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# Efficient Modified Adomian Decomposition Method for Solving Nonlinear Fractional Differential Equations

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**Abstract.** Based on the application of the standard Adomian method, the current paper proposes two modification approaches for the classes of the fractional differential equations and the system of fractional differential equations, which are featured through initial-value problems. Certainly, the constructed iterative schemes for the two classes are shown to be reliable, considering a number of test problems for demonstration, and upon deploying other existing numerical approaches for contrasting. In fact, the proposed schemes are found to portray less error, rapidity, accuracy and consume less computational time among others.

## 1. Introduction

Fractional calculus is a branch of mathematics that investigates the properties of derivatives and integrals of non-integer orders (also known as fractional derivatives and integrals; briefly fractional calculus). The differential equations of fractional order are generalizations of classical differential equations of integer order. Compared with the integer order calculus, many real-world phenomena can be better described by using the fractional operator [1,2]. They are increasingly used in such fields as fluid flow, control theory of dynamical systems, diffusive transport akin to diffusion, probability, and statistics [3,4]. Moreover, since most fractional differential equations do not have exact analytical solutions, approximate and numerical techniques are used extensively to treat the fractional differential models, including, for example, the homotopy analysis method [5], homotopy perturbation approach [6], variational iteration technique [7], Chebyshev spectral method [8], orthogonal polynomial method [9], Grunwald-Letnikov method [10], fractional Adams method [11], and several other methods; read [12,13] and the given references therewith.

Moreover, the base method of the present paper is the Adomian decomposition method (ADM) [14], which is a potent semi-analytical approach that was initiated by George Adomian in the mid 1980's. For more on the convergence of the ADM when applied to different types of functional equations, one may read the famous work of Abbaoui and Cherruault [15], and the comprehensive review work by Duan et al. [16] with regards to the method and its applications to fractional functional equations; see also Sadeghinia and Kumar [17] for the extension of the method to the class of multi-term fractional differential equations. Besides, this method is widely used in solving real-life problems that are modeled in both differential equations and fractional differential equations; in particular, Guo [18], and Afreen and Raheem [19] recently employed ADM to solve fractional differential equations and a system of nonlinear fractional differential equations, respectively. In addition, the ADM has seen several modifications with the view to improve its accuracy, rapidity, reduction in computational time, or even improvement to fit various classes of functional equations to mention a few. What is more, the powerful modifications of ADM demonstrated a rapid convergence of the series solution if compared with the standard ADM, and therefore, presented major progress. The modification of ADM has been shown to be computationally efficient in several models, which are important to researchers in applied science [20, 21].

However, the purpose of this study is to extend the modification of the ADM, which was recently given in [22–25] for different types of initial-value problems of the classical differential equations to the classes of fractional differential equations and systems of fractional differential equations. Indeed, with the modification method to be proposed, promising iterative schemes are set to be constructed with regard to both the fractional differential equations and systems of fractional differential equations; in addition, certain numerical test models will be considered to illustrate the application of the devised modified approach. Furthermore, the study will equally beseech some efficient numerical methods from the open literature to assess the competence of the proposed approach in order to establish a comparative study. Finally, the organization of the manuscripts follows the following order: Section 2 recalls some basics on fractional calculus; Section 3 outlines the two modified approaches to the fractional differential equations and systems of fractional differential equations; Section 4 applies the constructed scheme on certain test models, while Section 5 presents some concluding notes.

## 2. Basic definitions and properties of the fractional calculus

Some basic definitions and properties of the fractional calculus theory are given in this section. Indeed, these definitions and properties are reported in renowned books, read [1, 2].

**Definition 2.1.** *For a function $y(x) \in C([a, b])$ and $a < x < b$, the Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as*

$$I_a^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x y(t)(x - t)^{\alpha - 1} dt. \tag{2.1}$$

*Note from the above that for $\alpha = 0$, we have $I_a^0 y(x) = y(x)$, which is the identity operator. Additionally, when $\alpha \in \mathbb{N}$, then $I_a^\alpha y(x)$ coincides with that classical integral.*

**Definition 2.2.** *For $y(x) \in C([a,b])$, the Caputo fractional derivative of order $\alpha > 0$ is defined by*

$$D_*^\alpha y(x) = \begin{cases} I^{m-\alpha} y^{(m)}(x), & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} y(t), & \alpha = m. \end{cases} \tag{2.2}$$

**Property 2.3.** In fact, based on the aforementioned definitions, we outline some important properties for the fractional deferential and integral operators in what follows.

   i).
$$I^\alpha I^\beta y(x) = I^{\alpha+\beta} y(x), \; \alpha, \beta \geq 0. \tag{2.3}$$

   ii).
$$I^\alpha D_*^\alpha y(x) = y(x) - \sum_{k=0}^{m-1} y^{(k)}(0^+) \frac{x^k}{\Gamma(k+1)}, \tag{2.4}$$

where $m - 1 < \alpha \leq m$.

   iii).
$$I^\alpha D_*^\beta y(x) = I^{\beta-\alpha} y(x) - \sum_{k=0}^{m-1} y^{(k)}(0^+) \frac{x^{k-\alpha+\beta}}{\Gamma(k-\alpha+\beta+1)}, \tag{2.5}$$

where $m - 1 < \beta, \; \alpha \leq m$, and $\beta < \alpha$.

   iv).
$$I^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} x^{n+\alpha}, \; x > 0, \tag{2.6}$$

where $m - 1 < \alpha \leq m, \; n > -1$.

   v).
$$D_*^\alpha C = 0, \; \text{where } C \text{ is a real constant.} \tag{2.7}$$

   vi).
$$D_*^\alpha x^n = \begin{cases} 0, & n < m-1, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \geq m-1. \end{cases} \tag{2.8}$$

Moreover, the above outline properties of the fractional calculus will be used in deriving the resulting iterative schemes for the fractional differential equations and systems of fractional differential equations. In addition, Caputo's fractional derivative [1,2] would particularly be used in the present study. Indeed, the choice of Caputo's fractional definition is owing to its application in perfectly modeling a variety of real-life scenarios; besides, the initial conditions of the classical calculus are rightly used in Caputo's fractional models without the need to infuse fractional orders in the prescribed initial and boundary data.

## 3. Modified Adomian decomposition method

This section presents the modified Adomian decomposition methods (MADMs) for the solution of initial-value problems (IVPs) for fractional differential equations and systems of fractional differential equations.

### 3.1. **MADM for fractional differential equations.**  To exhibit the MADM for fractional differential equations, we make consideration of the IVP for the nonlinear fractional differential equation as follows

$$D_*^\alpha y(x) + Ry(x) + Ny(x) = g(x), \text{ ff} > 0. \tag{3.1}$$

$$y^k(0) = c_k, \quad k = 0, 1, 2, ..., m-1, \quad m-1 < \alpha < m.$$

In the above equation, $y(x)$ is the unknown function, $D_*^\alpha$ is the Caputo fractional derivative operator of high-order $\alpha$, which is easily invertible, $N$ is the nonlinear operator, $R$ is the remaining linear operator which might include other Caputo fractional derivative operators $D_*^v$ ($v < \alpha$) and $g$ is the given source function.

Next, to solve the governing IVP, we operate on Eq. $I^\alpha$ on both sides of (3.1), which is indeed the inverse operator of $D_*^\alpha$, and further make use of (2.4) to obtain

$$I^\alpha[D_*^\alpha y(x) + Ry(x) + Ny(x)] = I^\alpha[g(x)]. \tag{3.2}$$

Thus, from the above equation, one gets

$$y(x) = \sum_{n=0}^{m-1} c_k \frac{x^k}{k!} + [I^\alpha g(x)] - [I^\alpha Ry(x)] - [I^\alpha Ny(x)]. \tag{3.3}$$

or equally upon writing the latter equation using the standard Adomian decomposition method way the following

$$\sum_{n=0}^{\infty} y_n(x) = \sum_{n=0}^{m-1} c_k \frac{x^k}{k!} + [I^\alpha g(x)] - [I^\alpha R \sum_{n=0}^{\infty} y_n(x)] - [I^\alpha \sum_{n=0}^{\infty} A_n(x)], \tag{3.4}$$

where $A_n$'s are the Adomian polynomials, which are to be computed using the following compacted formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ N\left( \sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots.$$

Notably, from (3.4), the solution of the equation $y(x)$ is decomposed into a sum of infinite components $y_n$, for $n = 0, 1, 2, ...$, that it, $y(x) = \sum_{n=0}^{\infty} y_n(x)$. Further, MADM requires the addition of the expression

$$I^\alpha[\sum_{n=0}^{\infty} a_n x^n] - p I^\alpha[\sum_{n=0}^{\infty} a_n x^n],$$

into (3.4), where $p$ is an artificial parameter and for all $n \in \mathbb{N} \cup 0$, and $a_n$ are unknown coefficients. Hence, on adding the above expression to (3.4), one obtains the following

$$\sum_{n=0}^{\infty} y_n(x) = \sum_{n=0}^{m-1} c_k \frac{x^k}{k!} + [I^\alpha g(x)] + I^\alpha [\sum_{n=0}^{\infty} a_n x^n] - p I^\alpha [\sum_{n=0}^{\infty} a_n x^n] - [I^\alpha R \sum_{n=0}^{\infty} y_n(x)] - [I^\alpha \sum_{n=0}^{\infty} A_n(x)], \quad (3.5)$$

which subsequently reveals the formal recursive scheme for the governing IVP as follows

$$\begin{cases} y_0(x) = \sum_{n=0}^{m-1} c_k \frac{x^k}{k!} + I^\alpha [\sum_{n=0}^{\infty} a_n x^n], \\ y_1(x) = [I^\alpha g(x)] - p I^\alpha [\sum_{n=0}^{\infty} a_n x^n] - I^\alpha R[y_0(x)] - I^\alpha [A_0], \\ y_{n+1}(x) = -I^\alpha R[y_n(x)] - I^\alpha [A_n], \quad n \geq 1. \end{cases} \quad (3.6)$$

In fact, to avoid the calculation of $A_n$, for $n = 0, 1, 2, ...$, we calculate the values of the coefficients $a_n$ for $n = 0, 1, 2, ...$, by setting $y_1(x) = 0$ for all $n \geq 1$. More precisely, setting $p = 1$, we get a hold of the solution of the governing IVP in the following form

$$y(x) = \sum_{n=0}^{m-1} c_k \frac{x^k}{k!} + I^\alpha [\sum_{n=0}^{\infty} a_n x^n]. \quad (3.7)$$

3.2. **MADM for systems of fractional differential equations.** Accordingly, we make consideration of the coupled system of IVP, featuring nonlinear fractional differential equations as follows

$$D_*^{\alpha_i} y_i(x) = L_i(y_1, y_2, .., y_m) + N_i(y_1, y_2, .., y_m) + g_i(x), \quad (3.8)$$

$$i = 1, 2, ..., n, \quad m_{i-1} \leq \alpha \leq m_i,$$

$$y_i^{k_i}(0) = c_k^i, \quad k_i = 0, 1, 2, ..., m_{i-1},$$

with $L_i$ and $N_i$ representing the linear and nonlinear operators, respectively.

As proceed, the application of MADM begins by expressing the governing coupled model in operator notation. Thus, we apply $I^{\alpha_i}$, the inverse operator of $D_*^{\alpha_i}$ for $i = 1, 2, ..., n$ on both sides of (3.8) to get

$$I^{\alpha_i}[D_*^{\alpha_i} y_i(x)] = I^{\alpha_i}[g_i(x)] + I^{\alpha_i}[L_i(y_1, y_2, .., y_m) + N_i(y_1, y_2, .., y_m)], \quad (3.9)$$

or equally upon using the initial data the following

$$y_i(x) = \sum_{n=0}^{m-1} c_k \frac{x^k}{k!} + [I^{\alpha_i} g_i(x)] + I^{\alpha_i}[L_i(y_1, y_2, .., y_m) + N_i(y_1, y_2, .., y_m)]. \quad (3.10)$$

Next, when using the standard Adomian's method, the above equation is re-expressed as follow

$$\sum_{n=0}^{\infty} y_{i,n}(x) = \sum_{n=0}^{m-1} c_k \frac{x^k}{k!} + [I^{\alpha_i} g_i(x)] + [I^{\alpha_i} \sum_{n=0}^{\infty} y_{i,n}(x)] + [I^{\alpha_i} \sum_{n=0}^{\infty} A_{i,n}(x)], \quad (3.11)$$

where in the above, the solution $y_i(x)$ is decomposed into a sum of infinite components $y_{i,n}(x)$, for $i = 0, 1, 2, ...n$, and $n = 0, 1, 2, ...$, that it, $y_i(x) = \sum_{n=0}^{\infty} y_{i,n}(x)$, while the Adomian polynomials $A_{i,n}$'s follow as in the preceding scenario.

Further, MADM requires the addition of the following expression

$$I^{\alpha_i}[\sum_{n=0}^{\infty} a_n x^n] - p I^{\alpha_i}[\sum_{n=0}^{\infty} a_n x^n],$$

into (3.11), where $p$ is equally an artificial parameter and for all $n \in \mathbb{N} \cup 0$, and $a_n$ are unknown coefficients. Hence, the latter equation transforms to the following

$$\sum_{n=0}^{\infty} y_{i,n}(x) = \sum_{n=0}^{m-1} c_k \frac{x^k}{k!} + [I^{\alpha_i} g_i(x)] + I^{\alpha_i}[\sum_{n=0}^{\infty} a_n x^n] - p I^{\alpha_i}[\sum_{n=0}^{\infty} a_n x^n] + [I^{\alpha_i} \sum_{n=0}^{\infty} y_{i,n}(x)] + [I^{\alpha_i} \sum_{n=0}^{\infty} A_{i,n}(x)],$$

(3.12)

which subsequently gives the formal recursive scheme for the governing coupled system of fractional models as follows

$$\begin{cases} y_{i,0}(x) = \sum_{n=0}^{m-1} c_k \frac{x^k}{k!} + I^{\alpha_i}[\sum_{n=0}^{\infty} a_n x^n], \\ y_{i,1}(x) = [I^{\alpha_i} g_i(x)] - p I^{\alpha_i}[\sum_{n=0}^{\infty} a_n x^n] + [I^{\alpha_i} y_{i,0}(x)] + [I^{\alpha_i} A_{i,0}(x)], \\ y_{i,n+1}(x) = [I^{\alpha_i} y_{i,n}(x)] + [I^{\alpha_i} A_{i,n}(x)], \quad n \geq 1. \end{cases}$$

(3.13)

Consequently, to suppress the computation of $A_n$, $n = 0, 1, 2, ...$, one finds only the coefficients $a_n$ for $n = 0, 1, 2, ...$, through setting $y_1(x) = 0$ for all $n \geq 1$. Specifically, with these assumptions, and upon setting $p = 1$, the overall recursive solution for the governing coupled system of the fractional differential equation is determined in the following series form

$$y_i(x) = \sum_{n=0}^{m-1} c_k \frac{x^k}{k!} + I^{\alpha_i}[\sum_{n=0}^{\infty} a_n x^n].$$

(3.14)

## 4. Numerical applications

This section makes use of some test IVPs to ascertain the effectiveness of the proposed recursive schemes by MADM numerically for the two classes of the fractional differential equations under consideration. Indeed, several test models would be considered and further compared with the approximate solutions reported in [26, 27]. In the same vein, additional numerical methods would be sought to assess the reliability of the proposed MADM.

**Example 4.1.** *Consider the linear nonhomogeneous IVP for Bagley-Torvik equation [26]*

$$D_*^2 y(x) + D_*^{0.5} y(x) + y(x) = x^2 + 2 + \frac{2.6666666667 x^{1.5}}{\Gamma(1/2)}, \quad y(0) = y'(0) = 0.$$

(4.1)

To solve (4.1) by using MADM we apply the inverse operator $I^2$ on both sides of the equation and thereafter makes use of MADM to obtain the following

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) &= I^2[x^2 + 2 + \frac{2.6666666667 x^{1.5}}{\Gamma(1/2)}] + I^2[\sum_{n=0}^{\infty} a_n x^n] \\ &- p I^2[\sum_{n=0}^{\infty} a_n x^n] - I^2[D_*^{0.5} \sum_{n=0}^{\infty} y_n(x)] - I^2[\sum_{n=0}^{\infty} y_n(x)]. \end{aligned}$$

Therefore, the formal recursive scheme for the IVP is revealed as follows

$$\begin{cases} y_0(x) = I^2[\sum_{n=0}^{\infty} a_n x^n]. \\ y_1(x) = I^2[x^2 + 2 + \frac{2.6666666667 x^{1.5}}{\Gamma(1/2)}] - pI^2[\sum_{n=0}^{\infty} a_n x^n] - I^2[D_*^{0.5} y_0(x)] - I^2[y_0(x)]. \end{cases}$$

which when expressed explicitly gives

$$y_0(x) = \frac{a_0 x^2}{2} + \frac{a_1 x^3}{6} + \frac{a_2 x^4}{12} + \dots,$$

$$y_1(x) = \frac{x^4}{12} + x^2 + 0.1719434921 x^{7/2} - \frac{pa_0 x^2}{2} - \frac{pa_1 x^3}{6} - \frac{pa_2 x^4}{12} + \dots - 0.085797174606 a_0 x^{7/2}$$

$$- 0.01910483245 a_1 x^{9/2} - 0.04168327078 a_2 x^{11/2} - \frac{a_0 x^4}{24} - \frac{a_1 x^5}{120} - \frac{a_2 x^6}{720} + \dots.$$

Next, putting $y_1(x) = 0$, we compute $a_i$, for $i = 0, 1, \dots$, that is

$$y_1(x) = (1 - \frac{pa_0}{2})x^2 + (\frac{-pa_1}{6})x^3 + (\frac{1}{12} - \frac{pa_2}{12} - \frac{a_0}{24})x^4 + (\frac{-a_1}{120})x^5 + (\frac{-a_2}{720})x^6 + \dots = 0, \qquad (4.2)$$

which when setting $p = 1$, reveals $a_0 = 2, a_1 = a_2 = 0$. Therefore, on substituting these values of $a_0, a_1$ and $a_2$ into $y_0(x)$, one gets the exact solution as follows

$$y(x) = x^2, \qquad (4.3)$$

which indeed gives the exact solution for the IVP (4.1).

Notably, Shiralashetti and Deshi [26] solved the linear fractional IVP expressed in (4.1) through the application of the Haar wavelet collocation method and obtained the resulting approximate solution by calculating 8 terms. However, the proposed MADM revealed the exact solution while using only 2 terms. Moreover, we establish a comparative analysis between the proposed solution and that of the Haar wavelet collocation method presented in [26] in Table (1). In addition, we besiege the approaches presented in [27], in the presence of the Haar wavelet collocation method presented in [26] to further affirm the accuracy of the proposed MADM approach; see Table (2) for the simulated error difference analysis.

TABLE 1. Comparison of absolute errors between the Haar wavelet collocation method's solution [26] and the proposed MADM's solution with respect to the exact solution for the fractional IVP (4.1).

| $x$ | Haar wavelets method [26] | Exact solution | MADM | Absolute error in [26] with exact solution | Absolute error MADM with exact solution |
|---|---|---|---|---|---|
| 1/16 | 0.0039062498 | 0.0039062500 | 0.0039062500 | $2.000 \times 10^{-10}$ | 0 |
| 3/16 | 0.0351562489 | 0.0351562500 | 0.0351562500 | $1.1000 \times 10^{-9}$ | 0 |
| 5/16 | 0.0976562477 | 0.0976562500 | 0.0976562500 | $2.3000 \times 10^{-9}$ | 0 |
| 7/16 | 0.1914062468 | 0.1914062500 | 0.1914062500 | $3.2000 \times 10^{-9}$ | 0 |
| 9/16 | 0.3164062465 | 0.3164062500 | 0.3164062500 | $3.5000 \times 10^{-9}$ | 0 |
| 11/16 | 0.4726562466 | 0.4726562500 | 0.4726562500 | $3.4000 \times 10^{-9}$ | 0 |
| 13/16 | 0.6601562460 | 0.6601562500 | 0.6601562500 | $4.0000 \times 10^{-9}$ | 0 |
| 15/16 | 0.8789062455 | 0.8789062500 | 0.8789062500 | $4.5000 \times 10^{-9}$ | 0 |

TABLE 2. Comparison of various absolute error differences for the fractional IVP (4.1)

| $x$ | Method (1a) [27] | Method (1b) [27] | Method (2) [27] | Method (3) [27] | Haar wavelets method [26] | MADM |
|---|---|---|---|---|---|---|
| 1/32 | $4.84 \times 10^{-3}$ | $1.30 \times 10^{-2}$ | $9.08 \times 10^{-4}$ | $4.84 \times 10^{-3}$ | $5.2585 \times 10^{-12}$ | 0 |
| 1/64 | $1.66 \times 10^{-3}$ | $7.40 \times 10^{-3}$ | $3.32 \times 10^{-4}$ | $1.66 \times 10^{-3}$ | $2.4630 \times 10^{-13}$ | 0 |
| 1/128 | $5.74 \times 10^{-4}$ | $4.02 \times 10^{-3}$ | $1.20 \times 10^{-4}$ | $5.74 \times 10^{-4}$ | $3.3482 \times 10^{-16}$ | 0 |
| 1/256 | $2.00 \times 10^{-4}$ | $2.12 \times 10^{-3}$ | $4.30 \times 10^{-5}$ | $2.00 \times 10^{-4}$ | $5.5057 \times 10^{-21}$ | 0 |
| 1/512 | $7.00 \times 10^{-5}$ | $1.10 \times 10^{-3}$ | $1.54 \times 10^{-5}$ | $7.00 \times 10^{-5}$ | $4.2352 \times 10^{-22}$ | 0 |

**Example 4.2.** *Consider the nonlinear fractional IVP [26]*

$$aD_*^2 y(x) + bD_*^\alpha y(x) + cD_*^\beta y(x) + ey^3(x) = 2ax + \frac{2bx^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{2cx^{3-\beta}}{\Gamma(4-\beta)} + \frac{ex^9}{27}, \tag{4.4}$$

*where $0 < \alpha, \beta \le 1$ and $a = 1$, $b = 2$, $c = 1/2$, $e = 1, \alpha = 0.07621$ and $\beta = 0.00196$; together with the following initial conditions*

$$y(0) = y'(0) = 0.$$

To begin with, we explicitly rewrite the equation by using the above constants as follows

$$D_*^2 y(x) + 2D_*^{0.07621} y(x) + \frac{1}{2}D_*^{0.00196} y(x) + y^3(x) = 2x + \frac{4x^{2.92374}}{\Gamma(3.92374)} + \frac{x^{2.99804}}{\Gamma(3.99804)} + \frac{x^9}{27}. \tag{4.5}$$

Therefore, to solve (4.5) using MADM with the prescribed initial data, we apply the inverse operator $I^2$ on both sides of the equation, and thus obtain the following

$$\sum_{n=0}^{\infty} y_n(x) = I^2[2x + \frac{4x^{2.92374}}{\Gamma(3.92374)} + \frac{x^{2.99804}}{\Gamma(3.99804)} + \frac{x^9}{27}] + I^2[\sum_{n=0}^{\infty} a_n x^n]$$

$$- pI^2[\sum_{n=0}^{\infty} a_n x^n] - 2I^2[D_*^{0.07621} \sum_{n=0}^{\infty} y_n(x)] - \frac{1}{2}I^2[D_*^{0.00196} \sum_{n=0}^{\infty} y_n(x)] - I^2[\sum_{n=0}^{\infty} A_n(x)],$$

where $A_n(x)$'s are the Adomian polynomial for nonlinear term $y^3(x)$. In fact, upon using the given formula in Section 3 for the acquisition of Adomian polynomials, one get the polynomials for the $y^3(x)$ as follows

$$
\begin{aligned}
A_0 &= y_0^3, \\
A_1 &= 3(y_0)^2 y_1, \\
A_2 &= 3(y_0)^2 y_2 + 3y_1^2 y_0, \\
A_3 &= 3(y_0)^2 y_3 + 6y_0 y_1 y_2 + y_1^3, \\
&\vdots
\end{aligned}
\tag{4.6}
$$

Accordingly, the recursive scheme is obtain as follows

$$
\begin{cases}
y_0(x) = I^2[\sum_{n=0}^{\infty} a_n x^n], \\
y_1(x) = I^2[2x + \frac{4x^{2.92374}}{\Gamma(3.92374)} + \frac{x^{2.99804}}{\Gamma(3.99804)} + \frac{x^9}{27}] - pI^2[\sum_{n=0}^{\infty} a_n x^n] - 2I^2[D_*^{0.07621} y_0(x)] \\
\quad - \frac{1}{2}I^2[D_*^{0.00196} y_0(x)] - I^2[A_0(x)],
\end{cases}
$$

or in more details as follows

$$
y_0(x) = \frac{a_0 x^2}{2} + \frac{a_1 x^3}{6} + \frac{a_2 x^4}{12} + \cdots,
$$

$$
\begin{aligned}
y_1(x) = {} & \frac{x^3}{3} + 0.03794496196 x^{4.9237} + 0.008361243657 x^{4.9980} + \frac{x^{11}}{2970} - \frac{pa_0 x^2}{2} - \frac{pa_1 x^3}{6} - \frac{pa_2 x^4}{12} + \cdots \\
& - 2I^2[D_*^{0.07621}[\frac{a_0 x^2}{2} + \frac{a_1 x^3}{6} + \frac{a_2 x^4}{12} + \cdots]] - \frac{1}{2}I^2[D_*^{0.07621}[\frac{a_0 x^2}{2} + \frac{a_1 x^3}{6} + \frac{a_2 x^4}{12} + \cdots]] \\
& - [\frac{a_0^3 x^8}{448} + \frac{a_1^3 x^{11}}{23760} + \frac{a_2^3 x^{14}}{314496} + \cdots].
\end{aligned}
$$

In addition, we determine the values of $a_i$ for $i = 0, 1, ...,$ from the above by setting $y_1(x) = 0$ in order to bypass the computations of the Adomian polynomials $A_n$. Thus, setting $y_1(x) = 0$, one gets

$$
y_1(x) = (\frac{-pa_0}{2})x^2 + (\frac{1}{3} - \frac{pa_1}{6})x^3 + (\frac{-pa_2}{12})x^4 + (\frac{-pa_0^3}{448})x^8 + (\frac{-pa_1^3}{23760} + \frac{1}{2970})x^{11} + \cdots = 0, \tag{4.7}
$$

upon which when fixing $p = 1$ reveals the values of the coefficients as follows $a_0 = 0, a_1 = 2$, and $a_2 = 0$. Therefore, on substituting these values back into $y_0(x)$, one obtains

$$
y(x) = \frac{1}{3}x^3, \tag{4.8}
$$

which is indeed the required exact solution for (4.5).

Accordingly, Shiralashetti and Deshi [26] equally solved the governing nonlinear fractional IVP (4.5) with the help of Haar wavelet collocation, where the approximate solution was reported by calculating the first 8 terms. However, MADM gives a better result by calculating only 2 terms. Moreover, we report the absolute error differences between the MADM's solution and the Haar

wavelet collocation method's solution in Table (3). Certainly, the proposed MADM outperformed the contending method.

TABLE 3. Comparison of absolute errors between the Haar wavelet collocation method's solution [26] and the proposed MADM's solution with respect to the exact solution for the fractional IVP (4.5)

| $x$ | Haar wavelets method [26] | Exact solution | ADM [28] | MADM | Absolute error in [26] with exact | Absolute error ADM with exact solution [28] | Absolute error MADM with exact solution |
|---|---|---|---|---|---|---|---|
| 1/16 | 0.000243 | 0.000081 | 0.000081 | 0.000081 | $1.61 \times 10^{-4}$ | 0 | 0 |
| 3/16 | 0.002674 | 0.002197 | 0.002197 | 0.002197 | $4.77 \times 10^{-4}$ | 0 | 0 |
| 5/16 | 0.010945 | 0.010172 | 0.010171 | 0.010172 | $7.72 \times 10^{-4}$ | $1 \times 10^{-6}$ | 0 |
| 7/16 | 0.028947 | 0.027913 | 0.027903 | 0.027913 | $1.034 \times 10^{-3}$ | $1 \times 10^{-5}$ | 0 |
| 9/16 | 0.060578 | 0.059326 | 0.059268 | 0.059326 | $1.252 \times 10^{-3}$ | $5.7 \times 10^{-5}$ | 0 |
| 11/16 | 0.109736 | 0.108317 | 0.108089 | 0.108317 | $1.419 \times 10^{-3}$ | $2.27 \times 10^{-4}$ | 0 |
| 13/16 | 0.180320 | 0.178792 | 0.178074 | 0.178792 | $1.527 \times 10^{-3}$ | $7.17 \times 10^{-4}$ | 0 |
| 15/16 | 0.276231 | 0.274658 | 0.272742 | 0.274658 | $1.573 \times 10^{-3}$ | $1.915 \times 10^{-3}$ | 0 |

**Example 4.3.** *Consider nonlinear fractional IVP [29]*

$$D_*^4 y(x) + D_*^{7/2} y(x) + y^3(x) = x^9, \ y(0) = y'(0) = y''(0) = 0, \ y'''(0) = 6. \tag{4.9}$$

As a result, solving the nonlinear fractional IVP expressed in (4.9) by MADM requires the expression of the IVP in an operator form. Therefore, on doing so, and upon applying the inverse operator $I^4$ on both sides of the equation, MADM thus gives

$$\sum_{n=0}^{\infty} y_n(x) = x^3 + I^4[x^9] + I^4[\sum_{n=0}^{\infty} a_n x^n] - pI^4[\sum_{n=0}^{\infty} a_n x^n] - I^4[D_*^{7/2} \sum_{n=0}^{\infty} y_n(x)] - I^4[\sum_{n=0}^{\infty} A_n(x)],$$

where $A_n(x)$ are the Adomian polynomial for nonlinear term $y^3(x)$; see (4.6) for the explicit expression for some of these polynomial for $y^3(x)$. Moreover, the resulting recursive scheme is thus obtained as follows

$$\begin{cases} y_0(x) = x^3 + I^4[\sum_{n=0}^{\infty} a_n x^n], \\ y_1(x) = I^4[x^9] - pI^2[\sum_{n=0}^{\infty} a_n x^n] - I^4[D_*^{7/2} \sum_{n=0}^{\infty} y_n(x)] - I^4[\sum_{n=0}^{\infty} A_n(x)], \end{cases}$$

or more explicitly as follows

$$y_0(x) = x^3 + \frac{a_0 x^4}{24} + \frac{a_1 x^5}{120} + \frac{a_2 x^6}{360} + \cdots,$$

$$y_1(x) = \frac{x^{13}}{17160} - \frac{pa_0 x^4}{24} - \frac{pa_1 x^5}{120} - \frac{pa_2 x^6}{360} - 0.5158304763 x^{7/2} - 0.01910483264 a_0 x^{9/2} \tag{4.10}$$

$$- \cdots - \frac{x^3}{17160} - \cdots.$$

Equally, we bypass the computation of the Adomian polynomial $A_n$ by computing the values of the coefficients $a_i$, for $i = 0, 1, ...$, by equating $y_1(x) = 0$, that is,

$$y_1(x) = (\frac{-pa_0}{24})x^4 + (-\frac{pa_1}{120})x^5 + (-\frac{pa_2}{360})x^6 + \cdots = 0. \tag{4.11}$$

What is more, on further setting $p = 1$ gives $a_0 = 0, a_1 = 0$ and $a_2 = 0$, upon which when substitute these values into $y_0(x)$ yields the following solution

$$y(x) = x^3, \tag{4.12}$$

which is indeed the required exact solution for (4.9). Notably, Yang [29] made use of the Legendre polynomial method to equally solve the governing nonlinear fractional IVP in (4.9). However, in Yang [29] approach, the approximate solution was acquired after calculating the first 4 terms of the series; which is against the present MADM, which perfectly gives the exact solution by calculating the first 2 terms.

**Example 4.4.** *Consider the system of linear fractional IVPs as follows [30]*

$$\begin{cases} D_*^{1.5}y_1 = y_2, \ y_1(0) = y_1'(0) = 1, \\ D_*^{0.5}y_2 = -y_2 - y_1 + 1 + x, \ y_2(0) = 0. \end{cases} \tag{4.13}$$

Consequently, upon applying the related inverse operators on both sides of the respective equations, alongside utilizing the MADM procedure, one gets the following equations

$$\begin{cases} \sum_{n=0}^{\infty} y_{n,1}(x) = I^{1.5}[\sum_{n=0}^{\infty} y_{n,2}] + 1 + x + I^{1.5}[\sum_{n=0}^{\infty} a_n x^n] - pI^{1.5}[\sum_{n=0}^{\infty} a_n x^n], \\ \sum_{n=0}^{\infty} y_{n,2}(x) = I^{0.5}[-\sum_{n=0}^{\infty} y_{n,2} - \sum_{n=0}^{\infty} y_{n,1} + 1 + x] + I^{0.5}[\sum_{n=0}^{\infty} a_n x^n] \\ \qquad - pI^{0.5}[\sum_{n=0}^{\infty} a_n x^n]. \end{cases} \tag{4.14}$$

Moreover, the recursive schemes in $y_{n,1}(x)$ and $y_{n,2}(x)$ for $n = 0, 1$ are accordingly obtained as follow

$$\begin{cases} y_{0,1}(x) = 1 + x + I^{1.5}[\sum_{n=0}^{\infty} a_n x^n] = 1 + x + \frac{a_0 x^{1.5}}{1.5\Gamma(1.5)} + \frac{a_1 x^{2.5}}{\Gamma(2+1.5)} + 2\frac{a_2 x^{3.5}}{1.5\Gamma(3+1.5)} + \cdots, \\ y_{0,2}(x) = I^{0.5}[\sum_{n=0}^{\infty} a_n x^n] + I^{0.5}[x+1] = \frac{x^{0.5}}{0.5\Gamma(0.5)} + \frac{x^{1.5}}{\Gamma(2.5)} + a_0 \frac{x^{0.5}}{0.5\Gamma(0.5)} + a_1 \frac{x^{1.5}}{\Gamma(2.5)} \\ \qquad + a_2 \frac{2x^{2.5}}{\Gamma(3.5)} + \cdots. \end{cases}$$

$$\begin{cases} y_{1,1}(x) = I^{1.5}[y_{0,2}] - pI^{1.5}[\sum_{n=0}^{\infty} a_n x^n] = -p\frac{a_0 x^{1.5}}{1.5\Gamma(1.5)} - p\frac{a_1 x^{2.5}}{\Gamma(2+1.5)} - 2p\frac{a_2 x^{3.5}}{1.5\Gamma(3+1.5)} - \cdots, \\ y_{1,2}(x) = I^{0.5}[-y_{0,2} - y_{0,1}] - pI^{0.5}[\sum_{n=0}^{\infty} a_n x^n] = -pa_0 \frac{x^{0.5}}{0.5\Gamma(0.5)} - pa_1 \frac{x^{1.5}}{\Gamma(2.5)} - pa_2 \frac{2x^{2.5}}{\Gamma(3.5)} \\ \qquad - \frac{x\Gamma(1.5)}{1} - \frac{x^2\Gamma(2.5)}{2} - a_0 \frac{x\Gamma(1.5)}{1} - a_1 \frac{x^2\Gamma(2.5)}{2} - \cdots. \end{cases}$$

Further, on equating $y_{1,1}(x)$ and $y_{1,2}(x)$ to zero, that is, $y_{1,1}(x) = 0$ and $y_{1,2}(x) = 0$, one gets

$$\begin{cases} y_{1,1}(x) = I^{1.5}[y_{0,2}] - pI^{1.5}[\sum_{n=0}^{\infty} a_n x^n] = -p\frac{a_0 x^{1.5}}{1.5\Gamma(1.5)} - p\frac{a_1 x^{2.5}}{\Gamma(2+1.5)} - 2p\frac{a_2 x^{3.5}}{1.5\Gamma(3+1.5)} - \cdots = 0, \\ y_{1,2}(x) = I^{0.5}[-y_{0,2} - y_{0,1}] - pI^{0.5}[\sum_{n=0}^{\infty} a_n x^n] = -pa_0 \frac{x^{0.5}}{0.5\Gamma(0.5)} - pa_1 \frac{x^{1.5}}{\Gamma(2.5)} - pa_2 \frac{2x^{2.5}}{\Gamma(3.5)} \\ \qquad - \frac{x\Gamma(1.5)}{1} - \frac{x^2\Gamma(2.5)}{2} - a_0 \frac{x\Gamma(1.5)}{1} - a_1 \frac{x^2\Gamma(2.5)}{2} - \cdots = 0, \end{cases}$$

which eventually yields $a_0 = a_1 = -1$ and $a_2 = 0$ from $y_{1,2}(x) = 0$, and $a_0 = a_1 = a_2 = 0$ from $y_{1,1}(x) = 0$. Therefore, on substituting the respective values into $y_{0,1}(x)$ and $y_{0,2}(x)$, one gets the overall solution of the system of coupled fractional IVPs as follows

$$\begin{cases} y_1(x) = 1 + x, \\ y_2(x) = 0. \end{cases} \tag{4.15}$$

**Example 4.5.** *Consider the system of nonlinear fractional IVPs as follows* [30],

$$\begin{cases} D_*^{2.5} y_1 = y_2, \ y_1(0) = y_1'(0) = 0, y''(0) = 2, \\ D_*^{0.5} y_2 = -y_2 - y_1^2 + x^4, \ y_2(0) = 0. \end{cases} \tag{4.16}$$

In the same passion, and as detailed above, MADM portrays the coupled recurrent equations

$$\begin{cases} \sum_{n=0}^{\infty} y_{n,1}(x) = I^{2.5}[\sum_{n=0}^{\infty} y_{n,2}] + x^2 + I^{2.5}[\sum_{n=0}^{\infty} a_n x^n] - pI^{2.5}[\sum_{n=0}^{\infty} a_n x^n], \\ \sum_{n=0}^{\infty} y_{n,2}(x) = I^{0.5}[-\sum_{n=0}^{\infty} y_{n,2} - \sum_{n=0}^{\infty} A_n + x^4] + I^{0.5}[\sum_{n=0}^{\infty} a_n x^n] - pI^{0.5}[\sum_{n=0}^{\infty} a_n x^n], \end{cases} \tag{4.17}$$

where $A_n(x)$'s are the Adomian polynomials corresponding to the nonlinear term $y^2(x)$, and which take the following explicit forms

$$A_0 = y_0^2,$$
$$A_1 = 2y_0 y_1,$$
$$A_2 = 2y_0 y_2 + y_1^2,$$
$$\vdots$$

$$\begin{cases} y_{0,1}(x) = x^2 + I^{2.5}[\sum_{n=0}^{\infty} a_n x^n] = x^2 + \frac{a_0 x^{2.5}}{2.5\Gamma(2.5)} + \frac{a_1 x^{3.5}}{\Gamma(4.5)} + 2\frac{a_2 x^{4.5}}{\Gamma(5.5)} + \cdots, \\ y_{0,2}(x) = I^{0.5}[\sum_{n=0}^{\infty} a_n x^n] + I^{0.5}[x^4] = \frac{x^{4.5}\Gamma(5)}{\Gamma(5.5)} + a_0\frac{x^{0.5}}{0.5\Gamma(0.5)} + a_1\frac{x^{1.5}}{\Gamma(2.5)} + a_2\frac{2x^{2.5}}{\Gamma(3.5)} + \cdots. \end{cases}$$

Accordingly, the recursive schemes take the following forms

$$\begin{cases} y_{1,1}(x) = -pI^{2.5}[\sum_{n=0}^{\infty} a_n x^n] + I^{2.5}[y_{0,2}] = -p\frac{a_0 x^{2.5}}{2.5\Gamma(2.5)} - p\frac{a_1 x^{3.5}}{\Gamma(4.5)} - 2p\frac{a_2 x^{4.5}}{\Gamma(5.5)} + \cdots, \\ y_{1,2}(x) = I^{0.5}[-y_{0,2} - A_1] - pI^{0.5}[\sum_{n=0}^{\infty} a_n x^n] = -pa_0\frac{x^{0.5}}{0.5\Gamma(0.5)} - pa_1\frac{x^{1.5}}{\Gamma(2.5)} - pa_2\frac{2x^{2.5}}{\Gamma(3.5)} \\ \qquad\qquad - \frac{x^{4.5}\Gamma(5)}{\Gamma(5.5)} - pa_0\frac{x^3\Gamma(3.5)}{\Gamma(4)} - \cdots. \end{cases}$$

In the same way, set $y_{1,1}(x) = 0$ and $y_{1,2}(x) = 0$, and obtain from the latter equations the following

$$\begin{cases} y_{1,1}(x) = -p\frac{a_0 x^{2.5}}{2.5\Gamma(2.5)} - p\frac{a_1 x^{3.5}}{\Gamma(4.5)} - 2p\frac{a_2 x^{4.5}}{\Gamma(5.5)} + \cdots = 0, \\ y_{1,2}(x) = -pa_0\frac{x^{0.5}}{0.5\Gamma(0.5)} - pa_1\frac{x^{1.5}}{\Gamma(2.5)} - pa_2\frac{2x^{2.5}}{\Gamma(3.5)} - \frac{x^{4.5}\Gamma(5)}{\Gamma(5.5)} - pa_0\frac{x^3\Gamma(3.5)}{\Gamma(4)} - \cdots = 0, \end{cases}$$

where $y_{1,2}(x) = 0$ yields $a_0 = a_1 = a_2 = 0$, while $y_{1,1}(x) = 0$ equally yields $a_0 = a_1 = a_2 = 0$. Thus, on substituting these values in the respective equation, one gets the solution of the coupled system

of fractional IVPs as follows

$$\begin{cases} y_1(x) = x^2, \\ y_2(x) = 0, \end{cases}$$

which indeed satisfies the governing coupled nonlinear model (4.16).

## 5. Conclusion

In this study, effective modification methods for the solution of the classes of fractional differential equations and system of fractional differential equations have been successfully proposed and further applied to several linear and nonlinear test initial-value problems. Certainly, it was observed that the proposed MADMs reduce the number of iterations greatly owing to the fact that only $y_0$ and $y_1$ are utilized; unlike the standard ADM, which uses uncountable iterates. Hence the size of the computation is hugely minimized in comparison with the standard ADM. Moreover, the proposed approaches were equally noted to reduce the number of Adomian polynomials to be constructed as only $A_0$ is relevant. Thus, we can conclude that the proposed MADMs, in comparison with the standard ADM and several other numerical approaches are efficient, especially looking at the revelation that they are rapid, accurate, less error, and consume less computational time among others.

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