Int. J. Anal. Appl. (2024), 22:42

International Journal of Analysis and Applications

A New Four-Step Iterative Approximation Scheme for Reich-Suzuki-Type Nonexpansive Operators in Banach Spaces

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Abstract. In this paper, we present a new four-step iterative scheme namely DH-iterative which is faster than many super algorithms in the literature for numerical reckoning fixed points. Under this algorithm, some fixed point convergence results and ω^2-stability for contractive-like and Reich-Suzuki-type nonexpansive mappings are proposed. Our results extend and improve several related results in the literature. Finally, some numerical examples are given to study the efficiency and effectiveness of our iterative method.

1. Introduction

Functional and integral equations arises from many problems in engineering and applied sciences. Such equations can be transferred to FP theorems in an easy manner. Moreover, we use the FP theory to prove the existence and uniqueness of solutions of such integral and differential equations, for example, see [1–4].

Determining fixed point by some schematic algorithms takes major of searches by studying the behaviors of fixed points as convergence, stability, data dependance, etc. The iterative processes have been modified to find approximate solutions.

The Picard iterative process is a first iterative processes used to approximate a fixed point of a contraction mapping F on a nonempty subset Δ of a Banach space Ξ.

Received: Dec. 30, 2023.

2020 *Mathematics Subject Classification.* 47H09, 65F05, 46L05, 11Y50.

Key words and phrases. DH-iterative scheme; contractive-like mapping; Reich-Suzuki-type nonexpansive mapping; ω^2-stability; Banach space.

https://doi.org/10.28924/2291-8639-22-2024-42

ISSN: 2291-8639

A mapping $F : \Delta \to \Delta$ is called a contraction if there exists a constant $\mu \in [0,1)$ such that $\|Fa - Fb\| \leq \mu\|a - b\|, \forall a, b \in \Delta$. If $\mu = 1$, then F is called a nonexpansive mapping. A point $p^* \in \Delta$ is said to be a fixed point of F if it satisfies $Fp^* = p^*$. We denote the set of all fixed points of F by $\mho(F)$. Berinde [5] introduced the class of weak contractions. This class of mappings is also known by many authors as almost contraction mappings.

Definition 1.1. *[5] A mapping $F : \Delta \to \Delta$ is called a weak contraction if there exist $\mu \in (0,1)$ and $\delta \geq 0$ such that*

$$\|Fp - Fq\| \leq \mu\|p - q\| + \delta\|p - Fp\|, \forall p, q \in \Delta. \tag{1.1}$$

In [6], Imoru and Olantiwo generalize the definition of a class of weak-contractions which studied by Berinde [5] as follows:

Definition 1.2. *[6] A mapping $F : \Delta \to \Delta$ is called contractive-like if there exist $\mu \in [0,1)$ and a strictly increasing continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\|Fp - Fq\| \leq \mu\|p - q\| + \varphi\|p - Fp\|, \forall p, q \in \Delta. \tag{1.2}$$

Remark 1.1. *If $\varphi(p) = \delta p$, then (1.2) reduces to (1.1).*

Several extensions and generalizations of nonexpansive mappings have been discussed by many authors due to their importance in terms of applications. For instance, in 2008, Suzuki [7] introduced an interesting generalization of nonexpansive mappings and presented some existence and convergence results. Another common name for such mappings are known as mappings satisfying condition (C).

Definition 1.3. *[7] A mapping $F : \Delta \to \Delta$ is said to satisfy condition (C) if*

$$\frac{1}{2}\|p - Fp\| \leq \|p - q\| \text{ implies } \|Fp - Fq\| \leq \|p - q\|, \forall p, q \in \Delta. \tag{1.3}$$

Also, Pant and Pandey [8] in 2019 introduced the class of Reich-Suzuki-type nonexpansive mappings as follows:

Definition 1.4. *[8] A mapping $F : \Delta \to \Delta$ is said to be Reich-Suzuki-type nonexpansive if there exists a real number $\mu \in [0,1)$ such that for each $p, q \in \Delta$,*

$$\begin{aligned} \frac{1}{2}\|p - Fp\| &\leq \|p - q\| \text{ implies} \\ \|Fp - Fq\| &\leq \mu\|p - Fp\| + \mu\|q - Fq\| + (1 - 2\mu)\|p - q\|, \forall p, q \in \Delta. \end{aligned} \tag{1.4}$$

Remark 1.2. *[8]) Every mapping satisfying condition (C) is a Reich-Suzuki-type nonexpansive mapping with $\mu = 0$ but the converse is not true.*

On the other hand, many authors tended to create several iterative methods for approximating fixed points in terms of improving the performance and convergence behavior of algorithms for nonexpansive mappings [9–13] .

Some of these iterative schemes are: Mann [14], Ishikawa [15], Noor [16], Agarwal et al. [17], Abbas and Nazir [18], CR [19], Normal-S [20], Picard-S [21], Thakur et al. [22], and M iterative schemes [23].

The iteration of Thakur et al. [22] mentioned below

$$\begin{aligned} p_0 &\in \Delta, \\ z_\kappa &= (1-\alpha_\kappa)p_\kappa + \alpha_\kappa F p_\kappa, \\ w_\kappa &= F[(1-\beta_\kappa)p_\kappa + \beta_\kappa z_\kappa], \\ p_{\kappa+1} &= F(w_\kappa), \end{aligned} \tag{1.5}$$

where $\alpha_\kappa, \beta_\kappa$ are sequences in $(0,1)$, converges faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration for Suzuki generalized nonexpansive mappings by a numerical example. Ahmad et al. [24] introduced an iterative scheme known as the JK iterative scheme as follows:

$$\begin{aligned} p_0 &\in \Delta, \\ z_\kappa &= (1-\alpha_\kappa)p_\kappa + \alpha_\kappa F p_\kappa, \\ w_\kappa &= F z_\kappa, \\ p_{\kappa+1} &= F[(1-\beta_\kappa)F z_\kappa + \beta_\kappa F w_\kappa], \end{aligned} \tag{1.6}$$

where $\alpha_\kappa, \beta_\kappa$ are sequences in $(0,1)$. The authors showed numerically that the JK iterative scheme converges faster than the Agarwal [17] and Thakur [22] iterative schemes.

Very recently, Hammad et al. [25] construct a new algorithm to get a better affinity rate of almost contraction mappings and Suzuki generalized nonexpansive mappings defined as follows:

$$\begin{aligned} v_0 &\in \Delta, \\ \varpi_\kappa &= (1-\alpha_\kappa)v_\kappa + \alpha_\kappa F v_\kappa, \\ u_\kappa &= F[(1-\beta_\kappa)\varpi_\kappa + \beta_\kappa F\varpi_\kappa], \\ J_\kappa &= F[(1-\gamma_\kappa)u_\kappa + \gamma_\kappa F u_\kappa], \\ v_{\kappa+1} &= F J_\kappa, \end{aligned} \tag{1.7}$$

for $\kappa \geq 1$, where $\alpha_\kappa, \beta_\kappa$ and γ_κ are sequences in $[0,1]$.

Motivated and inspired by the above results, we introduced new four-step iterative methods called the DH-iterative scheme, to approximate the fixed points of contractive-like mappings and

Reich-Suzuki-type nonexpansive mappings as follows:

$$\begin{aligned} p_0 &\in \Delta, \\ z_\kappa &= F[(1-\alpha_\kappa)p_\kappa + \alpha_\kappa F p_\kappa], \\ w_\kappa &= F[(1-\beta_\kappa)z_\kappa + \beta_\kappa F z_\kappa], \\ q_\kappa &= F[(1-\gamma_\kappa)w_\kappa + \gamma_\kappa F w_\kappa], \\ p_{\kappa+1} &= F(Fq_\kappa), \end{aligned} \tag{1.8}$$

for $\kappa \geq 1$, where $\alpha_\kappa, \beta_\kappa$ and γ_κ are sequences in $(0,1)$.

In this article, we prove that the DH-iterative scheme (1.8) converges faster than the iterative scheme (1.7) in [25] for contractive-like mappings. Numerically, we further show that the iterative scheme (1.8) converges faster than a number of existing iterative schemes. Also, we prove that our proposed iterative scheme defined by (1.8) is w^2-stable and the stability result is supported with an example. Again, we establish weak and strong convergence results of the DH-iterative scheme (1.8) for Reich-Suzuki-type nonexpansive mappings. Further, we use a new example of Reich-Suzuki-type nonexpansive mappings to show that the DH iterative scheme (1.8) outperforms some existing prominent iterative schemes. Finally, we show by examples that this new iterative process gives better approximations as compared to other methods

2. Preliminaries

Definition 2.1. *A Banach space Ξ is called a uniformly convex if for each $\varepsilon \in (0,2]$; there exists $\delta > 0$ such that for $p, q \in \Delta$ satisfying $\|p\| \leq 1, \|q\| \leq 1$ and $\|p-q\| > \varepsilon$, we get $\|\frac{p+q}{2}\| < 1-\delta$.*

Definition 2.2. *A Banach space Ξ is called satisfy Opial's condition if for any sequence $\{p_\kappa\} \in \Delta$ so that $p_\kappa \to p \in \Delta$ weakly, implies*

$$\limsup_{\kappa\infty} \|p_\kappa - p\| < \limsup_{\kappa\to\infty} \|p_\kappa - q\|, \forall q \in \Delta, p \neq q.$$

Definition 2.3. *Let Δ be a nonempty closed convex subset of a Banach space Ξ, and $\{p_\kappa\} \in \Delta$ is a bounded sequence in Ξ. For $p \in \Xi$, we put*

$$r(p, \{p_\kappa\}) = \limsup_{\kappa\to\infty} \|p_\kappa - p\|. \tag{2.1}$$

The asymptotic radius of $\{p_\kappa\}$ relative to Δ is defined by

$$r(\Delta, \{p_\kappa\}) = \inf\{r(p, \{p_\kappa\}) : p \in \Delta\}. \tag{2.2}$$

The asymptotic center of $\{p_\kappa\}$ relative to Δ is given as:

$$A(\Delta, \{p_\kappa\}) = \{p \in \Delta : r(p, \{p_\kappa\}) = r(\Delta, \{p_\kappa\})\}. \tag{2.3}$$

In a uniformly convex Banach space, it is well known that $A(\Delta, \{p_\kappa\})$ consists of exactly one point.

Definition 2.4. *Let Δ be a nonempty closed convex subset of a Banach space Ξ. A mapping $F : \Delta \to \Delta$ is said to be demiclosed with respect to $p \in \Xi$, if for each sequence $\{p_\kappa\}$ that is weakly convergent to $p \in \Delta$ and $\{Fp_\kappa\}$ converges strongly to q implies that $Fp = q$.*

Definition 2.5. *[26] Let $\{\beta_\kappa\}$ and $\{\gamma_\kappa\}$ be two sequences of real numbers that converge to β and γ , respectively, and assume that there exists*

$$\ell = \lim_{\kappa\to\infty} \frac{\|\beta_\kappa - \beta\|}{\|\gamma_\kappa - \gamma\|}. \tag{2.4}$$

Then,
(t_1) We say that $\{\beta_\kappa\}$ converges to β faster than $\{\gamma_\kappa\}$ does to γ, if $\ell = 0$.
(t_2) We say that $\{\beta_\kappa\}$ and $\{\gamma_\kappa\}$ have the same rate of convergence, if $0 < \ell < \infty$.

Definition 2.6. *A sequence $\{p_\kappa\}$ in Δ is said to be an approximate fixed-point sequence for a mapping $F : \Delta \to \Delta$ if*

$$\lim_{\kappa\to\infty} \|Fp_\kappa - p_\kappa\| = 0. \tag{2.5}$$

Definition 2.7. *[27] A mapping $F : \Delta \to \Delta$ is said to be a satisfied condition (I) if a nondecreasing function $f : [0,\infty) \to [0,\infty)$ exists with $f(0) = 0$ and for all $t > 0$, then $f(t) > 0$ such that*

$$\|p - Fp\| \geq f(d(p, \mho(F))), \forall p \in \Delta, \tag{2.6}$$

where $d(p, \mho(F)) = \inf_{p^* \in \mho(F)} \|p - p^*\|$.

Lemma 2.1. *[28] Let $\{\theta_\kappa\}$ and $\{\lambda_\kappa\}$ be nonnegative real sequences satisfying the following inequalities:*

$$\theta_{\kappa+1} \leq (1 - \eta_\kappa)\theta_\kappa + \lambda_\kappa, \tag{2.7}$$

where $\eta_\kappa \in (0,1)$ for all $\kappa \in \mathbb{N}$, $\sum_{\kappa=0}^{\infty} \eta_\kappa = \infty$, and $\lim_{\kappa\to\infty} \frac{\lambda_\kappa}{\eta_\kappa} = 0$, then $\lim_{\kappa\to\infty} \theta_\kappa = 0$.

Lemma 2.2. *[29] Suppose Ξ is a uniformly convex Banach space and $\{\zeta_\kappa\}$ is any sequence satisfying $0 < p \leq \zeta_\kappa \leq q < 1$ for all $\kappa \geq 1$. Suppose $\{p_\kappa\}$ and $\{q_\kappa\}$ are any sequences of Ξ such that*

$$\begin{aligned} \limsup_{\kappa\to\infty} \|p_\kappa\| &\leq \nu, \\ \limsup_{\kappa\to\infty} \|q_\kappa\| &\leq \nu, \\ \limsup_{\kappa\to\infty} \|\zeta_\kappa p_\kappa + (1 - \zeta_\kappa) q_\kappa\| &= \nu, \end{aligned}$$

hold for some $\nu \geq 0$. Then, $\lim_{\kappa\to\infty} \|p_\kappa - q_\kappa\| = 0$.

Lemma 2.3. *[30] Let $F : \Delta \to \Delta$ be a mapping. If F is a Reich-Suzuki-type nonexpansive mapping with $\mho(F) \neq \phi$, then the following hold:*

1. *If F is a Reich-Suzuki-type nonexpansive mapping, then for every choice of $p \in \Delta$ and $p^* \in \mho(F)$, it follows that $\|Fp - Fp^*\| \leq \|p - p^*\|$.*
2. *If F satisfies condition (C), then F is a Reich-Suzuki-type nonexpansive mapping.*

Lemma 2.4. *Let $F : \Delta \to \Delta$ be a mapping. If F is a Reich-Suzuki-type nonexpansive mapping, then for all $p, q \in \Delta$, the following inequality holds:*

$$\|p - Fq\| \leq \left(\frac{3+\tau}{1-\tau} \right) \|p - Fp\| + \|p - q\|, \tag{2.8}$$

for all $\tau \in [0, 1)$.

Now, we present some definitions that related to stability.
The concept of stability of a fixed-point iteration process was firstly studied by Harder in her Ph.D thesis that was published in 1987.

Definition 2.8. *[31] Let $F : \Delta \to \Delta$ be a mapping. Define a fixed-point iteration method by $p_{\kappa+1} = f(F, p_\kappa)$ such that p_κ converges to a fixed point p^* of F. Let $\{M_\kappa\}$ be an arbitrary sequence in Ξ. Define*

$$\delta_\kappa = \|M_\kappa - f(F, M_\kappa)\|, \forall \kappa \in \mathbb{N}. \tag{2.9}$$

A fixed-point iterative method is said to be F-stable if the following condition is satisfied:

$$\lim_{\kappa\to\infty} \delta_\kappa = 0 \text{ if and only if } \lim_{\kappa\to\infty} M_\kappa = p^*. \tag{2.10}$$

The notion of stability mentioned in Definition 2.8 has recently been studied by several authors for different classes of contraction mappings (see e. g. [32]- [37]), and the references in them). Berinde [38] showed that the concept of stability in Definition 2.8 is not precise because of the sequence $\{M_\kappa\}$ that is arbitrarily taken. Thus for overcoming this limitation, Berinde [38] showed that it would bemore natural if $\{M_\kappa\}$ were an approximate sequence of $\{p_\kappa\}$. Therefore, any iteration process that is stable will also be weakly stable but the converse is generally not true.

Definition 2.9. *[38] Let $\{p_\kappa\} \subset \Delta$ be a given sequence. Then, a sequence $\{M_\kappa\} \subset \Delta$ is an approximate sequence of $\{p_\kappa\}$ if, for any $n \in \mathbb{N}$, there exists $\lambda = \lambda(n)$ such that*

$$\|p_\kappa - M_\kappa\| \leq \lambda, \forall \kappa \geq n. \tag{2.11}$$

Definition 2.10. *[38] Let $F : \Delta \to \Delta$ be a mapping and $\{p_\kappa\}$ be an iterative procedure defined for $p_1 \in \Delta$ and*

$$p_{\kappa+1} = f(F, p_\kappa), \kappa \geq 0. \tag{2.12}$$

Let $\{p_\kappa\}$ converge to a fixed point p^* of F. Suppose for any approximate sequence $\{M_\kappa\} \subset \Delta$ of $\{p_\kappa\}$

$$\lim_{\kappa\to\infty} \delta_\kappa = \lim_{\kappa\to\infty} \|M_{\kappa+1} - f(F, M_\kappa)\| = 0 \Rightarrow \lim_{\kappa\to\infty} M_\kappa = p^*, \tag{2.13}$$

then we say that 2.12 is weakly F-stable or weakly stable with respect to F.

Because some contractive conditions are very strictly and the associated fixed point iteration is not weakly stable, Timiş [39] introduced a new concept of weakly stability namely ω^2-stability by replacing the approximate sequence with the notion of the equivalent sequence that is more general. Following the previous job, Timiş in [40], gave some examples of ω^2-stable but not weak stable nor stable iterations.

Definition 2.11. *[41] Let $\{p_\kappa\}$ and $\{M_\kappa\}$ be two sequences.We say that these sequences are equivalent if*

$$\lim_{\kappa\to\infty} \|p_\kappa - M_\kappa\| = 0. \tag{2.14}$$

Definition 2.12. *[39] Let $F : \Delta \to \Delta$ be a mapping and $\{p_\kappa\}$ be an iterative procedure defined for $p_1 \in \Delta$ and*

$$p_{\kappa+1} = f(F, p_\kappa), \kappa \geq 0. \tag{2.15}$$

Let $\{p_\kappa\}$ converge to a fixed point p^* of F. Suppose for any equivalent sequence $\{M_\kappa\} \subset \Delta$ of $\{p_\kappa\}$

$$\lim_{\kappa\to\infty} \delta_\kappa = \|M_{\kappa+1} - f(F, M_\kappa)\| = 0 \Rightarrow \lim_{\kappa\to\infty} M_\kappa = p^*. \tag{2.16}$$

then we say that (2.15) is weakly ω^2-stable or weakly ω^2 stable with respect to F.

Remark 2.1. *Any equivalent sequence is an approximative sequence but the reverse is not true. (See [39]).*

3. Rate of convergence

In this section, we show the rate of convergence of DH-iterative (1.8) and show that it converges faster than the iterative (1.7) for the contractive -like mappings.

Theorem 3.1. *Let F be a contractive -like mapping satisfying (1.2) defined on a nonempty closed convex subset Δ of a Banach space Ξ. Then the sequence $\{p_\kappa\}$ generated by the DH-iterative scheme (1.8) converges strongly to a unique fixed point of F.*

Proof. beginning with the definition of the contractive-like mapping (1.2) and the DH-iterative defind by (1.8), we have

$$\begin{aligned}
\|z_\kappa - p^*\| &= \|F[(1-\alpha_\kappa)p_\kappa + \alpha_\kappa F p_\kappa] - Fp^*\| \\
&= \|Fp^* - F[(1-\alpha_\kappa)p_\kappa + \alpha_\kappa F p_\kappa]\| \\
&\leq \mu\|p^* - [(1-\alpha_\kappa)p_\kappa + \alpha_\kappa F p_\kappa]\| + \varphi\|p^* - Fp^*\| \\
&\leq \mu(1-\alpha_\kappa)\|(p_\kappa - p^*)\| + \mu\alpha_\kappa\|Fp_\kappa - p^*\| \\
&\leq \mu(1-\alpha_\kappa)\|p_\kappa - p^*\| + \mu\alpha_\kappa[\mu\|p_\kappa - p^*\| + \varphi\|p^* - Fp^*\|] \\
&= \mu(1-\alpha_\kappa)\|p_\kappa - p^*\| + \mu^2\alpha_\kappa\|p_\kappa - p^*\| \\
&= \mu(1-\alpha_\kappa(1-\mu))\|p_\kappa - p^*\|.
\end{aligned} \tag{3.1}$$

Again using (1.8) and (3.1), we obtain

$$\begin{aligned}
\|w_\kappa - p^*\| &= \|F[(1-\beta_\kappa)z_\kappa + \beta_\kappa F z_\kappa] - p^*\| \\
&\leq \mu(1-\beta_\kappa(1-\mu))\|z_\kappa - p^*\| \\
&\leq \mu^2(1-\beta_\kappa(1-\mu))(1-\alpha_\kappa(1-\mu))\|p_\kappa - p^*\|.
\end{aligned} \tag{3.2}$$

Similarly, applying the same steps above to obtain the following:

$$\begin{aligned}\|q_\kappa - p^*\| &\leq \mu(1-(1-\mu)\gamma_\kappa)\|w_\kappa - p^*\| \\ &\leq \mu^3(1-\gamma_\kappa(1-\mu))(1-\beta_\kappa(1-\mu))(1-\alpha_\kappa(1-\mu))\|p_\kappa - p^*\|. \end{aligned} \tag{3.3}$$

Finally, from (1.8) and (3.3)

$$\begin{aligned}\|p_{\kappa+1} - p^*\| &= \|F(Fq_\kappa) - p^*\| \\ &\leq \mu\|Fq_\kappa - p^*\| \\ &\leq \mu^2\|q_\kappa - p^*\| \\ &\leq \mu^5(1-\gamma_\kappa(1-\mu))(1-\beta_\kappa(1-\mu))(1-\alpha_\kappa(1-\mu))\|p_\kappa - p^*\|. \end{aligned} \tag{3.4}$$

Since $0 < \mu < 1$ and $\gamma_\kappa, \beta_\kappa, \alpha_\kappa \in (0,1)$, it follows that $(1-\gamma_\kappa(1-\mu)\gamma_\kappa) < 1$, $(1-\beta_\kappa(1-\mu)) < 1$, and $(1-\alpha_\kappa(1-\mu)) < 1$.
The inequality (3.4) becomes,

$$\|p_{\kappa+1} - p^*\| \leq \mu^5\|p_\kappa - p^*\|.$$

By induction, we obtain

$$\|p_{\kappa+1} - p^*\| \leq \mu^{5(\kappa+1)}\|p_0 - p^*\|. \tag{3.5}$$

Since $0 < \mu < 1$, we have $p_\kappa \to p^*$ as $\kappa \to \infty$.
For the uniqueness, let $p^*, q^* \in \mho(F)$ such that $p^* \neq q^*$. Since F be the contractive -like mapping, we can write

$$\begin{aligned}\|p^* - q^*\| = \|Fp^* - Fq^*\| &\leq \mu\|p^* - q^*\| + \varphi\|p^* - Fp^*\| \\ &= \mu\|p^* - q^*\| < \|p^* - q^*\|. \end{aligned} \tag{3.6}$$

A contradiction. Thus $p^* = q^*$ □

Theorem 3.2. *Let F be a contractive-like mapping satisfying (1.2) defined on a nonempty closed convex subset Δ of a Banach space Ξ with $\mho(F) \neq \phi$. If $\{p_\kappa\}$ is the sequence generated by the DH-iterative scheme (1.8), then $\{p_\kappa\}$ converges faster than $\{v_\kappa\}$ generated by the (1.7) iterative scheme*

Proof. From theorem (3.1), we have

$$\|p_{\kappa+1} - p^*\| \leq \mu^{5(\kappa+1)}\|p_0 - p^*\|, \kappa \in \mathbb{N}.$$

Also, from (1.7), we have

$$\begin{aligned}\|\varpi_\kappa - p^*\| &= \|(1-\alpha_\kappa)v_\kappa + \alpha_\kappa F v_\kappa - p^*\| \\ &\leq (1-\alpha_\kappa)\|v_\kappa - p^*\| + \alpha_\kappa\|Fv_\kappa - p^*\| \\ &\leq (1-\alpha_\kappa)\|v_\kappa - p^*\| + \mu\alpha_\kappa\|v_\kappa - p^*\| \\ &= (1-(1-\mu)\alpha_\kappa)\|v_\kappa - p^*\|. \end{aligned} \tag{3.7}$$

Using (3.7) and (1.7), one can write

$$\begin{aligned}\|u_\kappa - p^*\| &= \|F((1-\beta_\kappa)\varpi_\kappa + \beta_\kappa F\varpi_\kappa) - p^*\| \\ &= \|Fp^* - F[(1-\beta_\kappa)\varpi_\kappa + \beta_\kappa F\varpi_\kappa]\| \\ &\leq \mu\|p^* - ((1-\beta_\kappa)\varpi_\kappa + \beta_\kappa F\varpi_\kappa)\| + \varphi\|p^* - Fp^*\| \\ &= \mu\|(1-\beta_\kappa)(\varpi_\kappa - p^*) + \beta_\kappa(F\varpi_\kappa - p^*)\| \\ &\leq \mu(1-\beta_\kappa)\|\varpi_\kappa - p^*\| + \mu\beta_\kappa\|\varpi_\kappa - p^*\| \\ &= \mu(1-\beta_\kappa(1-\mu))\|\varpi_\kappa - p^*\| \\ &\leq \mu(1-\beta_\kappa(1-\mu))(1-\alpha_\kappa(1-\mu))\|v_\kappa - p^*\|. \end{aligned} \tag{3.8}$$

Similarly, by (3.8) and (1.7), we have

$$\begin{aligned}\|J_\kappa - p^*\| &\leq \mu(1-\gamma_\kappa(1-\mu))\|u_\kappa - p^*\| \\ &\leq \mu^2(1-\gamma_\kappa(1-\mu))(1-\beta_\kappa(1-\mu))(1-\alpha_\kappa(1-\mu))\|v_\kappa - p^*\|. \end{aligned} \tag{3.9}$$

Lastly, from (3.9), we get

$$\begin{aligned}\|v_{\kappa+1} - p^*\| &= \|FJ_\kappa - p^*\| \\ &\leq \mu\|J_\kappa - p^*\| \\ &\leq \mu^3(1-\gamma_\kappa(1-\mu))(1-\beta_\kappa(1-\mu))(1-\alpha_\kappa(1-\mu))\|v_\kappa - p^*\| \\ &\leq \mu^3\|v_\kappa - p^*\|. \end{aligned} \tag{3.10}$$

Inductively, we have

$$\|v_{\kappa+1} - p^*\| \leq \mu^{3(\kappa+1)}\|v_0 - p^*\|. \tag{3.11}$$

Let $\Gamma_\kappa = \mu^{5(\kappa+1)}\|p_0 - p^*\|$ and $\Upsilon_\kappa = \mu^{3(\kappa+1)}\|v_0 - p^*\|$, then we have

$$\frac{\Gamma_\kappa}{\Upsilon_\kappa} = \frac{\mu^{5(\kappa+1)}\|p_0 - p^*\|}{\mu^{3(\kappa+1)}\|v_0 - p^*\|} = \mu^{2(\kappa+1)}\frac{\|p_0 - p^*\|}{\|v_0 - p^*\|} \to 0 \ \ as\ \kappa \to \infty. \tag{3.12}$$

Thus, the sequence $\{p_\kappa\}$ converges faster to p^* than $\{v_\kappa\}$. □

4. Convergence results

In this section , we prove weak and strong covergence theorems of the DH-iterative scheme (1.8) for the Reich-Suzuki type nonexpansive mappings.

Lemma 4.1. *Let F be a self Reich-Suzuki type nonexpansive mappings defined on a nonempty closed convex subset Δ of a Banach space Ξ with $\mho(F) \neq \emptyset$. Let $\{p_\kappa\}$ be the sequence generated by the DH-iterative scheme (1.8). Then $\lim_{\kappa\to\infty}\|p_\kappa - p^*\|$ exists for each $p^* \in \mho(F)$.*

Proof. Let $p^* \in \mho(F)$, using Lemma 2.3, we have

$$\begin{aligned} \|z_\kappa - p^*\| &= \|F[(1-\alpha_\kappa)p_\kappa + \alpha_\kappa F p_\kappa] - Fp^*\| \\ &\leq \|(1-\alpha_\kappa)p_\kappa + \alpha_\kappa F p_\kappa - p^*\| \\ &\leq (1-\alpha_\kappa)\|p_\kappa - p^*\| + \alpha_\kappa \|Fp_\kappa - Fp^*\| \\ &\leq (1-\alpha_\kappa)\|p_\kappa - p^*\| + \alpha_\kappa \|p_\kappa - p^*\| \\ &= \|p_\kappa - p^*\|. \end{aligned} \tag{4.1}$$

Also, we have

$$\begin{aligned} \|w_\kappa - p^*\| &= \|F[(1-\beta_\kappa)z_\kappa + \beta_\kappa F z_\kappa] - Fp^*\| \\ &\leq \|z_\kappa - p^*\| \\ &\leq \|p_\kappa - p^*\|. \end{aligned} \tag{4.2}$$

Similarly, we have

$$\begin{aligned} \|q_\kappa - p^*\| &\leq \|F[(1-\gamma_\kappa)w_\kappa + \gamma_\kappa F w_\kappa] - Fp^*\| \\ &\leq \|w_\kappa - p^*\| \\ &\leq \|p_\kappa - p^*\|. \end{aligned} \tag{4.3}$$

At last, we obtain

$$\begin{aligned} \|p_{\kappa+1} - p^*\| &= \|F(Fq_\kappa) - p^*\| \\ &\leq \|Fq_\kappa - p^*\| \\ &\leq \|q_\kappa - p^*\| \\ &\leq \|p_\kappa - p^*\|. \end{aligned} \tag{4.4}$$

Thus, the real sequence $\{\|p_\kappa - p^*\|\}$ is a bounded and decreasing sequence. Hence $\lim_{n\to\infty} \|p_\kappa - p^*\|$ exists for each $p^* \in \mho(F)$. □

Lemma 4.2. *Let F be a self Reich-Suzuki type nonexpansive mappings defined on a nonempty closed convex subset Δ of a Banach space Ξ. Let $\{p_\kappa\}$ be the iterative sequence generated by the DH-iterative scheme (1.8). Then, $\mho(F) \neq \emptyset$ if and only if $\{p_\kappa\}$ is bounded and* $\lim_{\kappa\to\infty} \|Fp_\kappa - p_\kappa\| = 0$.

Proof. Let $\mho(F) \neq \emptyset$ and $p^* \in \mho(F)$. By Lemma 4.1, $\lim_{n\to\infty} \|p_\kappa - p^*\|$ exists and $\{p_\kappa\}$ is bounded. Consider the following:

$$\lim_{\kappa\to\infty} \|p_\kappa - p^*\| = L. \tag{4.5}$$

From (4.1) and (4.5)

$$\limsup_{\kappa\to\infty} \|z_\kappa - p^*\| \leq L. \tag{4.6}$$

Using Lemma 2.3, we get

$$\limsup_{\kappa\to\infty} \|Fp_\kappa - p^*\| \leq \limsup_{\kappa\to\infty} \|p_\kappa - p^*\| = L. \tag{4.7}$$

From (1.8) and (4.1), we have

$$\begin{aligned}
\|p_{\kappa+1} - p^*\| &= \|F(Fq_\kappa) - p^*\| \\
&\leq \|Fq_\kappa - p^*\| \\
&\leq \|q_\kappa - p^*\| \\
&\leq \|F[(1-\gamma_\kappa)w_\kappa + \alpha_\kappa F w_\kappa] - p^*\| \\
&\leq \|(1-\gamma_\kappa)w_\kappa + \gamma_\kappa F w_\kappa - p^*\| \\
&\leq (1-\gamma_\kappa)\|w_\kappa - p^*\| + \gamma_\kappa \|Fw_\kappa - p^*\| \\
&\leq (1-\gamma_\kappa)\|w_\kappa - p^*\| + \gamma_\kappa \|w_\kappa - p^*\| \\
&= \|w_\kappa - p^*\| \\
&= \|F[(1-\beta_\kappa)z_\kappa + \beta_\kappa F z_\kappa] - p^*\| \\
&\leq \|(1-\beta_\kappa)z_\kappa + \beta_\kappa F z_\kappa - p^*\| \\
&\leq (1-\beta_\kappa)\|z_\kappa - p^*\| + \beta_\kappa \|Fz_\kappa - p^*\| \\
&\leq (1-\beta_\kappa)\|p_\kappa - p^*\| + \beta_\kappa \|z_\kappa - p^*\| \\
&= \|p_\kappa - p^*\| - \beta_\kappa \|p_\kappa - p^*\| + \beta_\kappa \|z_\kappa - p^*\|.
\end{aligned}$$

Since $\beta_\kappa \in (0,1)$, The last inequality leads to

$$\|p_{\kappa+1} - p^*\| - \|p_\kappa - p^*\| \leq \frac{\|p_{\kappa+1} - p^*\| - \|p_\kappa - p^*\|}{\beta_\kappa} \leq \|z_\kappa - p^*\| - \|p_\kappa - p^*\|,$$

which implies that

$$\|p_{\kappa+1} - p^*\| \leq \|z_\kappa - p^*\|.$$

Thus by (4.5), we obtain

$$L \leq \liminf_{\kappa\to\infty} \|z_\kappa - p^*\|. \tag{4.8}$$

Both (4.8) and (4.6) implies that

$$L = \|z_\kappa - p^*\|. \tag{4.9}$$

From (4.1), we have

$$\|z_\kappa - p^*\| \leq \|(1-\alpha_\kappa)(p_\kappa - p^*) + \alpha_\kappa(Fp_\kappa - p^*)\| \leq \|p_\kappa - p^*\|.$$

Using the inequalities (4.9) and (4.5), it follows that:

$$\lim_{\kappa\to\infty} \|(1-\alpha_\kappa)(p_\kappa - p^*) + \alpha_\kappa(Fp_\kappa - p^*)\| = L. \tag{4.10}$$

Lastly, from (4.5), (4.7), (4.10) and Lemma 2.2, one can write

$$\lim_{\kappa\to\infty} \|Fp_\kappa - p_\kappa\| = 0.$$

Conversely, assume that $\{p_\kappa\}$ is bounded and $\lim_{\kappa\to\infty} \|Fp_\kappa - p_\kappa\| = 0$. Let $p^* \in A(\Delta, \{p_\kappa\})$, using Lemma 2.4, we have

$$\begin{aligned} r(Fp^*, \{p_\kappa\}) &= \limsup_{\kappa\to\infty} \|p_\kappa - Fp^*\| \\ &\leq \left(\tfrac{3+\tau}{1-\tau}\right) \limsup_{\kappa\to\infty} \|Fp_\kappa - p_\kappa\| + \limsup_{\kappa\to\infty} \|p_\kappa - p^*\| \\ &= \limsup_{\kappa\to\infty} \|p_\kappa - p^*\| \\ &= r(p^*, \{p_\kappa\}). \end{aligned}$$

Thus, $Fp^* \in A(\Delta, \{p_\kappa\})$. Since Ξ is uniformly convex, then $A(\Delta, \{p_\kappa\})$ contains only one element, so we have $Fp^* = p^*$. □

Now, we present the following weak convergence result.

Theorem 4.1. *Let F, Δ and $\{p_\kappa\}$ as in Lemma 4.2. Let Ξ be a uniformly convex Banach space. Suppose that Ξ satisfies Opial's condition and $1 - F$ is demiclosed with respect to zero*
If $\mho(F) \neq \emptyset$, then the sequence $\{p_\kappa\}$ converges weakly to a point of F.

Proof. From Lemma 4.1, we have that $\lim_{\kappa\to\infty} \|p_\kappa - p^*\|$ exists. Now, it is sufficient to prove that $\{p_\kappa\}$ have a unique weak subsequential limit in $\mho(F)$. Let $\{p_{\kappa_n}\}$ and $\{p_{\kappa_m}\}$ are two subsequences of $\{p_\kappa\}$, which converge weakly to p, q respectively.
Now, suppose that Ξ satisfies Opial's condition and $1 - F$ is demiclosed with respect to zero, then by Lemma 4.2, we have $\lim_{\kappa\to\infty} \|Fp_\kappa - p_\kappa\| = 0$, and since $1 - F$ is demiclosed at zero, we have $(1 - F)p = 0$, that is $p = Fp$ and similarly, $q = Fq$.
For the uniqueness, suppose that $p, q \in \mho(F), p \neq q$, by Opial's property, we get

$$\begin{aligned} \lim_{\kappa\to\infty} \|p_\kappa - p\| &= \lim_{\kappa_n\to\infty} \|p_{\kappa_n} - p\| < \lim_{\kappa_n\to\infty} \|p_{\kappa_n} - q\| = \lim_{\kappa\to\infty} \|p_\kappa - q\| \\ &= \lim_{\kappa_m\to\infty} \|p_{\kappa_m} - q\| < \lim_{\kappa_m\to\infty} \|p_{\kappa_m} - p\| = \lim_{\kappa\to\infty} \|p_\kappa - p\|, \end{aligned}$$

which is a contradiction, so $p = q$.

□

We now present the following strong convergence results:

Theorem 4.2. *Let F, Δ and Ξ as in Lemma 4.2. Let $\{p_\kappa\}$ be the sequence generated by the DH iterative scheme (1.8). Then*

$$\{p_\kappa\} \to p^* \in \mho(F) \Leftrightarrow \liminf_{\kappa\to\infty} d(p_\kappa, \mho(F)) = 0,$$

where $d(p_\kappa, \mho(F)) = \inf\{\|p_\kappa - p^*\| : p^* \in \mho(F)\}$.

Proof. The necessity is clear. Conversely, let $\liminf_{\kappa\to\infty} d(p_\kappa, \mho(F)) = 0, p^* \in \mho(F)$. Using Lemma 4.1, $\lim_{\kappa\to\infty} \|p_\kappa - p^*\|$ exists for all $p^* \in \mho(F)$. It is sufficient to prove that the sequence $\{p_\kappa\}$ is Cauchy in Δ. Since $\lim_{\kappa\to\infty} d(p_\kappa, \mho(F)) = 0$, then $\forall \varepsilon > 0$, there exists $\iota_0 \in \mathbb{N}$ such that

$$\forall \kappa \geq \iota_0, d(p_\kappa, \mho(F)) < \frac{\varepsilon}{2}, \ \inf\{\|p_\kappa - p^*\| : p^* \in \mho(F)\} < \frac{\varepsilon}{2}.$$

In particular, $\inf\{\|p_{\iota_0} - p^*\| : p^* \in \mho(F)\} < \frac{\varepsilon}{2}$. Thus , there exists $p^* \in \mho(F)$ such that $\|p_{\iota_0} - p^*\| < \frac{\varepsilon}{2}$. For $\kappa, \ell \geq \iota_0$,we obtain

$$\begin{aligned} \|p_{\kappa+\ell} - p_\kappa\| &\leq \|p_{\kappa+\ell} - p^*\| + \|p_\kappa - p^*\| \\ &\leq \|p_{\iota_0} - p^*\| + \|p_{\iota_0} - p^*\| \\ &= 2\|p_{\iota_0} - p^*\| < \varepsilon. \end{aligned}$$

Hence, the sequence $\{p_\kappa\}$ is Cauchy in Δ and since Δ is closed , there is an element $q \in \Delta$ such that $\lim_{\kappa\to\infty} p_\kappa = q$. Since $\lim_{\kappa\to\infty} d(p_\kappa, \mho(F)) = 0$, then $d(q, \mho(F)) = 0$, which means $q \in \mho(F)$. □

In a compact domain, we establish a strong convergence in the following way:

Theorem 4.3. *Let F and Ξ as in Lemma 4.2. Let Δ be a nonempty compact convex subset of Ξ. Then, the sequence $\{p_\kappa\}$ generated by the iterative scheme (1.8) converges strongly to a fixed point of F.*

Proof. By Lemma 4.2, we have $\lim_{n\to\infty} \|Fp_\kappa - p_\kappa\| = 0$. Since Δ is convex and compact, the iterative sequence $\{p_\kappa\}$ which contained in the set Δ has a convergent subsequence, say $\{p_{\kappa_i}\}$ has a strong limit, namely, $p \in \Delta$. Applying Lemma 2.4 with $\{p_{\kappa_i}\}$ and p,

$$\|p_{\kappa_i} - Fp\| \leq \left(\tfrac{3+\tau}{1-\tau}\right)\|p_{\kappa_i} - Fp_{\kappa_i}\| + \|p_{\kappa_i} - p\| \tag{4.11}$$

Letting $i \to \infty$, we obtain $p_{\kappa_i} \to Fp$, which means $q \in \mho(F)$. Using Lemma 4.1, $\lim_{\kappa\to\infty} \|p_\kappa - p\|$ exists, that is, p is a strong limit for $\{p_\kappa\}$. □

In the following, A condition (I) is using for strong convergence theorem.

Theorem 4.4. *Let F, Δ and Ξ as in Lemma 4.2. Let F satisfies condition (I), then the sequence $\{p_\kappa\}$ generated by the DH iterative scheme (1.8) converges strongly to a fixed point of F.*

Proof. As in Lemma (4.2), we have shown that

$$\lim_{\kappa\to\infty} \|Fp_\kappa - p_\kappa\| = 0 \Rightarrow f(d(p_\kappa, \mho(F))) = 0. \tag{4.12}$$

By (4.12) and the definition (2.7), one can write

$$0 \leq \lim_{\kappa\to\infty} f(d(p_\kappa, \mho(F))) \leq \lim_{\kappa\to\infty} \|p_\kappa - Fp_\kappa\| = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function that satisfies the condition $f(0) = 0$, and $f(t) > 0$, for all $t > 0$, we obtain

$$\lim_{\kappa\to\infty} d(p_\kappa, \mho(F)) = 0.$$

All the requirements of Theorem 4.3 are satisfied, thus, the sequence $\{p_\kappa\}$ is strongly convergent in the fixed-point set of F. □

5. Stability results

In this section, we show that the DH-iterative scheme defined in (1.8) is ω^2-stability with respect to F for contractive- like mappings.

Theorem 5.1. *Let Δ be a nonempty closed convex subset of a Banach metric space Ξ, $F : \Delta \to \Delta$ be a contractive-like mapping such that $\mho(F) \neq \phi$, and $\{p_\kappa\}$ be the DH-iterative sequence defined in (1.8). Then, the sequence $\{p_\kappa\}$ is w^2-stable with respect to F.*

Proof. Let $\{M_\kappa\}$ be an equivalent sequence of $\{p_\kappa\}$ in Δ. put $\delta_\kappa = \|M_{\kappa+1} - F(Fa_\kappa)\|$, where $a_\kappa = F[(1-\gamma_\kappa)b_\kappa - \gamma_\kappa F b_\kappa]$, $b_\kappa = F[(1-\beta_\kappa)c_\kappa - \beta_\kappa F c_\kappa]$, $c_\kappa = F[(1-\alpha_\kappa)M_\kappa - \alpha_\kappa F M_\kappa]$. Let $\lim_{n\to\infty} \delta_\kappa = 0$. Applying triangle inequality and using 1.8, we obtain

$$\begin{aligned}
\|M_{\kappa+1} - p^*\| &\leq \|M_{\kappa+1} - p_{\kappa+1}\| + \|p_{\kappa+1} - p^*\| \\
&\leq \|M_{\kappa+1} - F(Fa_\kappa)\| + \|F(Fa_\kappa) - p_{\kappa+1}\| + \|p_{\kappa+1} - p^*\| \\
&= \delta_\kappa + \|F(Fa_\kappa) - F(Fq_\kappa)\| + \|p_{\kappa+1} - p^*\| \\
&\leq \delta_\kappa + \mu\|Fq_\kappa - Fa_\kappa\| + \varphi(\|Fq_\kappa - F(Fq_\kappa)\|) + \|p_{\kappa+1} - p^*\| \\
&\leq \delta_\kappa + \mu(\mu\|q_\kappa - a_\kappa\| + \varphi(\|q_\kappa - Fq_\kappa\|) \\
&+ \varphi(\mu\|q_\kappa - p^*\| + \mu\|Fq_\kappa - p^*\|) + \|p_{\kappa+1} - p^*\| \\
&\leq \delta_\kappa + \mu^2\|q_\kappa - a_\kappa\| + \mu\varphi((1+\mu)\|q_\kappa - p^*\|) \\
&+ \varphi(\mu\|q_\kappa - p^*\| + \mu^2\|q_\kappa - p^*\|) + \|p_{\kappa+1} - p^*\| \\
&= \delta_\kappa + \mu^2\|q_\kappa - a_\kappa\| + \mu\varphi((1+\mu)\|q_\kappa - p^*\|) \\
&+ \varphi(\mu(1+\mu)\|q_\kappa - p^*\|) + \|p_{\kappa+1} - p^*\| \qquad (5.1)
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|q_\kappa - a_\kappa\| &= \|F[(1-\gamma_\kappa)\omega_\kappa - \gamma_\kappa F\omega_\kappa] - F[(1-\gamma_\kappa)b_\kappa - \gamma_\kappa F b_\kappa]\| \\
&\leq \mu\|(1-\gamma_\kappa)\omega_\kappa + \gamma_\kappa F\omega_\kappa - ((1-\gamma_\kappa)\omega_\kappa - \gamma_\kappa F\omega_\kappa)\| \\
&+ \varphi(\|(1-\gamma_\kappa)\omega_\kappa + \gamma_\kappa F\omega_\kappa - F[(1-\gamma_\kappa)\omega_\kappa - \gamma_\kappa F\omega_\kappa]\|) \\
&= \mu[(1-\gamma_\kappa)\|\omega_\kappa - b_\kappa\| + \gamma_\kappa\|F\omega_\kappa - Fb_\kappa\|] \\
&+ \varphi(\|(1-\gamma_\kappa)\omega_\kappa + \gamma_\kappa F\omega_\kappa - p^*\| + \|Fp^* - F[(1-\gamma_\kappa)\omega_\kappa - \gamma_\kappa F\omega_\kappa]\|) \\
&\leq \mu(1-\gamma_\kappa)\|\omega_\kappa - b_\kappa\| + \mu\gamma_\kappa[\mu\|\omega_\kappa - b_\kappa\| - \varphi(\|\omega_\kappa - F\omega_\kappa\|) \\
&+ \varphi((1-\gamma_\kappa(1-\mu))\|w_\kappa - p^*\| + \mu((1-\gamma_\kappa(1-\mu))\|w_\kappa - p^*\|)) \\
&= \mu(1-\gamma_\kappa)\|\omega_\kappa - b_\kappa\| + \mu^2\gamma_\kappa\|\omega_\kappa - b_\kappa\| - \mu\gamma_\kappa\varphi((1+\mu)\|\omega_\kappa - p^*\|) \\
&+ \varphi((1+\mu)(1-(1-\mu)\gamma_\kappa)\|w_\kappa - p^*\|) \\
&= \mu(1-(1-\mu)\gamma_\kappa)\|\omega_\kappa - b_\kappa\| - \mu\gamma_\kappa\varphi((1+\mu)\|\omega_\kappa - p^*\|) \\
&+ \varphi((1+\mu)(1-(1-\mu)\gamma_\kappa)\|\omega_\kappa - p^*\|). \qquad (5.2)
\end{aligned}$$

Since $\mu \in [0,1)$, $(1-\gamma_\kappa(1-\mu)) < 1$, then $\mu(1-(1-\mu)\gamma_\kappa) < 1$ and since φ is a strictly increasing continuous function, then from (5.2), we get

$$\begin{aligned}\|q_\kappa - a_\kappa\| &\leq \|\omega_\kappa - b_\kappa\| - \mu\gamma_\kappa\varphi((1+\mu)\|\omega_\kappa - p^*\|)\\ &+ \varphi((1+\mu)\|\omega_\kappa - p^*\|).\end{aligned} \tag{5.3}$$

Similarly

$$\begin{aligned}\|\omega_\kappa - b_\kappa\| &\leq \|z_\kappa - c_\kappa\| - \mu\beta_\kappa\varphi((1+\mu)\|z_\kappa - p^*\|)\\ &+ \varphi((1+\mu)\|z_\kappa - p^*\|).\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}\|z_\kappa - c_\kappa\| &\leq \|p_\kappa - M_\kappa\| - \mu\alpha_\kappa\varphi((1+\mu)\|p_\kappa - p^*\|)\\ &+ \varphi((1+\mu)\|p_\kappa - p^*\|).\end{aligned} \tag{5.5}$$

Finally, using (5.3), (5.4) and (5.5)

$$\begin{aligned}\|M_{\kappa+1} - p^*\| &\leq \delta_\kappa + \mu^2\|q_\kappa - a_\kappa\| + \mu\varphi((1+\mu)\|q_\kappa - p^*\|)\\ &\quad +\varphi(\mu(1+\mu)\|q_\kappa - p^*\|) + \|p_{\kappa+1} - p^*\|\\ &\leq \delta_\kappa + \mu^2[\|\omega_\kappa - b_\kappa\| - \mu\gamma_\kappa\varphi((1+\mu)\|\omega_\kappa - p^*\|)\\ &\quad +\varphi((1+\mu)\|\omega_\kappa - p^*\|)] + \mu\varphi((1+\mu)\|q_\kappa - p^*\|)\\ &\quad +\varphi(\mu(1+\mu)\|q_\kappa - p^*\|) + \|p_{\kappa+1} - p^*\|\\ &= \delta_\kappa + \mu^2\|\omega_\kappa - b_\kappa\| - \mu^3\gamma_\kappa\varphi((1+\mu)\|\omega_\kappa - p^*\|)\\ &\quad +\mu^2\varphi((1+\mu)\|\omega_\kappa - p^*\|) + \mu\varphi((1+\mu)\|q_\kappa - p^*\|)\\ &\quad +\varphi(\mu(1+\mu)\|q_\kappa - p^*\|) + \|p_{\kappa+1} - p^*\|.\end{aligned}$$

which implies that

$$\begin{aligned}\|M_{\kappa+1} - p^*\| &\leq \delta_\kappa + \mu^2[\|z_\kappa - c_\kappa\| - \mu\beta_\kappa\varphi((1+\mu)\|z_\kappa - p^*\|)\\ &\quad +\varphi((1+\mu)\|z_\kappa - p^*\|] - \mu^3\gamma_\kappa\varphi((1+\mu)\|\omega_\kappa - p^*\|)\\ &\quad +\mu^2\varphi((1+\mu)\|\omega_\kappa - p^*\|) + \mu\varphi((1+\mu)\|q_\kappa - p^*\|)\\ &\quad +\varphi(\mu(1+\mu)\|q_\kappa - p^*\|) + \|p_{\kappa+1} - p^*\|\\ &= \delta_\kappa + \mu^2\|z_\kappa - c_\kappa\| - \mu^3\beta_\kappa\varphi((1+\mu)\|z_\kappa - p^*\|)\\ &\quad +\mu^2\varphi((1+\mu)\|z_\kappa - p^*\|] - \mu^3\gamma_\kappa\varphi((1+\mu)\|\omega_\kappa - p^*\|)\\ &\quad +\mu^2\varphi((1+\mu)\|\omega_\kappa - p^*\|) + \mu\varphi((1+\mu)\|q_\kappa - p^*\|)\\ &\quad +\varphi(\mu(1+\mu)\|q_\kappa - p^*\|) + \|p_{\kappa+1} - p^*\|,\end{aligned}$$

yields,

$$\begin{aligned}
\|M_{\kappa+1}-p^*\| &\leq \delta_\kappa+\mu^2[\|p_\kappa-M_\kappa\|-\mu\alpha_\kappa\varphi((1+\mu)\|p_\kappa-p^*\|) \\
&+ \varphi((1+\mu)\|p_\kappa-p^*\|)]-\mu^3\beta_\kappa\varphi((1+\mu)\|z_\kappa-p^*\|) \\
&+ \mu^2\varphi((1+\mu)\|z_\kappa-p^*\|]-\mu^3\gamma_\kappa\varphi((1+\mu)\|\omega_\kappa-p^*\|) \\
&+ \mu^2\varphi((1+\mu)\|\omega_\kappa-p^*\|)+\mu\varphi((1+\mu)\|q_\kappa-p^*\|) \\
&+ \varphi(\mu(1+\mu)\|q_\kappa-p^*\|)+\|p_{\kappa+1}-p^*\|,
\end{aligned}$$

it follows that

$$\begin{aligned}
\|M_{\kappa+1}-p^*\| &\leq \delta_\kappa+\mu^2\|p_\kappa-M_\kappa\|-\mu^3\alpha_\kappa\varphi((1+\mu)\|p_\kappa-p^*\|) \\
&+ \mu^2\varphi((1+\mu)\|p_\kappa-p^*\|)-\mu^3\beta_\kappa\varphi((1+\mu)\|z_\kappa-p^*\|) \\
&+ \mu^2\varphi((1+\mu)\|z_\kappa-p^*\|]-\mu^3\gamma_\kappa\varphi((1+\mu)\|\omega_\kappa-p^*\|) \\
&+ \mu^2\varphi((1+\mu)\|\omega_\kappa-p^*\|)+\mu\varphi((1+\mu)\|q_\kappa-p^*\|) \\
&+ \varphi(\mu(1+\mu)\|q_\kappa-p^*\|)+\|p_{\kappa+1}-p^*\|. \qquad (5.6)
\end{aligned}$$

From Theorem 3.1, we get that $\lim_{k\to\infty}\|p_\kappa-p^*\|=0$, and since φ is a strictly increasing continuous function with $\varphi(0)=0$, then $\lim_{k\to\infty}\|p_{\kappa+1}-p^*\|=0$, The equivalence of $\{p_\kappa\}$ and $\{M_\kappa\}$ implies that $\lim_{k\to\infty}\|p_\kappa-M_\kappa\|=0$. Taking the limit of both sides of (5.6) and since $\lim_{k\to\infty}\delta_\kappa=0$, we get $\lim_{k\to\infty}\|M_\kappa-p^*\|=0$, that is the sequence $\{p_\kappa\}$ is ω^2-stable with respect to F. □

6. Numerical experiments

This section presents a sequence of numerical experiments designed to demonstrate the effectiveness of the proposed methods. The main goal of these experiments is to provide insights into the selection of optimal control settings and to conduct a comprehensive investigation of control parameter configurations. In this section, the error term is consistently identified as D_k, while crucial parameters, including the total number of iterations and the necessary execution time, are denoted by k and t, respectively.

Example 6.1. *Consider the sets $\Xi=\mathbb{R}$ and $\Delta=[0,50]$. Let $F:\Delta\to\Delta$ be a mapping defined by the expression*

$$F(\nu)=\sqrt{\nu^2-9\nu+54}.$$

It is evident that 6.0000 serves as a fixed point (FP) for the mapping F. In this experiment, we assess the numerical efficiency of Algorithm (7) and Algorithm (8) by varying the initial value for x_0. The chosen termination condition is defined as $\|x_{k+1}-x_k\|\leq 10^{-15}$. Let:

$$\alpha_k=\frac{1}{2k+2},\beta_k=\frac{1}{2k+2},\gamma_k=\frac{1}{2k+2}.$$

Our primary objective is to accurately determine the number of iterations and the corresponding execution time required for convergence. We are particularly interested in understanding how the initial choice of starting points influences the algorithm's performance.

Figures 1–5 depict graphs illustrating the numerical results. It is crucial to note that the computational performance in each scenario is intricately linked to the initial starting point choice. This underscores the significance of initial conditions in shaping the overall numerical performance of the algorithm. To conduct this experiment, we will initiate the process with the following parameters:

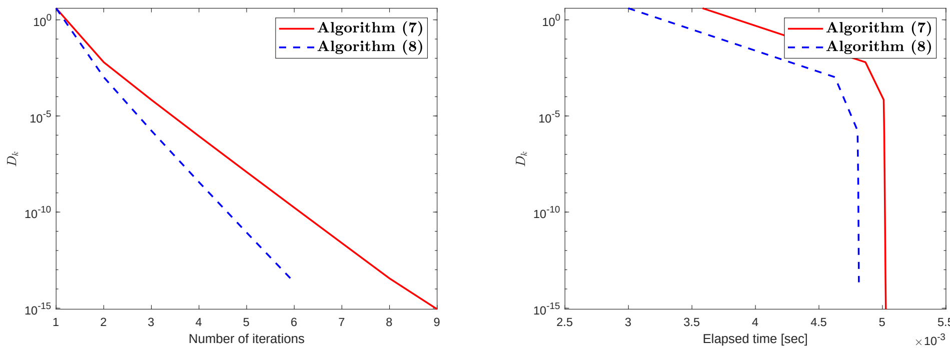

FIGURE 1. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 9, t = 0.0050268$] and Algorithm (8) [$k = 7, t = 0.0048174$] with $p_0 = 2$.

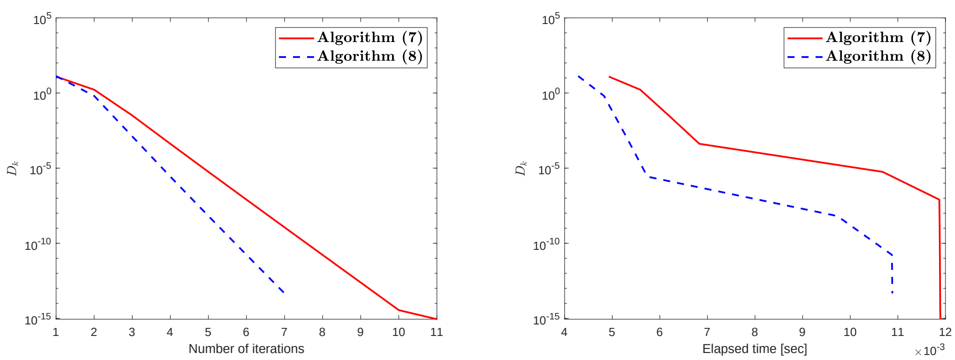

FIGURE 2. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 11, t = 0.0118947$] and Algorithm (8) [$k = 8, t = 0.0108899$] with $p_0 = 20$.

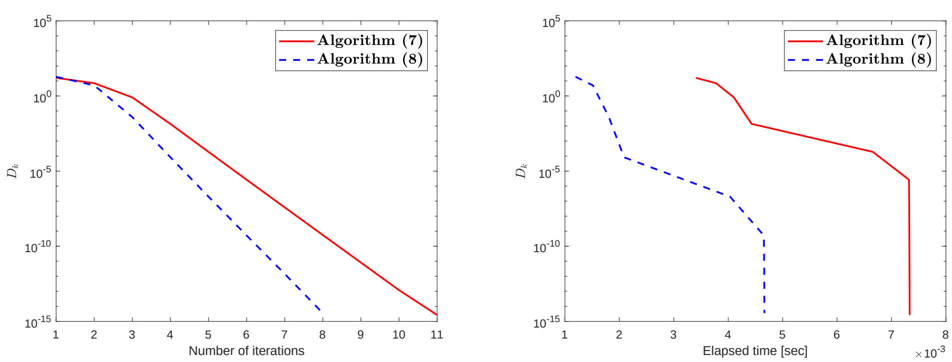

FIGURE 3. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 12, t = 0.0073363$] and Algorithm (8) [$k = 9, t = 0.0046669$] with $p_0 = 30$.

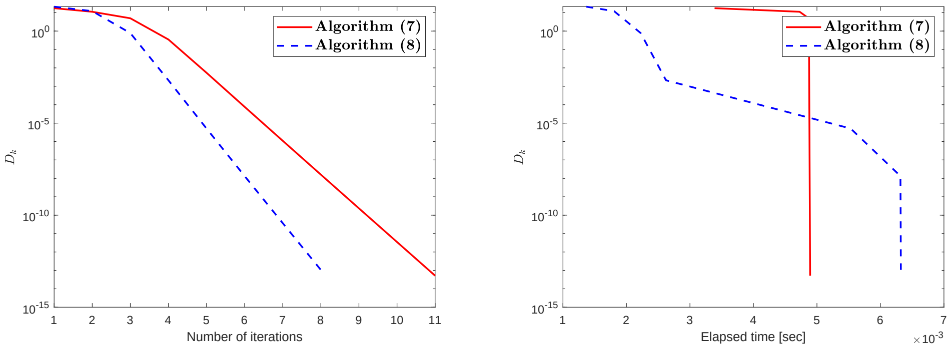

FIGURE 4. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 12$, $t = 0.0048951$] and Algorithm (8) [$k = 9$, $t = 0.0063244$] with $p_0 = 40$.

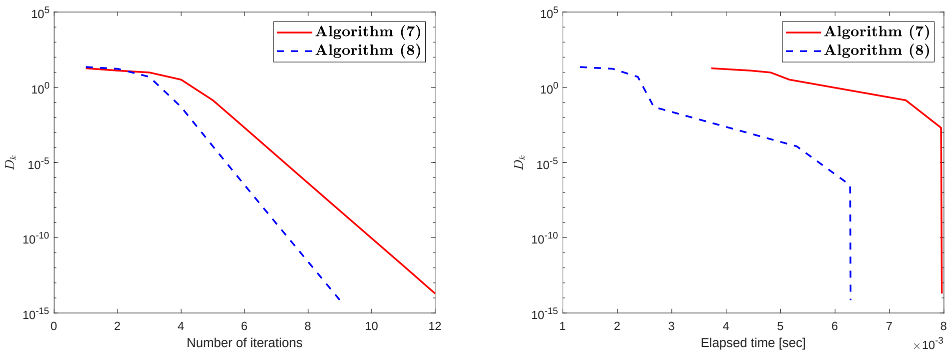

FIGURE 5. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 13$, $t = 0.0079629$] and Algorithm (8) [$k = 10$, $t = 0.0062882$] with $p_0 = 50$.

Example 6.2. *Consider a mapping $F : \Delta \to \Xi$ defined as*

$$F(\nu) = \max\{0, -\nu\}.$$

The mapping F is non-expansive and possesses a unique fixed point at $\nu = 0$. The set Δ is characterized by

$$\Delta := \{\nu : -100 \leq \nu \leq 100\}.$$

In this experiment, we evaluate the numerical efficiency of Algorithm (7) and Algorithm (8) by varying the initial value for x_0. The selected termination condition is defined as $\|x_{k+1} - x_k\| \leq 10^{-10}$. Figures 6–9 present graphs illustrating the numerical results. It is imperative to recognize that the computational performance in each scenario is intricately tied to the initial starting point choice. Let the control conditions for both algorithms be specified as follows:

$$\alpha_k = \frac{1}{2k+2}, \beta_k = \frac{1}{2k+2}, \gamma_k = \frac{1}{2k+2}.$$

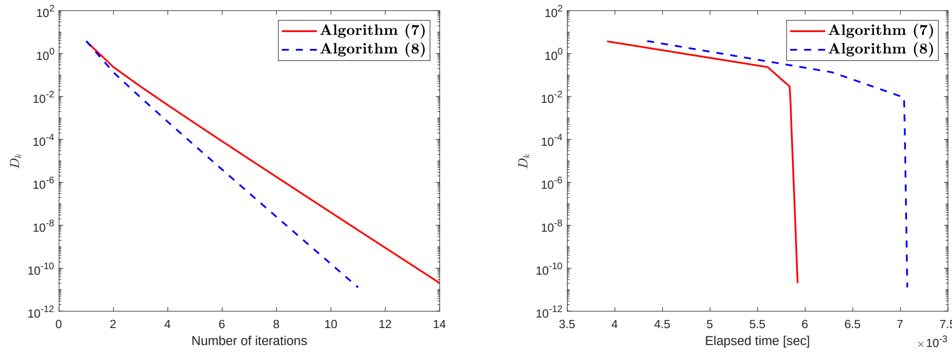

FIGURE 6. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 14$, $t = 0.0059199$] and Algorithm (8) [$k = 11$, $t = 0.0070727$] with $p_0 = 10$.

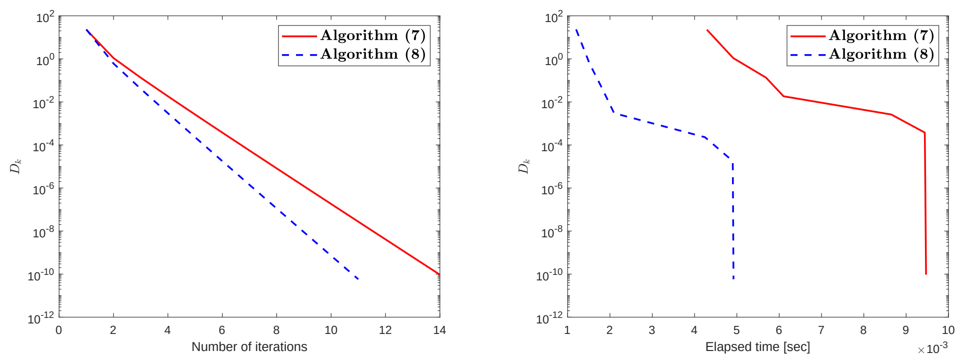

FIGURE 7. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 14$, $t = 0.0094708$] and Algorithm (8) [$k = 11$, $t = 0.0049198$] with $p_0 = 30$.

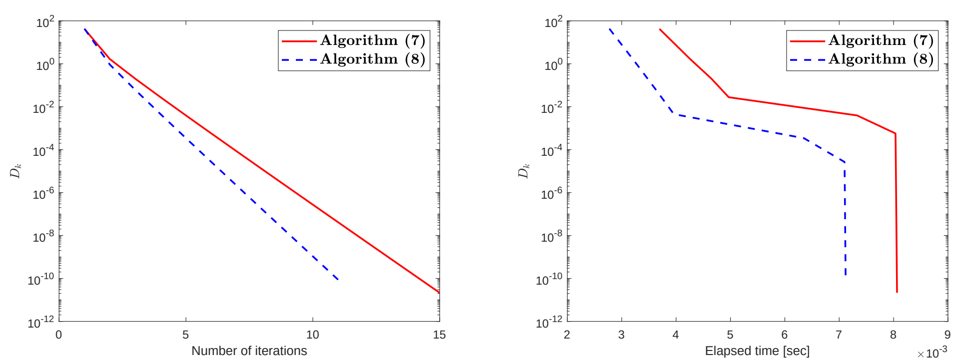

FIGURE 8. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 15$, $t = 0.0080635$] and Algorithm (8) [$k = 11$, $t = 0.0071185$] with $p_0 = 50$.

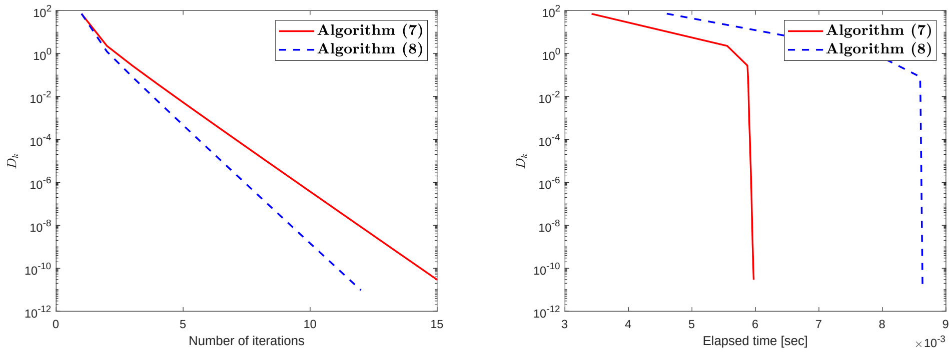

FIGURE 9. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 15$, $t = 0.0059722$] and Algorithm (8) [$k = 12$, $t = 0.0086321$] with $p_0 = 70$.

Example 6.3. *Consider an operator $\mathcal{F} : \Delta \subset \Xi \to \Xi$, and define the variational inequality problem as follows:*

$$\text{Find } v^* \in \Delta \text{ such that } \left\langle \mathcal{F}(v^*), v - v^* \right\rangle \geq 0, \text{ for all } v \in \Delta.$$

Let $F : \Delta \subset \Xi \to \Xi$ be a mapping defined by

$$F := P_\Delta(I - \lambda\mathcal{F}),$$

where $0 < \lambda < \frac{2}{L}$ and L is the Lipschitz constant of the mapping $\mathcal{F}$. The constraint set Δ is defined as

$$\Delta = \{v \in \mathbb{R}^4 : 1 \leq v_i \leq 5,\ i = 1, 2, 3, 4\}.$$

The mapping $F : \mathbb{R}^4 \to \mathbb{R}^4$ is evaluated as

$$F(v) = \begin{pmatrix} v_1 + v_2 + v_3 + v_4 - 4v_2v_3v_4 \\ v_1 + v_2 + v_3 + v_4 - 4v_1v_3v_4 \\ v_1 + v_2 + v_3 + v_4 - 4v_1v_2v_4 \\ v_1 + v_2 + v_3 + v_4 - 4v_1v_2v_3 \end{pmatrix}.$$

In this experiment, we assess the numerical efficiency of Algorithm (7) and Algorithm (8) by varying the initial value for x_0. The chosen termination condition is defined as $\|x_{k+1} - x_k\| \leq 10^{-10}$. Figures 10–12 illustrate graphs depicting the numerical results. It is crucial to note that the computational performance in each scenario is intricately linked to the initial starting point choice. The control conditions for both algorithms are specified as follows:

$$\alpha_k = \frac{1}{2k+2}, \beta_k = \frac{1}{2k+2}, \gamma_k = \frac{1}{2k+2}.$$

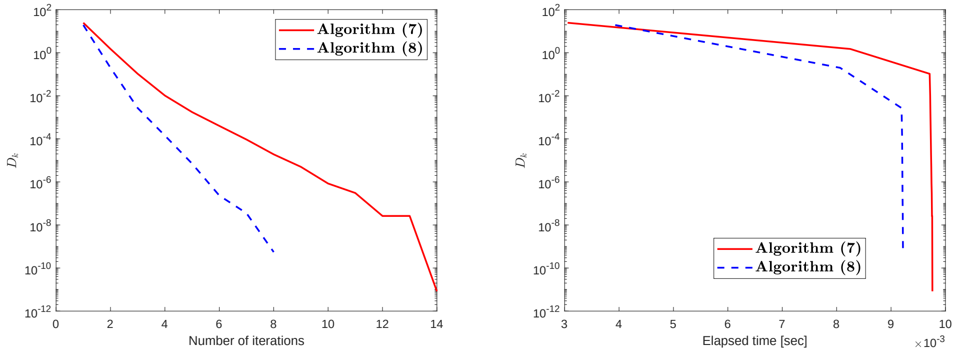

FIGURE 10. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 14$, $t = 0.0097602$] and Algorithm (8) [$k = 8$, $t = 0.0092196$] with $p_0 = [1;1;1;1]$.

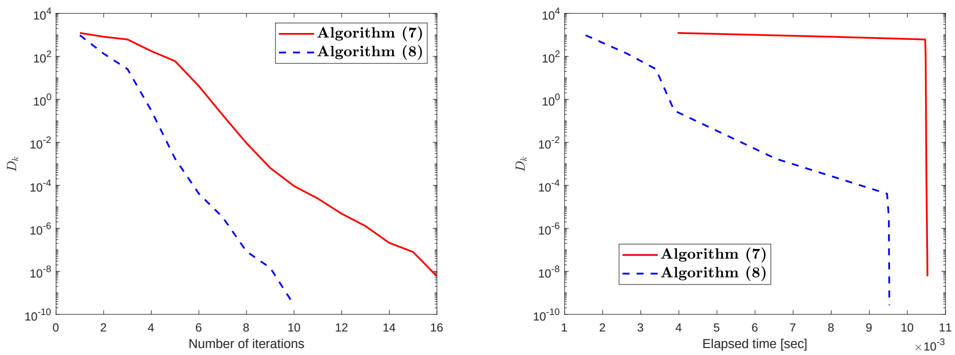

FIGURE 11. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 16$, $t = 0.0105268$] and Algorithm (8) [$k = 10$, $t = 0.0095261$] with $p_0 = [2;2;2;2]$.

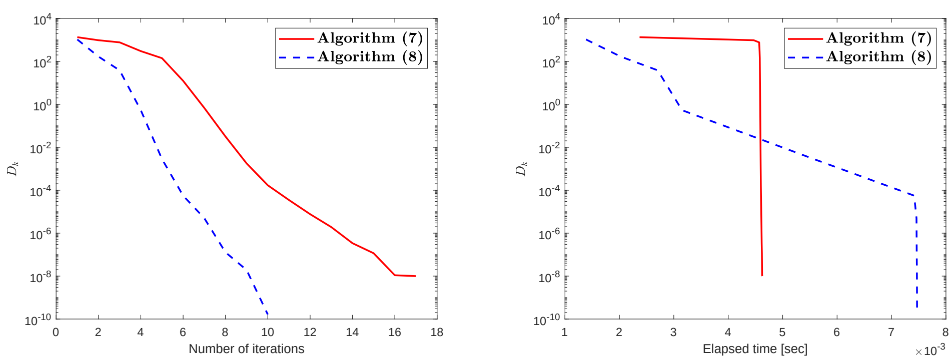

FIGURE 12. A numerical graph with iteration count and execution time of Algorithm (7) [$k = 17$, $t = 0.0046268$] and Algorithm (8) [$k = 10$, $t = 0.0074699$] with $p_0 = [3;3;3;3]$.

Author's contributions: All authors contributed equally and significantly in writing this article.

Funding: This work was supported in part by the Basque Government under Grant IT1555-22.

Acknowledgments: The authors thank the Basque Government for Grant IT1555-22.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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