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Dynamical Analysis of Thirtieth-Order Difference Equations

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Abstract. The main goal of this paper is to determine exact solutions of a family of thirtieth-order difference equations with variable coefficients. We use similarity variables obtained via symmetries to lower the order of the equations. We then reverse the transformations and obtain closed form solutions. We compare our solutions to those found in the literature for special cases. We investigate the periodic nature of the solutions and present some numerical examples to confirm the results. Finally, we analyze the stability of the equilibrium points. The method employed in this work can be applied to equations of higher order provided that they admit non zero characteristics.

1. Introduction

Difference equations are used to investigate the evolution of a state of a particular system over a period of time. In essence, difference equations are used to determine the future value of a system in relation to its past value and this is done through a recurrence relationship. Difference equations are useful in various mathematical disciplines such as control theory, integrable systems theory, and number theory. However, these equations can also be applied when solving real-life problems in areas such as economics, biology, physics, engineering, computer science, and others. Applications in economics include using difference equations to model market trends and economic growth and determining the price of certain commodities. In biology, for example, researchers use difference equations to build models to study population dynamics epidemiology to understand and predict the spread of diseases. In physics, difference equations are particularly useful for simplifying complex computations that would result from investigating systems that involve many elements. Difference equations have also been used to solve numerical solutions to some differential equations and this is done by discretizing the given differential equation.
Various techniques can be used to solve difference equations such as numerical techniques or iterative methods. Analytical methods involve finding a closed form expression satisfying the given

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difference equation. Other methods involve using a combination of analytical and numerical techniques. For instance, Maeda's method for solving difference equations uses both analytical and iterative techniques to solve non-homogeneous difference equations with linear-constant coefficients [12]. Some other methods include direct substitution, matrix methods, generating functions, and difference calculus. In this study, we use a symmetry-based method to find the analytic solution of the following equation

$$u_{n+1} = \frac{u_{n-29}}{A_n + B_n u_{n-4} u_{n-9} u_{n-14} u_{n-19} u_{n-24} u_{n-29}} \tag{1.1}$$

where A_n and B_n are sequences with initial conditions $u_{-29}, u_{-28}, \ldots, u_0$.

Lama [1] studied the solutions and the dynamical properties, that is, boundedness, local and global points, and the equilibrium points of the difference equation

$$\zeta_{r+1} = \frac{\zeta_{r-29}}{\pm 1 \pm \zeta_{r-29}\zeta_{r-24}\zeta_{r-19}\zeta_{r-14}\zeta_{r-9}\zeta_{r-4}}, \qquad r = 0, 1, 2, \cdots . \tag{1.2}$$

where the initial conditions $\zeta_{-29}, \zeta_{-28}, \ldots, \zeta_0$ are real numbers.

Our main objective is to find the solutions of (1.1) using its symmetry. In our investigation, we expect our solutions to be more general with less restrictions on A_n and B_n. For definiteness, we study the equivalent equation

$$u_{n+30} = \frac{u_n}{A_n + B_n u_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}} \tag{1.3}$$

and demonstrate how one uses the solution of (1.3) to recover those of (1.1). For more results on difference equations, the reader can refer to [2–8, 10]

1.1. **Preliminaries.** In this section, we define the terms used and the notation according to [9]. Let

$$u_{n+30} = \omega(n, u_n, u_{n+5}, u_{n+10}, u_{n+15}, u_{n+20}, u_{n+25}) \tag{1.4}$$

be a difference equation of order thirty. We assume that the partial derivative of ω with respect to u_n is non-zero.

Definition 1.1. *The forward shift operator is defined as* $S : n \mapsto n+1$ *and so,* $S^i(u_n) = u_{n+i}$.

Consider the point transformations

$$\Gamma_\epsilon : (n, u_n) \mapsto (\tilde{n}, \tilde{u}_n). \tag{1.5}$$

It is known [9] that Γ_ε is a one-parameter Lie group of transformations if:

- Γ_0 is the identity map, so that $\tilde{\mathbf{x}} = \mathbf{x}$ when $\epsilon = 0$.
- $\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0.
- Each $\tilde{x}_i$ can be represented as a Taylor series in ϵ (in a neighborhood of $\epsilon = 0$ that is determined by $\mathbf{x}$), and therefore $\tilde{x}_i(\mathbf{x}; \epsilon) = x_i + \epsilon \xi_i(\mathbf{x}) + O(\epsilon^2), i = 1, \ldots, p$.

In this paper, we shall assume that the Lie group of point transformations is of the form

$$\tilde{n} = n; \quad \tilde{u}_n = u_n + \epsilon Q(n, u_n) \tag{1.6}$$

and that the corresponding (prolonged) infinitesimal generator is given by

$$\begin{aligned} X =& Q(n,u_n)\frac{\partial}{\partial u_n} + S^5 Q(n,u_n)\frac{\partial}{\partial u_{n+5}} + S^{10} Q(n,u_n)\frac{\partial}{\partial u_{n+10}} + S^{15} Q(n,u_n)\frac{\partial}{\partial u_{n+15}} \\ &+ S^{20} Q(n,u_n)\frac{\partial}{\partial u_{n+20}} + S^{25} Q(n,u_n)\frac{\partial}{\partial u_{n+25}}. \end{aligned} \tag{1.7}$$

The characteristics $Q(n, u_n)$ can be found by solving the linearized symmetry condition

$$\mathcal{S}^{(30)} Q(n, u_n) - X\omega = 0 \tag{1.8}$$

as long as (1.4) holds.

2. Main results

2.1. Symmetries. In the first part of the investigation, we focus on finding the symmetries of (1.3). Applying the symmetry condition given in (1.8), we obtain the following:

$$\begin{aligned} &S^{30} Q(n,u_n) - Q(n,u_n)\frac{\partial \omega}{\partial u_n} - S^5 Q(n,u_n)\frac{\partial w}{\partial u_{n+5}} - S^{10} Q(n,u_n)\frac{\partial w}{\partial u_{n+10}} - \\ &S^{15} Q(n,u_n)\frac{\partial w}{\partial u_{n+15}} - S^{20} Q(n,u_n)\frac{\partial w}{\partial u_{n+20}} - S^{25} Q(n,u_n)\frac{\partial w}{\partial u_{n+25}} = 0. \end{aligned} \tag{2.1}$$

We apply the differential operator $\frac{\partial}{\partial u_n} - \frac{\omega_{,u_n}}{\omega_{,u_{n+5}}}\frac{\partial}{\partial u_{n+5}}$ to (2.1) and obtain

$$\begin{aligned} &Q'(n,u_n)\omega_{,u_n} - Q(n,u_n)\omega_{,u_n u_n} - S^5 Q(n,u_n)\omega_{,u_n u_{n+5}} - S^{10} Q(n,u_n)\omega_{,u_n u_{n+10}} - S^{15} Q(n,u_n)\omega_{,u_n u_{n+15}} \\ &- S^{20} Q(n,u_n)\omega_{,u_n u_{n+20}} - S^{25} Q(n,u_n)\omega_{,u_n u_{n+25}} - \frac{\omega_{,u_n}}{\omega_{,u_n+5}}\Bigg[- Q(n,u_n)\omega_{,u_n u_{n+5}} - S^5 Q'(n,u_n)\omega_{,u_{n+5}} \\ &- S^5 Q(n,u_n)\omega_{,u_{n+5} u_{n+5}} - S^{10} Q(n,u_n)\omega_{,u_{n+5} u_{n+10}} - S^{15} Q(n,u_n)\omega_{,u_{n+5} u_{n+15}} - S^{20} Q(n,u_n)\omega_{,u_{n+5} u_{n+20}} \\ &- S^{25} Q(n,u_n)\omega_{,u_{n+5} u_{n+25}} \Bigg] = 0. \end{aligned} \tag{2.2}$$

Note that $f_{,x}$ denotes the partial derivative of f with respect to x. Substituting the partial derivatives into (2.2) with a bit of simplification, we get

$$\begin{aligned} &2Q(n,u_n)u_{n+10}u_{n+15}u_{n+20}u_{n+25} - (Q'(n,u_n) - S^5 Q'(n,u_n))u_n u_{n+10}u_{n+15}u_{n+20}u_{n+25} \\ &+ S^{10} Q(n,u_n)u_n u_{n+15}u_{n+20}u_{n+25} + S^{15} Q(n,u_n)u_n u_{n+10}u_{n+20}u_{n+25} + S^{20} Q(n,u_n)u_n u_{n+10}u_{n+15}u_{n+25} \\ &+ S^{25} Q(n,u_n)u_n u_{n+10}u_{n+15}u_{n+20} = 0. \end{aligned} \tag{2.3}$$

Now, differentiating (2.3) twice with respect to u_n and a bit of simplification, we get

$$Q'''(n, u_n) = 0. \tag{2.4}$$

It follows that the characteristic must be of the form

$$Q(n,u_n) = \alpha_n + \beta_n u_n + \gamma_n u_n^2, \tag{2.5}$$

where α_n, β_n and γ_n are some functions of n. Using (2.5) in (2.1) and employing the method of separation (see [3,9] for more details), we obtain the following system of constraints:

$$\alpha_n = 0, \tag{2.6}$$

$$\beta_{n+5} + \beta_{n+10} + \beta_{n+15} + \beta_{n+20} + \beta_{n+25} + \beta_{n+30} = 0, \tag{2.7}$$

$$\beta_n - \beta_{n+30} = 0, \tag{2.8}$$

$$\gamma_n = 0. \tag{2.9}$$

Consequently,

$$\beta_n + \beta_{n+5} + \beta_{n+10} + \beta_{n+15} + \beta_{n+20} + \beta_{n+25} = 0. \tag{2.10}$$

This is a linear difference equation with constant coefficients. It follows that

$$\beta_n = \left(\exp \frac{i(2k\pi)}{30}\right)^n, \tag{2.11}$$

$k = 1, \dots, 29$ and $k \notin \{6, 12, 18, 24\}$. Hence, the 25 symmetries are as follows:

$$X_k = \exp\left(\frac{2k\pi n}{30} i\right) u_n \frac{\partial}{\partial u_n}, \tag{2.12}$$

$k = 1, \dots, 29$ and $k \notin \{6, 12, 18, 24\}$.

2.2. **Reduction and exact solutions.** We introduce the canonical coordinate

$$S_n = \int \frac{du_n}{Q(n,u_n)} = \frac{1}{\beta_n} \ln |u_n| \tag{2.13}$$

and let

$$\begin{aligned} \tilde{V_n} =& S_n\beta_n + S_{n+5}\beta_{n+5} + S_{n+10}\beta_{n+10} + S_{n+15}\beta_{n+15} + S_{n+20}\beta_{n+20} + S_{n+25}\beta_{n+25} \\ =& \ln\left(u_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}\right). \end{aligned} \tag{2.14}$$

Setting $V_n = \exp(-\tilde{V_n})$, we obtain

$$V_n = \frac{1}{u_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}}. \tag{2.15}$$

Now shifting equation (2.15) five times and substituting for u_{n+30}, we get

$$V_{n+5} = A_n V_n + B_n. \tag{2.16}$$

By iterating (2.16), we get

$$V_{5n+j} = V_j \left(\prod_{t}^{n-1} A_{5t+j}\right) + \sum_{i=0}^{n-1}\left(B_{5i+j} \prod_{k_2=i+1}^{n-1} A_{5k_2+j}\right), \tag{2.17}$$

$j = 0, 1, 2, 3, 4$. From (2.15), we have that

$$u_{n+30} = \frac{V_n}{V_{n+5}} u_n. \tag{2.18}$$

By iteration, we get

$$\begin{aligned} u_{30n+j} &= u_j \prod_{t=0}^{n-1} \frac{V_{30t+j}}{V_{30t+j+5}} \\ &= u_j \prod_{t=0}^{n-1} \frac{V_{30t+5\lfloor \frac{j}{5} \rfloor + \tau(j)}}{V_{30t+5+5\lfloor \frac{j}{5} \rfloor + \tau(j)}} \\ &= u_j \prod_{t=0}^{n-1} \frac{V_{5(6t+\lfloor \frac{j}{5} \rfloor) + \tau(j)}}{V_{5(6t+1+\lfloor \frac{j}{5} \rfloor) + \tau(j)}}. \end{aligned} \tag{2.19}$$

Here, $\lfloor \cdot \rfloor$ denotes the floor function and $\tau(j)$ is the remainder when j is divided by 5. Substituting (2.17) in equation (2.19), we obtain

$$\begin{aligned} u_{30n+j} &= u_j \prod_{t=0}^{n-1} \frac{V_{\tau(j)} \left(\prod_{s=0}^{6t+\lfloor \frac{j}{5} \rfloor - 1} A_{5s+\tau(j)} \right) + \sum_{i=0}^{6t+\lfloor \frac{j}{5} \rfloor - 1} B_{5i+\tau(j)} \left(\prod_{k_2=i+1}^{6t+\lfloor \frac{j}{5} \rfloor - 1} A_{5k_2+\tau(j)} \right)}{V_{\tau(j)} \left(\prod_{s=0}^{6t+\lfloor \frac{j}{5} \rfloor} A_{5s+\tau(j)} \right) + \sum_{i=0}^{6t+\lfloor \frac{j}{5} \rfloor} B_{5i+\tau(j)} \left(\prod_{k_2=i+1}^{6t+\lfloor \frac{j}{5} \rfloor} A_{5k_2+\tau(j)} \right)} \\ &= u_j \prod_{t=0}^{n-1} \frac{\left(\prod_{s=0}^{6t+\lfloor \frac{j}{5} \rfloor - 1} A_{5s+\tau(j)} \right) + \sum_{i=0}^{6t+\lfloor \frac{j}{5} \rfloor - 1} \frac{B_{5i+\tau(j)}}{V_{\tau(j)}} \left(\prod_{k_2=i+1}^{6t+\lfloor \frac{j}{5} \rfloor - 1} A_{5k_2+\tau(j)} \right)}{\left(\prod_{s=0}^{6t+\lfloor \frac{j}{5} \rfloor} A_{5s+\tau(j)} \right) + \sum_{i=0}^{6t+\lfloor \frac{j}{5} \rfloor} \frac{B_{5i+\tau(j)}}{V_{\tau(j)}} \left(\prod_{k_2=i+1}^{6t+\lfloor \frac{j}{5} \rfloor} A_{5k_2+\tau(j)} \right)}, \end{aligned} \tag{2.20}$$

where $1/V_h = u_h u_{h+5} u_{h+10} u_{h+15} u_{h+20} u_{h+25}$. Equation (2.20) gives the solution of (1.3) in terms of n and the initial conditions. Consequently, the solution of (1.1) is obtained by shifting back (2.20) 29 times as follows:

$$u_{30n+j-29} = u_{j-20} \prod_{t=0}^{n-1} \frac{\left(\prod_{s=0}^{6t+\lfloor \frac{j}{5} \rfloor - 1} a_{5s+\tau(j)} \right) + \sum_{i=0}^{6t+\lfloor \frac{j}{5} \rfloor - 1} \frac{b_{5i+\tau(j)}}{V_{\tau(j)-29}} \left(\prod_{k_2=i+1}^{6t+\lfloor \frac{j}{5} \rfloor - 1} a_{5k_2+\tau(j)} \right)}{\left(\prod_{s=0}^{6t+\lfloor \frac{j}{5} \rfloor} a_{5s+\tau(j)} \right) + \sum_{i=0}^{6t+\lfloor \frac{j}{5} \rfloor} \frac{b_{5i+\tau(j)}}{V_{\tau(j)-29}} \left(\prod_{k_2=i+1}^{6t+\lfloor \frac{j}{5} \rfloor} a_{5k_2+\tau(j)} \right)}. \tag{2.21}$$

2.3. **The case where the sequences A_n and B_n are one-periodic.** Here, we set $A_n = A$ and $B_n = B$. Equation (2.20) simplifies to

$$u_{30n+j} = u_j \prod_{t=0}^{n-1} \frac{A^{6t+\lfloor \frac{j}{5} \rfloor} + \frac{B}{V_{\tau(j)}} \sum_{i=0}^{6t+\lfloor \frac{j}{5} \rfloor - 1} A^i}{A^{6t+\lfloor \frac{j}{5} \rfloor+1} + \frac{B}{V_{\tau(j)}} \sum_{i=0}^{6t+\lfloor \frac{j}{5} \rfloor} A^i}. \tag{2.22}$$

Now, using (2.15) in the equation above, we get the closed form solution of (1.3) for constant coefficients given by

$$u_{30n+j} = u_j \prod_{t=0}^{n-1} \frac{A^{6t+\lfloor \frac{j}{5} \rfloor} + Bu_{\tau(j)}u_{\tau(j)+5}u_{\tau(j)+10}u_{\tau(j)+15}u_{\tau(j)+20}u_{\tau(j)+25} \sum_{i=0}^{6t+\lfloor \frac{j}{5} \rfloor - 1} A^i}{A^{6t+\lfloor \frac{j}{5} \rfloor+1} + Bu_{\tau(j)}u_{\tau(j)+5}u_{\tau(j)+10}u_{\tau(j)+15}u_{\tau(j)+20}u_{\tau(j)+25} \sum_{i=0}^{6t+\lfloor \frac{j}{5} \rfloor} A^i} \tag{2.23}$$

for $j = 0, 1, \ldots, 29$. Note that $\tau(j) \in \{0, 1, 2, 3\}$ when $j \in \{0, 1, \ldots, 29\}$.

2.3.1. *Case where* $A = 1$. When we replace $A = 1$ in (2.23), we get

$$u_{30n+j} = u_j \prod_{t=0}^{n-1} \frac{1 + (6t + \lfloor \frac{j}{5} \rfloor)Bu_{\tau(j)}u_{\tau(j)+5}u_{\tau(j)+10}u_{\tau(j)+15}u_{\tau(j)+20}u_{\tau(j)+25}}{1 + (6t + \lfloor \frac{j}{5} \rfloor + 1)Bu_{\tau(j)}u_{\tau(j)+5}u_{\tau(j)+10}u_{\tau(j)+15}u_{\tau(j)+20}u_{\tau(j)+25}}. \tag{2.24}$$

2.3.2. *Case where* $A \neq 1$. In this instance, (2.23) becomes

$$u_{30n+j} = u_j \prod_{t=0}^{n-1} \frac{A^{6t+\lfloor \frac{j}{5} \rfloor} + \frac{1-A^{6t+\lfloor \frac{j}{5} \rfloor}}{1-A} Bu_{\tau(j)}u_{\tau(j)+5}u_{\tau(j)+10}u_{\tau(j)+15}u_{\tau(j)+20}u_{\tau(j)+25}}{A^{6t+\lfloor \frac{j}{5} \rfloor+1} + \frac{1-A^{6t+\lfloor \frac{j}{5} \rfloor+1}}{1-A} Bu_{\tau(j)}u_{\tau(j)+5}u_{\tau(j)+10}u_{\tau(j)+15}u_{\tau(j)+20}u_{\tau(j)+25}}. \tag{2.25}$$

Recall that we shifted equation (1.1) forward 29 times to obtain (1.3) whose solution is given by (2.24) and (2.25). Now we shift backwards 29 times the equations (2.24) and (2.25) to obtain the solution of the difference equation (1.1) which is given by

$$u_{30n+j-29} = u_{j-29} \prod_{t=0}^{n-1} \frac{1 + (6t + \lfloor \frac{j}{5} \rfloor)Bu_{\tau(j)-29}u_{\tau(j)-24}u_{\tau(j)-19}u_{\tau(j)-14}u_{\tau(j)-9}u_{\tau(j)-4}}{1 + (6t + \lfloor \frac{j}{5} \rfloor + 1)Bu_{\tau(j)-29}u_{\tau(j)-24}u_{\tau(j)-19}u_{\tau(j)-14}u_{\tau(j)-9}u_{\tau(j)-4}} \tag{2.26}$$

when $a = 1$; and

$$u_{30n+j-29} = u_{j-29} \prod_{t=0}^{n-1} \frac{A^{6t+\lfloor \frac{j}{5} \rfloor} + \left(\frac{1-A^{6t+\lfloor \frac{j}{5} \rfloor}}{1-A}\right) Bu_{\tau(j)-29}u_{\tau(j)-24}u_{\tau(j)-19}u_{\tau(j)-14}u_{\tau(j)-9}u_{\tau(j)-4}}{A^{6t+\lfloor \frac{j}{5} \rfloor+1} + \left(\frac{1-A^{6t+\lfloor \frac{j}{5} \rfloor+1}}{1-A}\right) Bu_{\tau(j)-29}u_{\tau(j)-24}u_{\tau(j)-19}u_{\tau(j)-14}u_{\tau(j)-9}u_{\tau(j)-4}} \tag{2.27}$$

when $a \neq 1$.

3. Some special cases leading to some results in the existing literature

Setting $k := 29 - j$, we have that $\lfloor \frac{j}{5} \rfloor = 5 - \lfloor \frac{k}{5} \rfloor$ and $\tau(j) = 4 - \tau(k)$ when $j = 0, 1, \ldots, 20$, $k = j - 29$. Hence, equations (2.26) and (2.27) become

$$u_{30n-k} = u_{-k} \prod_{t=0}^{n-1} \frac{1 + (6t + 5 - \lfloor \frac{k}{5} \rfloor) B u_{-\tau(k)-25} u_{-\tau(k)-20} u_{-\tau(k)-15} u_{-\tau(k)-10} u_{-\tau(k)-5} u_{-\tau(k)}}{1 + (6t + 6 - \lfloor \frac{k}{5} \rfloor) B u_{-\tau(k)-25} u_{-\tau(k)-20} u_{-\tau(k)-15} u_{-\tau(k)-10} u_{-\tau(k)-5} u_{-\tau(k)}} \tag{3.1}$$

when $a = 1$; and

$$u_{30n-k} = u_{-k} \prod_{t=0}^{n-1} \frac{A^{6t+5-\lfloor \frac{k}{5} \rfloor} + \left(\frac{1 - A^{6t+5-\lfloor \frac{k}{5} \rfloor}}{1-A} \right) B u_{-\tau(k)-25} u_{-\tau(k)-20} u_{-\tau(k)-15} u_{-\tau(k)-10} u_{-\tau(k)-5} u_{-\tau(k)}}{A^{6t+6-\lfloor \frac{k}{5} \rfloor} + \left(\frac{1 - A^{6t+6-\lfloor \frac{k}{5} \rfloor}}{1-A} \right) B u_{-\tau(k)-25} u_{-\tau(k)-20} u_{-\tau(k)-15} u_{-\tau(k)-10} u_{-\tau(k)-5} u_{-\tau(k)}} \tag{3.2}$$

when $A \neq 1$. If we replace $B = 1$ in (3.1), we get the results in [1] (see Theorems 1 and 6). In fact, for $B = 1$, using (2.26), we have

$$\begin{aligned} u_{30n-k} &= u_{-k} \prod_{t=0}^{n-1} \left(\frac{1 + (6t + 5 - \lfloor \frac{k}{5} \rfloor) u_{-\tau(k)-25} u_{-\tau(k)-20} u_{-\tau(k)-15} u_{-\tau(k)-10} u_{-\tau(k)-5} u_{-\tau(k)}}{1 + (6t + 6 - \lfloor \frac{k}{5} \rfloor) u_{-\tau(k)-25} u_{-\tau(k)-20} u_{-\tau(k)-15} u_{-\tau(k)-10} u_{-\tau(k)-5} u_{-\tau(k)}} \right) \\ &= \epsilon_k \prod_{t=0}^{n-1} \left(\frac{1 + (6t + \eta_k - 1) \mu_k}{1 + (6t + \eta_k) \mu_k} \right), \end{aligned} \tag{3.3}$$

where $\mu_k = \prod_{j=0}^{5} \epsilon_{\bmod (k,5)+5j}$, $\eta_k = 6 - \lfloor \frac{k}{5} \rfloor$ and $u_{-k} = \epsilon_k$ with $k = 0, 1, \ldots, 29$.

Similarly, for $B = -1$, using (2.26), we have

$$\begin{aligned} u_{30n-k} &= u_{-k} \prod_{t=0}^{n-1} \left(\frac{1 - (6t + 5 - \lfloor \frac{k}{5} \rfloor) u_{-\tau(k)-25} u_{-\tau(k)-20} u_{-\tau(k)-15} u_{-\tau(k)-10} u_{-\tau(k)-5} u_{-\tau(k)}}{1 - (6t + 6 - \lfloor \frac{k}{5} \rfloor) u_{-\tau(k)-25} u_{-\tau(k)-20} u_{-\tau(k)-15} u_{-\tau(k)-10} u_{-\tau(k)-5} u_{-\tau(k)}} \right) \\ &= \epsilon_k \prod_{t=0}^{n-1} \left(\frac{1 - (6t + \eta_k - 1) \mu_k}{1 - (6t + \eta_k) \mu_k} \right), \end{aligned} \tag{3.4}$$

where $\mu_k = \prod_{j=0}^{5} \epsilon_{\bmod (k,5)+5j}$, $\eta_k = 6 - \lfloor \frac{k}{5} \rfloor$ and $u_{-k} = \epsilon_k$ with $k = 0, 1, \ldots, 29$.

If we replace $A = -1$ and $B = \pm 1$ in (3.2), we get the results in [1] (see Theorems 8 and 12). In fact, for $A = -1$ and $B = 1$, using (3.2), we have

$$\begin{aligned} u_{30n-k} &= u_{-k} \prod_{t=0}^{n-1} \frac{(-1)^{1+\lfloor \frac{k}{5} \rfloor} + \left(\frac{1 - (-1)^{1+\lfloor \frac{k}{5} \rfloor}}{2} \right) u_{-\tau(k)-25} u_{-\tau(k)-20} u_{-\tau(k)-15} u_{-\tau(k)-10} u_{-\tau(k)-5} u_{-\tau(k)}}{-(-1)^{1+\lfloor \frac{k}{5} \rfloor} + \left(\frac{1 + (-1)^{1+\lfloor \frac{k}{5} \rfloor}}{2} \right) u_{-\tau(k)-25} u_{-\tau(k)-20} u_{-\tau(k)-15} u_{-\tau(k)-10} u_{-\tau(k)-5} u_{-\tau(k)}} \\ &= \frac{\epsilon_k}{(-1 + \mu_k)^{t\alpha_k}}, \end{aligned} \tag{3.5}$$

where $\mu_k = \prod_{j=0}^{5} \epsilon_{\bmod (k,5)+5j}$, $\alpha_k = (-1)^{\lfloor \frac{k}{5} \rfloor + 1}$ and $u_{-k} = \epsilon_k$ with $k = 0, 1, \ldots, 29$.

Similarly, for $A=-1$ and $B=-1$, using (3.2), we have

$$\begin{aligned} u_{30n-k} &= u_{-k}\prod_{t=0}^{n-1}\frac{(-1)^{1+\lfloor\frac{k}{5}\rfloor}-\left(\frac{1-(-1)^{1+\lfloor\frac{k}{5}\rfloor}}{2}\right)u_{-\tau(k)-25}u_{-\tau(k)-20}u_{-\tau(k)-15}u_{-\tau(k)-10}u_{-\tau(k)-5}u_{-\tau(k)}}{-(-1)^{1+\lfloor\frac{k}{5}\rfloor}-\left(\frac{1+(-1)^{1+\lfloor\frac{k}{5}\rfloor}}{2}\right)u_{-\tau(k)-25}u_{-\tau(k)-20}u_{-\tau(k)-15}u_{-\tau(k)-10}u_{-\tau(k)-5}u_{-\tau(k)}}\\ &= \frac{\epsilon_k}{(-1-\mu_k)^{t\alpha_k}}, \end{aligned} \tag{3.6}$$

where $\mu\prod_{j=0}^{5}\epsilon_{\text{mod }(k,5)+5j}$, $\alpha_k=(-1)^{\lfloor\frac{k}{5}\rfloor+1}$ and $u_{-k}=\epsilon_k$ with $k=0,1,\dots,29$.

4. Behavior of the solutions

In this section, we give the conditions for periodic solutions with periods 2, 3, 6, 10, 15 and 30. Additionally, we show that the equilibrium points are non-hyperbolic.

Theorem 4.1. *Let u_n be a solution of*

$$u_{n+30}=\frac{u_n}{A+Bu_nu_{n+5}u_{n+10}u_{n+15}u_{n+20}u_{n+25}} \tag{4.1}$$

with initial conditions $u_i,\ i=0,\dots,29$; and $A\neq 1$ and B some non-zero constants. If the initial conditions satisfy the conditions

$$u_iu_{i+5}u_{i+10}u_{i+15}u_{i+20}u_{i+25}=\frac{1-A}{B} \tag{4.2}$$

and

$$u_i\neq u_{i+2}, u_i\neq u_{i+3}, u_i\neq u_{i+6}, u_i\neq u_{i+10}, u_i\neq u_{i+15}, \tag{4.3}$$

then we have a 30-periodic solution.

Proof. Using the assumption $u_iu_{i+5}u_{i+10}u_{i+15}u_{i+20}u_{i+25}=\frac{1-A}{B}$ stated in Theorem 4.1, equation (2.25) reduces to

$$\begin{aligned} u_{30n+j} =& u_j\prod_{t=0}^{n-1}\frac{A^{6t+\lfloor\frac{j}{5}\rfloor}+\left(\frac{1-A^{6t+\lfloor\frac{j}{5}\rfloor}}{1-A}\right)B\left(\frac{1-A}{B}\right)}{A^{6t+\lfloor\frac{j}{5}\rfloor+1}+\left(\frac{1-A^{6t+\lfloor\frac{j}{5}\rfloor+1}}{1-A}\right)B\left(\frac{1-A}{B}\right)} \qquad (4.4)\\ =& u_j. \qquad (4.5) \end{aligned}$$

Adding the condition $u_i\neq u_{i+2}, u_i\neq u_{i+3}, u_i\neq u_{i+6}, u_i\neq u_{i+10}, u_i\neq u_{i+15}$, we conclude that the solution is periodic with period 30. □

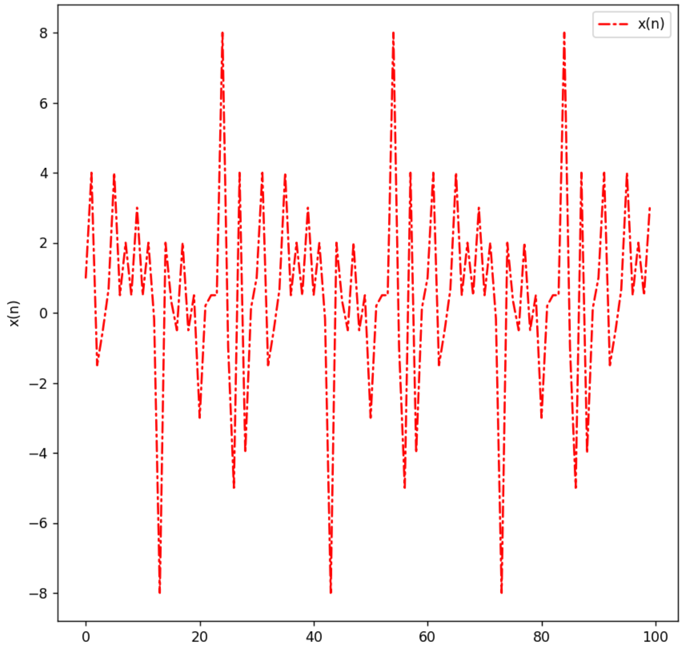

Figure 1. Graph of $u_{n+30} = \frac{u_n}{-3+2u_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}}$.

Figure 1 illustrates the graph of (1.3) with the initial conditions, $u_0 = 1; u_1 = 4; u_2 = -3/2; u_3 = -1/2; u_4 = 2/3; u_5 = 4; u_6 = 1/2; u_7 = 2; u_8 = 1/2; u_9 = 3; u_{10} = 1/2; u_{11} = 2; u_{12} = -1/6; u_{13} = -8; u_{14} = 2; u_{15} = 1/3; u_{16} = -1/2; u_{17} = 2; u_{18} = -1/2; u_{19} = 1/2; u_{20} = -3; u_{21} = 1/5; u_{22} = 1/2; u_{23} = 1/2; u_{24} = 8; u_{25} = -1; u_{26} = -5; u_{27} = 4; u_{28} = -4; u_{29} = 1/8$, satisfying the conditions in Theorem 4.1. As expected, the solution is 30-periodic.

Theorem 4.2. *Let u_n be a solution of*

$$u_{n+30} = \frac{u_n}{A + Bu_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}} \tag{4.6}$$

with initial conditions u_i, $i = 0, \ldots, 29$; and $A \neq 1$ and B some non-zero constants. If the initial conditions satisfy the conditions

$$u_i^2 u_{i+5}^2 u_{i+10}^2 = \frac{1-A}{B} \tag{4.7}$$

and

$$u_i \neq u_{i+2}, u_i \neq u_{i+3}, u_i \neq u_{i+6}, u_i \neq u_{i+10}, u_i = u_{i+15}, \tag{4.8}$$

then we have a periodic solution with period 15.

Proof. The proof is similar to the proof of Theorem 4.1 and is omitted. □

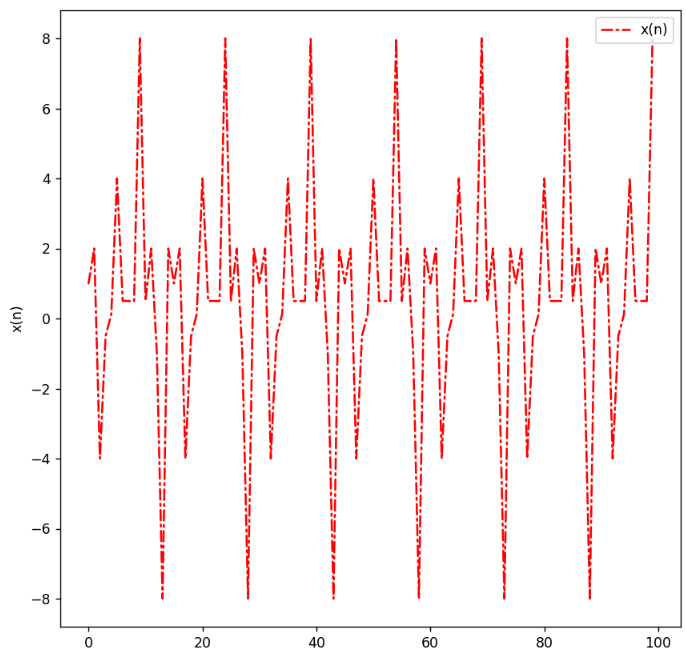

FIGURE 2. Graph of (1.3) when $a = 2$ and $b = -1/4$.

Figure 2 illustrates the graph of (1.3) with the initial conditions, $u_0 = 1; u_1 = 2; u_2 = -4; u_3 = -1/2; u_4 = 1/8; u_5 = 4; u_6 = 1/2; u_7 = 1/2; u_8 = 1/2; u_9 = 8u_{10} = 1/2; u_{11} = 2; u_{12} = -1; u_{13} = -8; u_{14} = 2; u_{15} = 1; u_{16} = 2; u_{17} = -4; u_{18} = -1/2; u_{19} = 1/8; u_{20} = 4; u_{21} = 1/2; u_{22} = 1/2; u_{23} = 1/2; u_{24} = 8; u_{25} = 1/2; u_{26} = 2; u_{27} = -1; u_{28} = -8; u_{29} = 2$, satisfying the conditions in Theorem 4.2. As expected, the solution is 15-periodic.

Theorem 4.3. *Let u_n be a solution of*

$$u_{n+30} = \frac{u_n}{A + Bu_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}} \tag{4.9}$$

with initial conditions u_i, $i = 0, \ldots, 29$; and $A \neq 1$ and B some non-zero constants. If the initial conditions satisfy the conditions

$$u_i^3 u_{i+5}^3 = \frac{1 - A}{B} \tag{4.10}$$

and

$$u_i \neq u_{i+2}, u_i \neq u_{i+3}, u_i \neq u_{i+6}, u_i = u_{i+10}, \tag{4.11}$$

then we have a periodic solution with period 10.

Proof. The proof is similar to the proof of Theorem 4.1 and is omitted. □

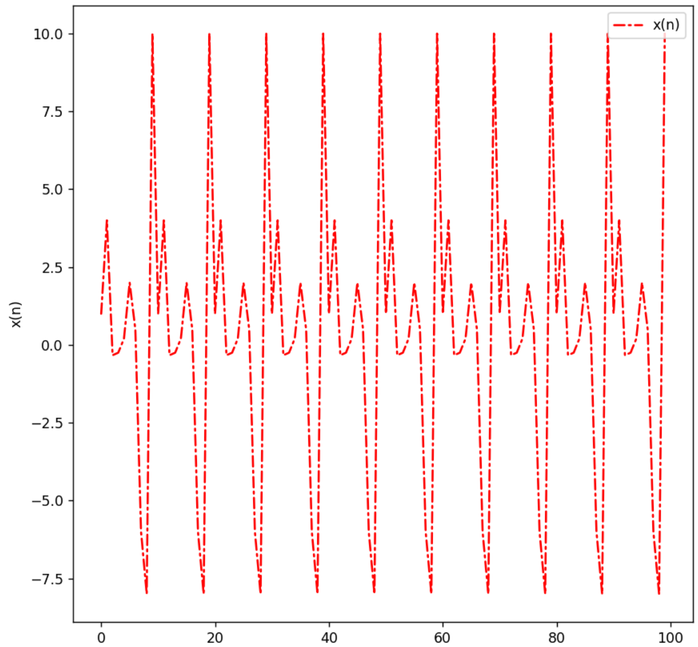

FIGURE 3. Graph of $u_{n+30} = \frac{u_n}{2-\frac{1}{8}u_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}}$.

Figure 3 illustrates the graph of (1.3) with the initial conditions, $u_0 = 1; u_1 = 4; u_2 = -1/3; u_3 = -1/4; u_4 = 1/5; u_5 = 2; u_6 = 1/2; u_7 = -6; u_8 = -8; u_9 = 10; u_{10} = 1; u_{11} = 4; u_{12} = -1/3; u_{13} = -1/4; u_{14} = 1/5; u_{15} = 2; u_{16} = 1/2; u_{17} = -6; u_{18} = -8; u_{19} = 10; u_{20} = 1; u_{21} = 4; u_{22} = -1/3; u_{23} = -1/4; u_{24} = 1/5; u_{25} = 2; u_{26} = 1/2; u_{27} = -6; u_{28} = -8; u_{29} = 10$, satisfying the following conditions:

$$u_{\tau(j)} u_{\tau(j)+5} u_{\tau(j)+10} u_{\tau(j)+15} u_{\tau(j)+20} u_{\tau(j)+25} = \frac{1-A}{B} \tag{4.12}$$

and

$$u_i = u_{i+10}. \tag{4.13}$$

As expected, the solution is 10-periodic.

Theorem 4.4. *Let u_n be a solution of*

$$u_{n+30} = \frac{u_n}{A + B u_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}} \tag{4.14}$$

with initial conditions u_i, $i = 0, \dots, 29$; and $A \neq 1$ and B some non-zero constants. If the initial conditions satisfy the conditions

$$u_0 u_1 u_2 u_3 u_4 u_5 = \frac{1-A}{B} \tag{4.15}$$

and

$$u_i \neq u_{i+2}, u_i \neq u_{i+3}, u_i = u_{i+6}, \tag{4.16}$$

then we have a periodic solution with period 6.

Proof. The proof is similar to the proof of Theorem 4.1 and is omitted. □

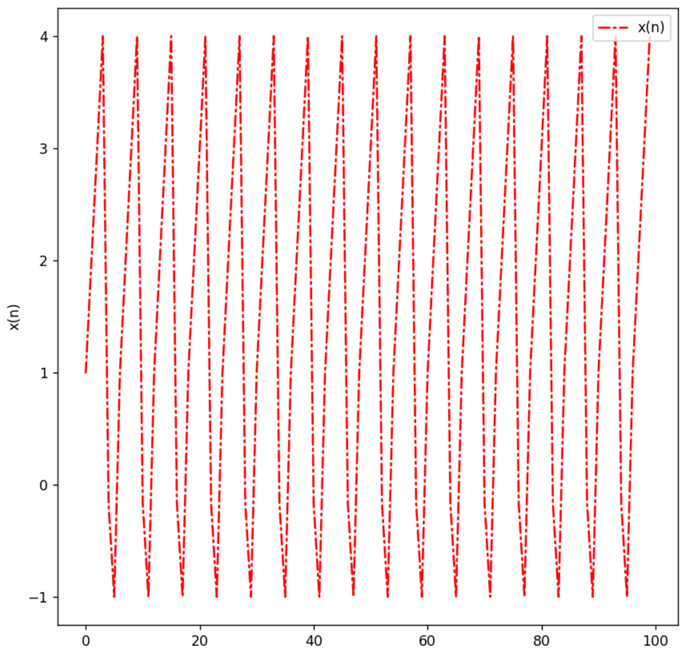

FIGURE 4. Graph of (1.3) when $a = 2$ and $b = -1/4$.

Figure 4 illustrates the graph of (1.3) with the initial conditions, $u_0 = 1; u_1 = 2; u_2 = 3; u_3 = 4; u_4 = -1/6; u_5 = -1; u_6 = 1; u_7 = 2; u_8 = 3; u_9 = 4; u_{10} = -1/6; u_{11} = -1; u_{12} = 1; u_{13} = 2; u_{14} = 3; u_{15} = 4; u_{16} = -1/6; u_{17} = -1; u_{18} = 1; u_{19} = 2; u_{20} = 3; u_{21} = 4; u_{22} = -1/6; u_{23} = -1; u_{24} = 1; u_{25} = 2; u_{26} = 3; u_{27} = 4; u_{28} = -1/6; u_{29} = -1$, satisfying the conditions in Theorem 4.4. As expected, the solution is 6-periodic.

Theorem 4.5. *Let u_n be a solution of*

$$u_{n+30} = \frac{u_n}{A + Bu_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}} \tag{4.17}$$

with initial conditions u_i, $i = 0, \ldots, 29$; and $A \neq 1$ and B some non-zero constants. If the initial conditions satisfy the conditions

$$u_0^2 u_1^2 u_2^2 = \frac{1-A}{B} \tag{4.18}$$

and

$$u_i \neq u_{i+2}, u_i = u_{i+3}, \tag{4.19}$$

then we have a periodic solution with period 3.

Proof. The proof is similar to the proof of Theorem 4.1 and is omitted. □

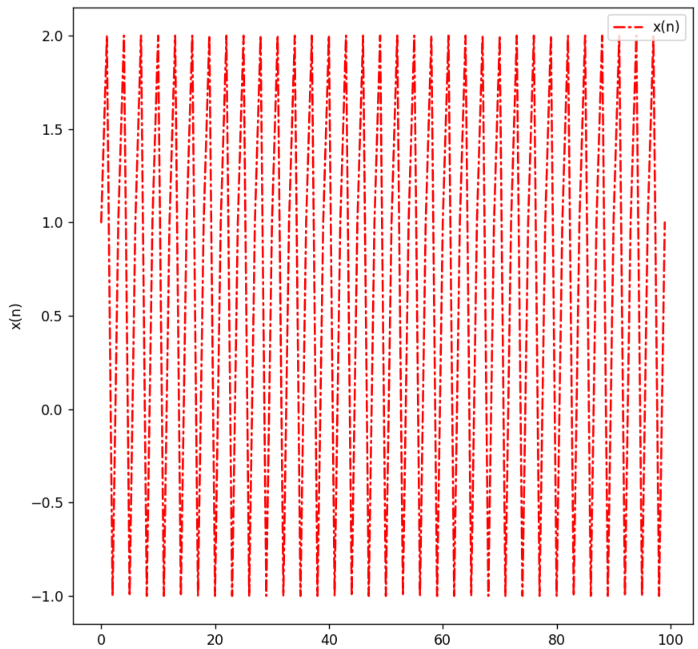

FIGURE 5. Graph of $u_{n+30} = \frac{u_n}{2-\frac{1}{4}u_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}}$.

Figure 5 illustrate the graph of (1.3) with the initial conditions, $u_0 = 1; u_1 = 2; u_2 = -1; u_3 = 1; u_4 = 2; u_5 = -1; u_6 = 1; u_7 = 2; u_8 = -1; u_9 = 1; u_{10} = 2; u_{11} = -1; u_{12} = 1; u_{13} = 2; u_{14} = -1; u_{15} = 1; u_{16} = 2; u_{17} = -1; u_{18} = 1; u_{19} = 2; u_{20} = -1; u_{21} = 1; u_{22} = 2; u_{23} = -1; u_{24} = 1; u_{25} = 2; u_{26} = -1; u_{27} = 1; u_{28} = 2; u_{29} = -1$, satisfying the conditions in Theorem 4.5. As expected, the solution is 3-periodic.

Theorem 4.6. *Let u_n be a solution of*

$$u_{n+30} = \frac{u_n}{A + Bu_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}} \tag{4.20}$$

with initial conditions u_i, $i = 0, \ldots, 29$; and $A \neq 1$ and B some non-zero constants. If the initial conditions satisfy the conditions

$$u_0^3 u_1^3 = \frac{1-A}{B} \tag{4.21}$$

and

$$u_i = u_{i+2}, \tag{4.22}$$

then we have a periodic solution with period 2.

Proof. The proof is similar to the proof of Theorem 4.1 and is omitted. □

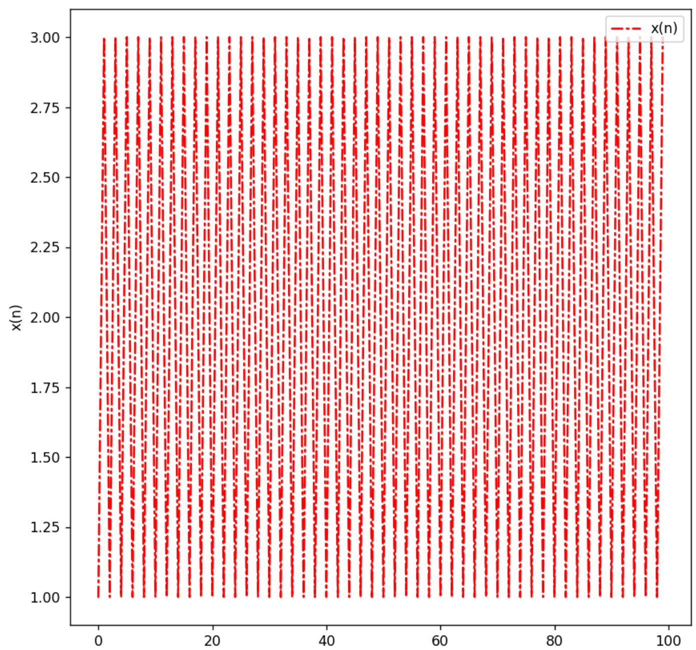

FIGURE 6. Graph of (1.3) when $a = 2$ and $b = -1/27$.

Figure 6 illustrate the graph of (1.3) with the initial conditions, $u_0 = 1; u_1 = 3; u_2 = 1; u_3 = 3; u_4 = 1; u_5 = 3u_6 = 1; u_7 = 3; u_8 = 1; u_9 = 3; u_{10} = 1; u_{11} = 3; u_{12} = 1; u_{13} = 3; u_{14} = 1; u_{15} = 3; x[16] = 1; u_{17} = 3; u_{18} = 1; u_{19} = 3; u_{20} = 1; u_{21} = 3; u_{22} = 1; u_{23} = 3u_{24} = 1; u_{25} = 3; u_{26} = 1; u_{27} = 3; u_{28} = 1; u_{29} = 3$, satisfying the conditions in Theorem 4.6. As expected, the solution is 2-periodic.

Theorem 4.7. *Given the equation*

$$u_{n+30} = \frac{u_n}{1 + Bu_n u_{n+5} u_{n+10} u_{n+15} u_{n+20} u_{n+25}}, \tag{4.23}$$

the sole equilibrium point $\bar{u} = 0$ *is non-hyperbolic.*

Proof. The fixed point condition $u = u/(1 + Bu^6)$ yields the point $u = 0$ and the characteristic equation of (4.23) near 0 reads $\lambda^{30} - 1 = 0$. The roots of $\lambda^{30} - 1 = 0$ are the thirtieth roots of unity, their moduli are all equal to 1. Thus, $u = 0$ is non-hyperbolic. □

Theorem 4.8. *The fixed point $\bar{u} = 0$ of* (4.1) *is (locally) asymptotically stable for $|A| > 1$. In addition, the non-zero fixed points of* (4.1) *are non-hyperbolic for all $A \neq 1$.*

Proof. Imposing the fixed points condition on (4.1) yields $\bar{u}(A + B\bar{u}^6 - 1) = 0$. For the first part of the proof, the characteristic equation of (4.1) near $\bar{u} = 0$ given by $\lambda^{30} - \frac{1}{A} = 0$. Consequently, the roots λ_i of the latter are in such a way that $|\lambda_i| < 1$ for $|A| > 1$. Thus, $u = 0$ is locally asymptotically stable when $|A| > 1$. For the second part, we find the non-zero fixed points by solving the equation $A + Bu^6 - 1 = 0$ for u. Here, the characteristic equation of (4.1) near a non-zero equilibrium point is given by

$$0 =\lambda^{30} - (A-1)\lambda^{25} - (A-1)\lambda^{20} - (A-1)\lambda^{15} - (A-1)\lambda^{10} - (A-1)\lambda^{5} - A \tag{4.24}$$

$$=(\lambda^{25} + \lambda^{20} + \lambda^{15} + \lambda^{10} + \lambda^{5} + 1)(\lambda^5 - A) \tag{4.25}$$

$$= \frac{1-\lambda^{30}}{1-\lambda^5}(\lambda^5 - A), \quad \text{with} \quad \lambda^5 \neq 1. \tag{4.26}$$

There exists a root of the characteristic equation with modulus equals 1, for instance, $\lambda = e^{i\pi/15}$. □

5. Conclusion

We investigated the solutions of the thirtieth-order difference equations (3) by the method of symmetry analysis. In order to use this method, the shift operator was applied to equation (3) thereby shifting the equation 29 times to give equation (5). Then the Lie symmetries of the shifted equation were found and they were used to lower the order of the reduced equation. Our results were further verified by substituting different values for A and B in (1.1) and then comparing our solutions to the solutions in the literature. We also studied the periodic nature of the solutions , stability of the fixed points and used graphical illustrations to confirm our results.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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