves are $\mathcal{B}$ curves.*

In a similar manner, all the outcomes of the tangential surface may be stated for the $\mathcal{SL}$ $\mathcal{BRS}$.

3.2. **Bertrand offsets with a constant Disteli-axis.** In this subsection, we are going to deal with and construct $\mathcal{BO}$ with a constant Disteli-axis. Therefore, via Equation 2.10, let $\widehat{\phi} = \phi + \varepsilon\phi^*$ be the dual radii of curvature from $\widehat{e}$ to $\widehat{g}_1$. Then, we gain

$$\widehat{e}(\widehat{u}) = \cos\widehat{\phi}\widehat{g}_1 - \sin\widehat{\phi}\widehat{g}_3, \text{ with } \cot\widehat{\phi} = \widehat{\chi}. \tag{3.16}$$

Then, we have:

$$\left.\begin{array}{l} \widehat{\chi}(\widehat{u}) = \chi + \varepsilon\left(F - \chi\varkappa\right) = \cot\widehat{\phi} = \cot\phi - \varepsilon\phi^*(1+\cot^2\phi), \\ \widehat{\kappa}(\widehat{u}) := \kappa + \varepsilon\kappa^* = \sqrt{1+\widehat{\chi}^2} = \frac{1}{\sin\widehat{\phi}} = \frac{1}{\widehat{\rho}(\widehat{u})}, \\ \widehat{\tau}(\widehat{u}) := \tau + \varepsilon\tau^* = \pm\widehat{\phi}' = \pm\widehat{\chi}'\left(1+\widehat{\chi}^2\right)^{-1}, \end{array}\right\} \tag{3.17}$$

where $\widehat{\kappa}(\widehat{u})$ is the dual curvature, and $\widehat{\tau}(\widehat{u})$ is the dual torsion of the $\mathcal{TL}$ $\mathcal{DC}$ $\widehat{g}(\widehat{u}) \in \mathcal{S}_1^2$.

Proposition 3.1. *If the dual geodesic curvature function $\widehat{\chi}(\widehat{u})$ is constant, $\widehat{g}(\widehat{u})$ is a $\mathcal{TL}$ dual circle on $\mathcal{S}_1^2$.*

Proof. From Equation 3.17 we can find that $\widehat{\chi}(\widehat{u})$ is dual constant yields that $\widehat{\tau}(\widehat{s}) = 0$, and $\widehat{\kappa}(\widehat{u})$ is dual constant, which reveals that $\widehat{g}(v)$ is a $\mathcal{TL}$ dual circle on $\mathcal{S}_1^2$. □

Definition 3.2. *A non-developable $\mathcal{RS}$ is a constant Disteli-axis $\mathcal{RS}$ if its dual geodesic curvature is constant.*

Via the E. Study map, the constant Disteli-axis $\mathcal{SL}$ $\mathcal{RS}$ $(\widehat{g})$ is traced by a $\mathcal{SL}$ line undergoing a Lorentzian helical locomotion of constant pitch h about the $\mathcal{SL}$ constant Disteli-axis $\widehat{e}$. The pencil of the constant Disteli-axis $\mathcal{SL}$ $\mathcal{RS}$ is necessary to the curvature theory of $\mathcal{RS}$. We therefore will hand its assets in somewhat detail.

3.2.1. *Height dual functions.* Via [20], a dual point $\widehat{e}_0 \in \mathcal{S}_1^2$ will be coined an $\widehat{e}_k$ evolute of the $\mathcal{DC}$ $\widehat{g}(\widehat{u}) \in \mathcal{S}_1^2$; for all $\widehat{u}$ such that $<\widehat{e}_0, \widehat{g}^k(\widehat{u})>= 0$, but $<\widehat{e}_0, \widehat{g}^{k+1}(\widehat{u})>\neq 0$. Here $\widehat{g}^{k+1}$ signalizes the k-th derivatives of $\widehat{g}(\widehat{u})$ with respect to $\widehat{u}$. For the 1st evolute $\widehat{e}$ of $\widehat{g}(\widehat{u})$, we have $<\widehat{e}, \widehat{g}'>= \pm <\widehat{e}, \widehat{g}_2>= 0$, and $<\widehat{e}, \widehat{g}''>= \pm <\widehat{e}, \widehat{g}_1 + \widehat{\chi}\widehat{g}_3>\neq 0$. So, $\widehat{e}$ is at least an $\widehat{e}_2$ evolute of $\widehat{g}(\widehat{u}) \in \mathcal{S}_1^2$.

We now describe a dual function $\widehat{\sigma} : I \times \mathcal{S}_1^2 \to \mathcal{D}$, by $\widehat{\sigma}(\widehat{u}, \widehat{e}_0) =<\widehat{e}_0, \widehat{g}>$. We call $\widehat{\sigma}$ a height dual function on $\widehat{g}(\widehat{u}) \in \mathcal{S}_1^2$. We employ the entry $\widehat{\sigma}(\widehat{u}) = \widehat{\sigma}(\widehat{u}, \widehat{e}_0)$ for any steady point $\widehat{e}_0 \in \mathcal{S}_1^2$. Hence, we state the following:

Proposition 3.2. *Under the above hypotheses, the following holds:*
i- $\widehat{\sigma}$ will be stable in the 1st estimation iff $\widehat{e}_0 \in Sp\{\widehat{g}_1, \widehat{g}_3\}$, that is,

$$\widehat{\sigma}' = 0 \Leftrightarrow <\widehat{g}', \widehat{e}_0>=0 \Leftrightarrow <\widehat{g}_2, \widehat{e}_0>=0 \Leftrightarrow \widehat{e}_0 = \widehat{c}_1\widehat{g}_1 + \widehat{c}_3\widehat{g}_3;$$

for some dual numbers $\widehat{c}_1, \widehat{c}_3 \in \mathcal{D}$, and $\widehat{a}_1^2 + \widehat{a}_3^2 = 1$.
ii- $\widehat{\sigma}$ will be stable in the 2nd estimation iff $\widehat{e}_0$ is $\widehat{e}_2$ evolute of $\widehat{g}(\widehat{u}) \in \mathcal{S}_1^2$, that is,

$$\widehat{\sigma}' = \widehat{\sigma}'' = 0 \Leftrightarrow \widehat{e}_0 = \pm\widehat{e}.$$

iii- $\widehat{\sigma}$ will be stable in the 3rd estimation iff $\widehat{e}_0$ is $\widehat{e}_3$ evolute of $\widehat{e}_0 \in \mathcal{S}_1^2$, that is,

$$\widehat{\sigma}' = \widehat{\sigma}'' = \widehat{\sigma}''' = 0 \Leftrightarrow \widehat{e}_0 = \pm\widehat{e}, \text{ and } \widehat{\chi}' \neq 0.$$

iv- $\widehat{\sigma}$ *will be stable in the 4th estimation iff* $\widehat{e_0}$ *is* $\widehat{e_4}$ *evolute of* $\widehat{e_0} \in \mathcal{S}_1^2$*, that is,*

$$\widehat{\sigma}' = \widehat{\sigma}'' = \widehat{\sigma}''' = \widehat{\sigma}^{iv} = 0 \Leftrightarrow \widehat{e_0} = \pm\widehat{e}, \widehat{\chi}' = 0, \text{ and } \widehat{\chi}'' \neq 0.$$

Proof. For the 1st derivation of $\widehat{a}$ we gain

$$\widehat{\sigma}' = <\widehat{g}', \widehat{e_0}>. \tag{3.18}$$

So, we gain

$$\widehat{\sigma}' = 0 \Leftrightarrow <\widehat{g_2}, \widehat{e_0}>=0 \Leftrightarrow \widehat{e_0} = \widehat{c_1}\widehat{g_1} + \widehat{c_3}\widehat{g_3};$$

for some dual numbers $\widehat{c_1}, \widehat{c_3} \in \mathcal{D}$, and $\widehat{c}_1^2 + \widehat{c}_3^2 = 1$, the result is evident.
2- Derivation of Equation 3.18 leads to:

$$\widehat{\sigma}'' = <\widehat{g}'', \widehat{e_0}> = <\widehat{g_1} + \widehat{\chi}\widehat{g_3}, \widehat{e_0}>. \tag{3.19}$$

By the Equations 3.18, and 3.19 we find:

$$\widehat{\sigma}' = \widehat{\sigma}'' = 0 \Leftrightarrow <\widehat{g}', \widehat{e_0}> = <\widehat{g}'', \widehat{e_0}>=0 \Leftrightarrow \widehat{e_0} = \pm\frac{\widehat{g}' \times \widehat{g}''}{\left\|\widehat{g}' \times \widehat{g}''\right\|} = \pm\widehat{e}.$$

3- Derivation of Equation 3.19 leads to:

$$\widehat{\sigma}''' = <\widehat{g}''', \widehat{e_0}> = \left(1 + \widehat{\chi}^2\right) <\widehat{g_2}, \widehat{e_0}> + \widehat{\chi}' <\widehat{g_3}, \widehat{e_0}>.$$

Hence, we have:

$$\widehat{\sigma}' = \widehat{\sigma}'' = \widehat{\sigma}''' = 0 \Leftrightarrow \widehat{e_0} = \pm\widehat{e}, \text{ and } \widehat{\chi}' \neq 0.$$

4- By the comparable controversy, we can also have:

$$\widehat{\sigma}' = \widehat{\sigma}'' = \widehat{\sigma}''' = \widehat{\sigma}'''' = 0 \Leftrightarrow \widehat{e_0} = \pm\widehat{e}, \widehat{\chi}' = 0, \text{ and } \widehat{\chi}'' \neq 0.$$

The proof is completed. □

In view of the Proposition 3.1, we have the following:
(a) The $\mathcal{TL}$ osculating circle $\mathcal{S}(\widehat{\rho}, \widehat{e_0})$ of $\widehat{g}(\widehat{u}) \in \mathcal{S}_1^2$ is showed by

$$<\widehat{e_0}, \widehat{g}> = \widehat{\rho}(\widehat{u}),\ <\widehat{g}', \widehat{e_0}> = <\widehat{g}'', \widehat{e_0}> = 0,$$

which are stated via the status that the osculating circle must have osculate of at least 3rd order at $\widehat{g}(\widehat{u_0})$ iff $\widehat{\chi}' \neq 0$.
(b) The $\mathcal{TL}$ osculating circle $\mathcal{S}(\widehat{\rho}, e)$ and the $\mathcal{TL}$ curve $\widehat{g}(\widehat{u}) \in \mathcal{S}_1^2$ have at least 4-th order at $\widehat{g}(u_0)$ iff $\widehat{\chi}' = 0$, and $\widehat{\chi}'' \neq 0$.

In this gate, by capturing into meditation the evolutes of $\widehat{g}(\widehat{u}) \in \mathcal{S}_1^2$, we can acquire a sequence of evolutes $\widehat{e_2}$, $\widehat{e_3}$,..., $\widehat{e_n}$. The ownerships and the joint relationships by these evolutes and their involutes are very important matters. For instance, it is easy to have that when $\widehat{e_0} = \pm\widehat{e}$, and $\widehat{\chi}' = 0$, $\widehat{e}(\widehat{u})$ is existing at $\widehat{\phi}$ is steady with respect to $\widehat{e_0}$. In this circumstances, the Disteli-axis is constant up to 2nd order, and the line $\widehat{g}$ change positions on it with constant pitch. Thus, the $\mathcal{SLRS}$ $(\widehat{g})$ with

$\mathcal{SL}$ constant Disteli-axis is created by $\mathcal{SL}$ line $\widehat{g}$ detected at a constant distance ϕ^* and constant angle ϕ on the $\mathcal{SL}$ Disteli-axis $\widehat{e}$, that is,

$$\widehat{\chi}(\widehat{u}) := \chi + \varepsilon\left(F - \chi\varkappa\right) = \cot\widehat{\phi} = \widehat{c},$$

where $\widehat{c} = c + \varepsilon c^* \in \mathcal{D}$. By separating the real and dual parts, the following theorem can be stated:

Theorem 3.4. *A non-developable $\mathcal{SL}$ $\mathcal{RS}$ $(\widehat{g})$ is a constant Disteli-axis iff $\chi(u)$=constant, and $F - \chi\varkappa$=constant.*

Furthermore, in view of Equation 3.6 and Theorem 3.4, we have:

Corollary 3.4. *The $\mathcal{BO}$ of a constant Disteli-axis $\mathcal{SL}$ $\mathcal{RS}$ is also a constant Disteli-axis $\mathcal{SL}$ $\mathcal{RS}$.*

However, if

$$\widehat{\chi}(\widehat{u}) := \chi + \varepsilon\left(F - \chi\varkappa\right) = 0 = \cot\phi - \varepsilon\phi^*(1 + \cot^2\phi),$$

then $\phi = \frac{\pi}{2}$, and $\phi^* = 0$, that is,

$$\mathcal{S}(1,\widehat{e}) = \{\widehat{g}\in\mathcal{S}_1^2 \,|< \widehat{g},\widehat{e} >= 0;\ \text{with}\ \left\|\widehat{e}\right\|^2 = 1\}.$$

In this case, all the rulings of $(\widehat{g})$ intersected orthogonally with the Disteli-axis. Thus, we have $\widehat{\chi}(\widehat{u}) := \chi + \varepsilon\left(F - \chi\varkappa\right) = 0 \Leftrightarrow (\widehat{g})$ is a $\mathcal{SL}$ helicoidal surface.

Theorem 3.5. *A non-developable $\mathcal{SL}$ $\mathcal{RS}$ $(\widehat{g})$ is a helicoidal ruled surface iff $\chi(u) = 0$, and $F(u) = 0$.*

In view of Equation 3.6 and Theorem 3.5, we have:

Corollary 3.5. *The $\mathcal{BO}$ of a $\mathcal{SL}$ helicoidal surface, in general, does not have to be a $\mathcal{SL}$ helicoidal surface.*

3.3. **Construction of the constant Disteli-axis $\mathcal{SL}$ $\mathcal{RS}$.** In this subsection, we consider the construction of the constant Disteli-axis $\mathcal{SL}$ $\mathcal{RS}$. In view of Equations 2.14 and 3.17, and since $\widehat{\chi}(\widehat{u})$ is a constant dual number we have the ODE, $\widehat{g}''' - \widehat{\kappa}^2\widehat{g}' = 0$. After several algebraic manipulations, the general solution of this equation is:

$$\widehat{g}(\widehat{\vartheta}) = \left(\sin\widehat{\phi}\cosh\widehat{\vartheta}, \sin\widehat{\phi}\sinh\widehat{\vartheta}, \cos\widehat{\phi}\right), \tag{3.20}$$

Here $\widehat{\kappa u} := \widehat{\vartheta} = \vartheta + \varepsilon\vartheta^*$; where $0 \le \vartheta \le 2\pi$, and $\vartheta^* \in \mathbb{R}$. It is readily seen that:

$$\begin{pmatrix} \widehat{g}_1 \\ \widehat{g}_2 \\ \widehat{g}_3 \end{pmatrix} = \begin{pmatrix} \sin\widehat{\phi}\cosh\widehat{\vartheta} & \sin\widehat{\phi}\sinh\widehat{\vartheta} & \cos\widehat{\phi} \\ \sinh\widehat{\vartheta} & \cosh\widehat{\vartheta} & 0 \\ \cos\widehat{\phi}\cosh\widehat{\vartheta} & \cos\widehat{\phi}\sinh\widehat{\vartheta} & -\sin\widehat{\phi} \end{pmatrix} \begin{pmatrix} \widehat{e}_1 \\ \widehat{e}_2 \\ \widehat{e}_3 \end{pmatrix}. \tag{3.21}$$

Furthermore, in view of Equations 3.16 and 3.21 the $\mathcal{SL}$ Disteli-axis $\widehat{e}$ is:

$$\widehat{e} := \cos\widehat{\phi}\widehat{g}_1 - \sin\widehat{\phi}\widehat{g}_3 = \widehat{e}_3. \tag{3.22}$$

This shows that the instantaneous screw of the mobile Blaschke frame is the $\mathcal{SL}$ constant $\mathcal{DUV}$ $\widehat{e}_3$. If we let $\widehat{\vartheta} = \vartheta(1 + \varepsilon h)$; where h being the constant pitch of the Blaschke frame.

By differentiation Eq (38) with respect to ϑ and after several algebraic manipulation, we have $\widehat{p}(\vartheta) = (1+\varepsilon h)\sin\widehat{\phi}$, $\widehat{q}(\vartheta) = (1+\varepsilon h)\cos\widehat{\phi}$ and then

$$\varkappa = \phi^{*}\cot\phi + h \text{ and } F = h\tan\phi - \phi^{*}. \tag{3.23}$$

From the real and dual parts of Equation 3.20, resp., we have:

$$g(\vartheta) = (\sin\phi\cosh\vartheta, \sin\phi\sinh\vartheta, \cos\phi),$$

and

$$g^{*}(\vartheta) = \begin{pmatrix} g^{*}_{11} \\ g^{*}_{12} \\ g^{*}_{13} \end{pmatrix} = \begin{pmatrix} \phi^{*}\cos\phi\cosh\vartheta + \vartheta^{*}\sin\phi\sinh\vartheta \\ \phi^{*}\cos\phi\sinh\vartheta + \vartheta^{*}\sin\phi\cosh\vartheta \\ -\phi^{*}\sin\phi \end{pmatrix}.$$

Let $r(r_1, r_2, r_3)$ be a point on $\widehat{g}$. Since $r \times g = g^{*}$ we have the system of linear equations in r_1, r_2, and r_3:

$$\left.\begin{array}{c} r_2\cos\phi - r_3\sin\phi\sinh\vartheta = g^{*}_{11}, \\ r_1\cos\phi - r_3\sin\phi\cosh\vartheta = g^{*}_{12}, \\ (r_1\sinh\vartheta - r_2\cosh\vartheta)\sin\phi = g^{*}_{13}. \end{array}\right\}$$

The matrix of coefficients of unknowns r_1, r_2, and r_3 is

$$A = \begin{pmatrix} 0 & \cos\phi & -\sin\phi\sinh\vartheta \\ \cos\phi & 0 & \sin\phi\cosh\vartheta \\ \sin\phi\sinh\vartheta & -\sin\phi\cosh\vartheta & 0 \end{pmatrix}.$$

It is clear that $rank(A)$=2; where $\phi \neq p\pi$ (p is an integer) and $\vartheta \neq 0$. The *rank* of the augmented matrix:

$$\begin{pmatrix} 0 & \cos\phi & -\sin\phi\sinh\vartheta & g^{*}_{11} \\ \cos\phi & 0 & \sin\phi\cosh\vartheta & g^{*}_{12} \\ \sin\phi\sinh\vartheta & -\sin\phi\cosh\vartheta & 0 & g^{*}_{13} \end{pmatrix},$$

is 2. Then, this set has infinitely numerous solutions specfied with

$$\begin{gathered} r_1 = \phi^{*}\cosh\vartheta + (\vartheta^{*} + r_3)\tan\phi\sinh\vartheta, \\ r_2 = \phi^{*}\sinh\vartheta + (\vartheta^{*} + r_3)\tan\phi\cosh\vartheta, \\ -r_1\sinh\vartheta + r_2\cosh\vartheta = \phi^{*}. \end{gathered} \tag{3.24}$$

Since r_3 is assumed at random, then we may set $\vartheta^{*} + r_3 = 0$. In this case, Equation 3.24 reads

$$r_1 = \phi^{*}\cosh\vartheta, \; m_2 = \phi^{*}\sinh\vartheta, \; \vartheta^{*} + r_3 = 0.$$

We now just find the base curve as;

$$r(\vartheta) = (\phi^{*}\cosh\vartheta, \phi^{*}\sinh\vartheta, -h\vartheta).$$

Since $< r', g' >= 0$; ($' = \frac{d}{d\vartheta}$), then $r(\vartheta)(= c(\vartheta))$ is the striction curve of $(\widehat{g})$. Also, it can be show that $c(\vartheta)$ is a $\mathcal{SL}$ (resp. $\mathcal{TL}$) if and only if $\left|\phi^{*}\right| \leq |h|$ (resp. $|h| \leq \left|\phi^{*}\right|$). The curvature $\kappa_c(\vartheta)$ and the torsion $\tau_c(\vartheta)$ are

$$\kappa_c(\vartheta) = \frac{\left\|c' \times c''\right\|}{\|c'\|^3} = \frac{\phi^*}{-\phi^{*2} + h^2}, \text{ and } \tau_c(\vartheta)\frac{\det(c', c'', c''')}{\|c' \times c''\|^2} = \frac{h}{-\phi^{*2} + h^2}.$$

Then $r(\vartheta)$ is a $\mathcal{SL}$ (resp. $\mathcal{TL}$) cylindrical helix if and only if $\left|\phi^*\right| \leq |h|$ (resp. $|h| \leq \left|\phi^*\right|$). Furthermore, we have

$$(\widehat{g}) : \Gamma(\vartheta, t) = \begin{pmatrix} \phi^* \cosh\vartheta + t\sin\phi\cosh\vartheta \\ \phi^* \sinh\vartheta + t\sin\phi\sinh\vartheta \\ -h\vartheta + t\cos\phi \end{pmatrix}, \tag{3.25}$$

where ϕ, ϕ^*, and h can control the shape of $(\widehat{g})$. In view of the striction curve the $\mathcal{SL}$ $\mathcal{RS}$ $(\widehat{g})$ can be disseminated as follow:

(1) A $\mathcal{SL}$ helicoidal surface of the 3rd kind; for $h = 3$, $\phi^* = 1$, $\phi = \frac{\pi}{4}$, $-3 \leq \vartheta \leq 3$ and $-3 \leq t \leq 3$ (Fig. 2),

(2) A $\mathcal{SL}$ helicoidal surface of the 2nd kind; for $h = 2$, $\phi^* = 1$, $\phi = \frac{\pi}{2}$, $-3 \leq \vartheta \leq 3$ and $-2 \leq t \leq 2$ (Fig. 3),

(3) A $\mathcal{SL}$ helicoidal surface of the 1st kind; for $h = 1$, $\phi^* = 0$, $\phi = \frac{\pi}{2}$, $-3 \leq \vartheta \leq 3$ and $-2 \leq t \leq 2$ (Fig. 4),

(5) (4) A $\mathcal{SL}$ cone; for $h = \phi^* = 0$, $\phi = \frac{\pi}{4}$, $-3 \leq \vartheta \leq 3$ and $-2 \leq t \leq 2$ (Fig. 5),

(6) A $\mathcal{SL}$ cylinder; for $h = \phi = 0$, $\phi^* = 1$, $-2 \leq \vartheta \leq 2$ and $-1 \leq t \leq 1$ (Fig. 6).

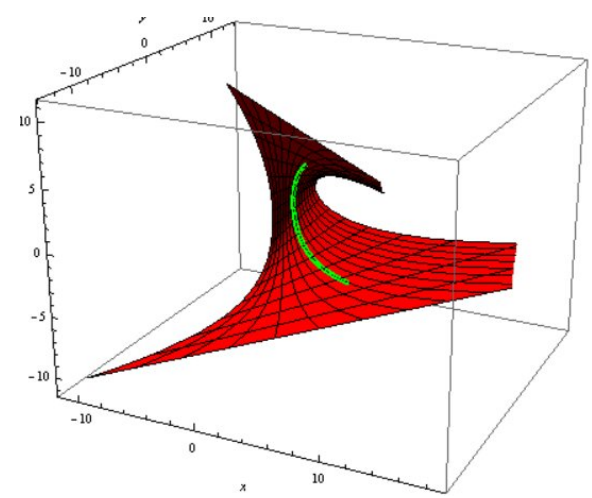

FIGURE 2. A $\mathcal{SL}$ helicoidal surface of the 3rd kind.

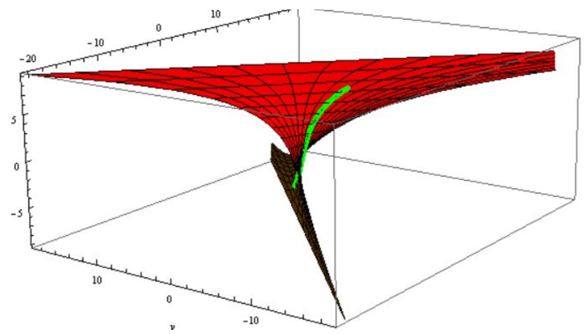

FIGURE 3. A $\mathcal{SL}$ helicoidal surface of the 2nd kind.

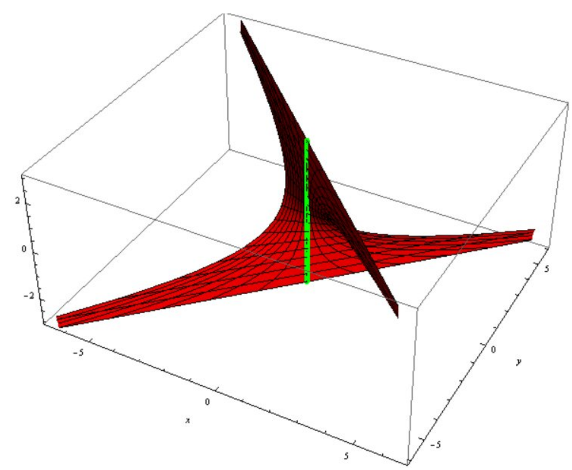

FIGURE 4. A $\mathcal{SL}$ helicoidal surface of the 1st kind.

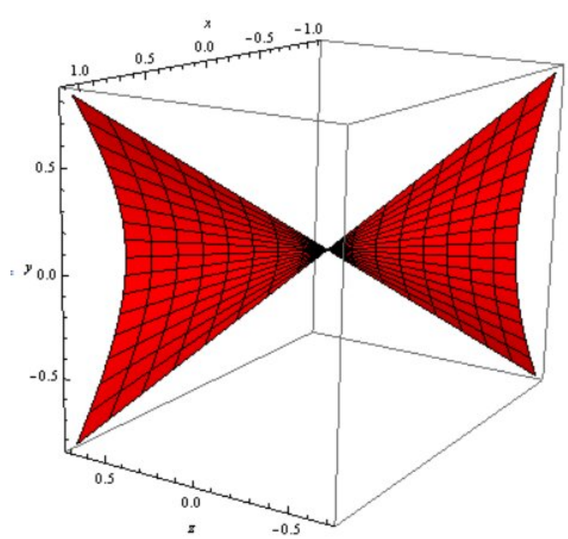

FIGURE 5. A $\mathcal{SL}$ cone.5

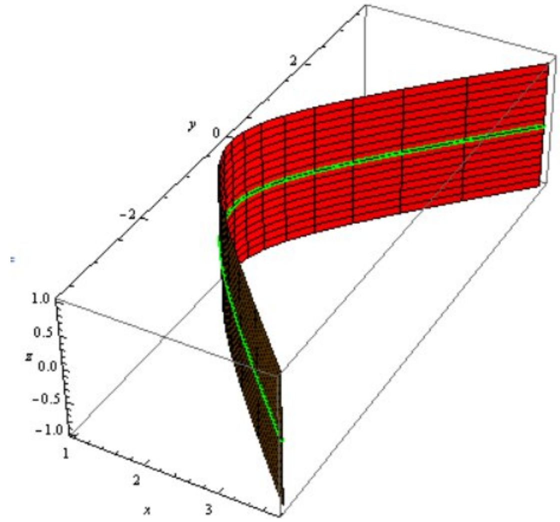

FIGURE 6. A spacelike cylinder.

On the other hand, the striction curve of $(\widehat{\overline{g}})$, in terms of $c(\vartheta)$, can be written as:

$$\bar{c}(\vartheta) := c(\vartheta) + \psi^* g_2(\theta) = (\phi^* \cosh\vartheta, \phi^* \sinh\vartheta, -h\vartheta) + \psi^* (\sinh\vartheta, \cosh\vartheta, 0). \tag{3.26}$$

With the help of the Equations 3.8, 3.21 and 3.26, we obtain

$$(\widehat{\overline{g}}) : \overline{\Gamma}(\vartheta, t) = \begin{pmatrix} \phi^* \cosh\vartheta + \psi^* \sinh\vartheta + t\sin(\phi+\psi)\cosh\vartheta \\ \phi^* \sinh\vartheta + \psi^* \cosh\vartheta + t\sin(\phi+\psi)\sinh\vartheta \\ -h\vartheta + t\cos(\phi+\psi) \end{pmatrix}. \tag{3.27}$$

Example 3.1. *In this example, we verify the idea of Corollary 3.5. In view of Equation 3.6 we have that:* $\widehat{\chi} = \cot\widehat{\phi} = 0$ *(*$\phi = \frac{\pi}{2}$, $\phi^* = 0$*)*$\Leftrightarrow 1 + \widehat{\overline{\chi}\chi} = 0$. *Then,*

$$(\widehat{\overline{g}}) : \overline{\Gamma}(\theta, t) = \begin{pmatrix} \psi^* \sinh\vartheta + t\cos\psi\cosh\vartheta \\ \psi^* \cosh\vartheta + t\cos\psi\sinh\vartheta \\ -h\vartheta - t\sin\psi \end{pmatrix}, \tag{3.28}$$

or,

$$(\widehat{\overline{g}}) : -\frac{x^2}{\psi^{*2}} + \frac{y^2}{\psi^{*2}} + \frac{Z^2}{\beta^2} = 1,$$

where $\beta = \psi^* \tan\psi$, *and* $Z = z + h\vartheta$. *The constants* h, ψ *and* ψ^* *can organize the constitute of the* $\mathcal{SL}$ $\mathcal{RS}$ $(\widehat{\overline{g}})$. *Hence,* $(\widehat{\overline{g}})$ *is a 3-parameter pencil of hyperbolic unit spheres which are the* $\mathcal{BO}$ *of* $\mathcal{SL}$ *helicoidal surface. The intersection of each hyperbolic unit sphere and the conformable* $\mathcal{TL}$ *plane* $z = h\vartheta$ *is a one-parameter pencil of hyperbola* $-x^2 + y^2 = \psi^{*2}$ *Therefore the envelope of* $(\widehat{\overline{g}})$ *is a one-parameter pencil of* $\mathcal{SL}$ *cylinders. Take* $\psi^* = 1$, $\psi = \pi/4$ *and* $h = 1$ *for example, the* $\mathcal{SL}$ $\mathcal{BO}$ *is shown Fig.6; where* $-3 \leq \vartheta \leq 3$ *and* $-2 \leq t \leq 2$. *The graph of the* $\mathcal{SL}$ *helicoidal surface of the 1st kind* $(\widehat{g})$ *and its* $\mathcal{SL}$ $\mathcal{BO}$ $(\widehat{\overline{g}})$ *is shown in Fig.7.*

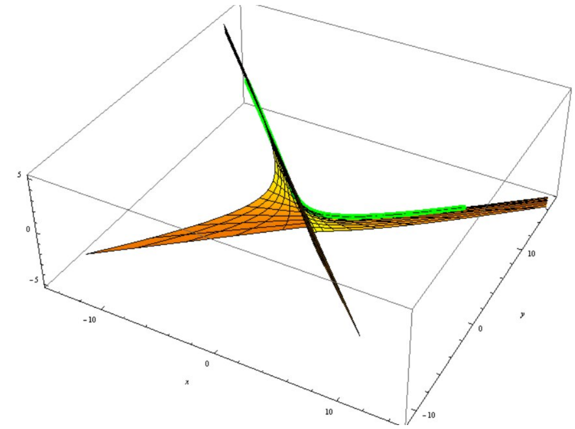

FIGURE 7. A $\mathcal{SL}$ $\mathcal{BO}$

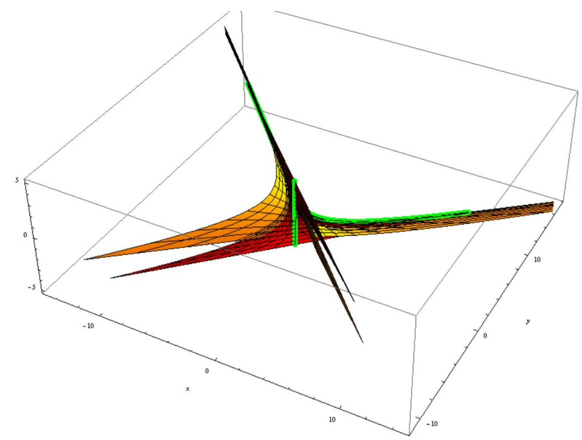

FIGURE 8. A $\mathcal{SL}$ helicoidal surface of the 1st kind $(\widehat{g})$ and its $\mathcal{SL}$ $\mathcal{BO}$ $(\widehat{\overline{g}})$.

4. Conclusion

In this study, an addendum of $\mathcal{B}$ offsets of curves for $\mathcal{SL}$ ruled and developable surfaces has been modulated. Noteworthy, there are many similarities meanwhile the $\mathcal{SL}$ $\mathcal{B}$ curves and the $\mathcal{BO}$ for $\mathcal{SL}$ ruled surfaces. For epitome, a $\mathcal{SL}$ $\mathcal{RS}$ can have an infinity of $\mathcal{BO}$ in identification via a plane curve can have an infinity of $\mathcal{B}$ mates. From this outcome the conclusions of some beneficial geometrical connections, epitomes and instructive figures of the $\mathcal{SL}$ ruled surfaces are ensured. For future work, we will attract by the novel ideas that Gaussian and mean curvatures of these Bertrand offsets can be acquired, when the W-map for the $\mathcal{SL}$ $\mathcal{BO}$ ruled surfaces is perceived. We will also consider integrating the study of singularity and submanifold theories and so forth, given in [21–23], with the consequences of this work to explore novel manners to find more theorems linked to symmetric possessions on this theme.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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