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SOME COUPLED COINCIDENCE POINTS RESULTS OF MONOTONE MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT. In this paper, we introduce the concepts of a monotone mappings and monotone mapping with respect to other mapping to obtain some coupled coincidence point results in partially ordered metric spaces. Our results generalize, extend and complement various comparable results in the existing literature.

1. Introduction and preliminaries

The existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [19], and then by Nieto and Lopez [13]. Further results in this direction were proved, e.g ([2], [3], [4], [7], [9], [17], [20]. Results on weak contractive mappings in such spaces, together with applications to differential equations, were obtained by Harjani and Sadarangani in [10].

The notion of a coupled fixed point was introduced and studied by Opoitsev ([14]-[16]) and then by Guo and Lakshmikantham [8]. Bhashkar and Lakshmikantham in [5] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. They also discussed an application of their result by investigating the existence and uniqueness of solution for a periodic boundary value problem. Choudhury and Kundu [6] obtained coupled coincidence point results in partially ordered metric spaces for compatible mappings.

Recently, Abbas et al. [1] proved coupled coincidence and coupled common fixed point results in cone metric spaces for $w-$ compatible mappings (see also, [11]). The aim of this paper is to prove some coupled coincidence points results for so-called monotone mappings or monotone mappings with respect some other mapping in partially ordered metric spaces. The results presented in this paper generalize, extend and complement various comparable results in the existing literature ([5, 6, 11, 12, 18]).

We start with the following.

Definition 1.1. [12] Let $(X, \preceq)$ be a partially ordered set. A mapping $F : X \times X \to X$ is a monotone with respect to $g : X \to X$, if for any $x, y \in X$,

$$x_1, x_2 \in X,\ gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y),$$

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and

$$y_1, y_2 \in X,\ gy_1 \preceq gy_2 \text{ implies } F(x, y_1) \preceq F(x, y_2).$$

If we take $g = I_X$ (an identity mapping on X), then F is a monotone mapping on X. ([5]).

Definition 1.2. [5] An element $(x, y) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.3. [1] An element $(x, y) \in X \times X$ is called:

a coupled coincidence point of mappings $F : X\times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called coupled point of coincidence;

a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 1.4. [18] Let $(X, \preceq)$ be an ordered set and d be a metric on X. We say that $(X, d, \preceq)$ is regular if it has the following properties:

(**i**) if for non-decreasing sequence $\{x_n\}$ holds $d\,(x_n, x) \to 0$, then $x_n \preceq x$ for all n,

(**ii**) if for non-increasing sequence $\{y_n\}$holds $d\,(x_n, x) \to 0$, then $y_n \succeq y$ for all n.

2. Main results

All the results in [2], [5], [6], [7], [9], [10], [18] are obtained for mixed monotone mappings, that is., for mappings $F : X \times X \to X$ which are increasing with respect to the first variable and decreasing with respect to the second variable.

It is our main aim in this paper to consider coupled coincidence points of mappings which are of the same monotonicity with respect to both variables.

Now, we start with the following result.

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that a mapping $F : X \times X \to X$ is a monotone with respect to $g : X \to X$ and*

$$\begin{aligned} & d(F(x,y),F(u,v)) + d\left(F\left(y,x\right),F\left(v,u\right)\right) \\ \le\ & \phi(\max\{\frac{d(gx,gu)+d(gy,gv)}{2},\frac{d(F(x,y),gx)+d(F(x,y),gu)}{2}, \\ & \frac{d(gy,gv)+d(F(x,y),gx)}{2},\frac{d(gy,gv)+d(F(x,y),gu)}{2}\}) \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$, for which $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$, where $\phi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing function such that $\phi(t) < t$ for all $t > 0$. If $F(X \times X)$ is contained in a complete set $g(X)$, $(X, d \preceq)$ is a regular and if there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0) \text{ and } g(y_0) \preceq F(y_0, x_0),$$

then there exist $x, y \in X$ such that $g\,(x) = F\,(x, y)$ and $g\,(y) = F\,(y, x)$.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \preceq F(y_0, x_0)$. Set $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, this can be done as $F(X \times X) \subseteq g(X)$. Similarly, $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$ because $F(X \times X) \subseteq g(X)$. Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \text{ for all } n \ge 0. \tag{2.2}$$

We shall show that $g(x_n) \preceq g(x_{n+1})$ and $g(y_n) \preceq g(y_{n+1})$ for all $n \ge 0$.

By induction, let $n = 0$. Since $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \preceq F(y_0, x_0)$ also $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, so that $gx_0 \preceq gx_1$ and $gy_0 \preceq gy_1$. Now, let it holds for some fixed $n \geq 0$. Since $gx_n \preceq gx_{n+1}$ and $gy_n \preceq gy_{n+1}$, and as F is monotone mapping with respect to g, so that $gx_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \preceq F(x_{n+1}, y_{n+2}) = gx_{n+2}$ and $gy_{n+1} = F(y_n, x_n) \preceq F(y_{n+1}, x_n) \preceq F(y_{n+1}, x_{n+1}) = gy_{n+2}$. Hence $gx_{n+1} \preceq gx_{n+2}$ and $gy_{n+1} \preceq gy_{n+2}$. Thus by the mathematical induction we conclude that for all $n \geq 0$,

$$\begin{aligned} gx_0 &\preceq gx_1 \preceq ... \preceq gx_n \preceq gx_{n+1} \preceq ..., \text{ and} \\ gy_0 &\preceq gy_1 \preceq ... \preceq gy_n \preceq gy_{n+1} \preceq .... \end{aligned}$$

We will suppose that $d(gx_n, gx_{n+1}) > 0$ and $d(gy_n, gy_{n+1}) > 0$ for all n, since if $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$ for some n, then by (2.2),

$$gx_n = F(x_n, y_n) \text{ and } gy_n = F(y_n, x_n),$$

that is, F and g have a coupled coincidence point (x_n, y_n), and so we have finished the proof. Now from (2.1), we have

$$\begin{aligned} & d(gx_n, gx_{n+1}) + d\,(gy_n, gy_{n+1}) \\ = \;& d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ \leq \;& \phi(\max\{\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}, \\ & \frac{d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + d(F(x_{n-1}, y_{n-1}), gx_n)}{2}, \\ & \frac{d(gy_{n-1}, gy_n) + d(F(x_{n-1}, y_{n-1}), gx_{n-1})}{2}, \frac{d(gy_{n-1}, gy_n) + d(F(x_{n-1}, y_{n-1}), gx_n)}{2}\}) \\ = \;& \phi(\max\{\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}, \frac{d(gx_n, gx_{n-1})}{2}, \frac{d(gy_{n-1}, gy_n)}{2}\}), \end{aligned}$$

and hence

$$\begin{aligned} d(gx_n, gx_{n+1}) + d\,(gy_n, gy_{n+1}) &\leq \phi(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}) \\ &< \phi\,(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)) \end{aligned} \tag{2.3}$$

Now

$$\begin{aligned} & d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \\ \leq \;& \phi(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)) \\ \leq \;& \phi^2(d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})) \\ \leq \;& ... \\ \leq \;& \phi^n(d(gx_0, gx_1) + d(gy_0, gy_1)). \end{aligned}$$

Since $\lim_{n\to\infty} \phi^n(d(gx_0, gx_1) + d(gy_0, gy_1)) = 0$, then for a given $\varepsilon > 0$, there is a positive integer n_0 such that for all $n \geq n_0$,

$$\phi^n(d(gx_0, gx_1) + d(gy_0, gy_1)) < \frac{\varepsilon - \phi(\varepsilon)}{2}. \tag{2.4}$$

Hence

$$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) < \varepsilon - \phi(\varepsilon), \tag{2.5}$$

for all $n \geq n_0$. That is,

$$d(gx_n, gx_{n+1}) < \varepsilon - \phi(\varepsilon) \text{ and } d(gy_n, gy_{n+1}) < \varepsilon - \phi(\varepsilon). \tag{2.6}$$

Now, for any $m, n \in N$ with $m > n \geq n_0$, we claim that

$$d(gx_n, gx_m) < \varepsilon \tag{2.7}$$

$$\text{and } d(gy_n, gy_m) < \varepsilon. \tag{2.8}$$

We prove the inequality (2.7) and (2.8) by induction on m. The inequality (2.7) and (2.8) hold for $m = n+1$ by using (2.6). Assume that (2.7) and (2.8) hold for $m = k$. Since $gx_n \preceq gx_k$ and $gy_n \preceq gy_k$, so that for $m = k+1$, we have

$$\begin{aligned}
& d(gx_n, gx_m) + d(gy_n, gy_m) = d(gx_n, gx_{k+1}) + d(gy_n, gy_{k+1}) \\
\leq\ & d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{k+1}) + d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{k+1}) \\
<\ & \varepsilon - \phi(\varepsilon) + d(gx_{n+1}, gx_{k+1}) + d(gy_{n+1}, gy_{k+1}) \\
=\ & \varepsilon - \phi(\varepsilon) + d(F(x_n, y_n), F(x_k, y_k)) + d\left(F\left(y_n, x_n\right), F\left(y_k, x_k\right)\right) \\
\leq\ & \varepsilon - \phi(\varepsilon) + \phi(\max\{\frac{d(gx_n, gx_k) + d(gy_n, gy_k)}{2}, \frac{d(F(x_n, y_n), gx_n) + d(F(x_n, y_n), gx_k)}{2}, \\
& \frac{d(gy_n, gy_k) + d(F(x_n, y_n), gx_n)}{2}, \frac{d(gy_n, gy_k) + d(F(x_n, y_n), gx_k)}{2}\}) \\
=\ & \varepsilon - \phi(\varepsilon) + \phi(\max\{\frac{d(gx_n, gx_k) + d(gy_n, gy_k)}{2}, \frac{d(gx_{n+1}, gx_n) + d(gx_{n+1}, gx_k)}{2}, \\
& \frac{d(gy_n, gy_k) + d(gx_{n+1}, gx_n)}{2}, \frac{d(gy_n, gy_k) + d(gx_{n+1}, gx_k)}{2}\}) \\
\leq\ & \varepsilon - \phi(\varepsilon) + \phi(\max\{\frac{\varepsilon + \varepsilon}{2}, \frac{\varepsilon - \phi(\varepsilon) + \varepsilon}{2}, \frac{\varepsilon + \varepsilon - \phi(\varepsilon)}{2}, \frac{\varepsilon + \varepsilon}{2}\}) \\
=\ & \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon.
\end{aligned}$$

By induction on m, we conclude that (2.7) and (2.8) hold for $m > n \geq n_0$. Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$, so there exists x and y in X such that $\{gx_n\}$ and $\{gy_n\}$ converges to gx and gy respectively. Now, we prove that $F(x, y) = gx$ and $F(y, x) = gy$.
Since $gx_n \preceq gx$ and $gy_n \preceq gy$ for all $n \geq 0$, so that we have

$$\begin{aligned}
& d(F(x, y), gx) + d\left(F\left(y, x\right), gy\right) \\
\leq\ & d(F(x, y), gx_{n+1}) + d(gx_{n+1}, gx) + d\left(F\left(y, x\right), gy_{n+1}\right) + d\left(gy_{n+1}, gy\right) \\
=\ & d(F(x_n, y_n), F(x, y)) + d\left(F\left(y_n, x_n\right), F\left(y, x\right)\right) + d(gx_{n+1}, gx) + d\left(gy_{n+1}, gy\right) \\
\leq\ & \phi(\max\{\frac{d(gx_n, gx) + d(gy_n, gy)}{2}, \frac{d(F(x_n, y_n), gx_n) + d(F(x_n, y_n), gx)}{2}, \\
& \frac{d(gy_n, gy) + d(F(x_n, y_n), gx_n)}{2}, \frac{d(gy_n, gy) + d(F(x_n, y_n), gx)}{2}\}) \\
& + d(gx_{n+1}, gx) + d\left(gy_{n+1}, gy\right) \\
=\ & \phi(\max\{\frac{d(gx_n, gx) + d(gy_n, gy)}{2}, \frac{d(gx_{n+1}, gx_n) + d(gx_{n+1}, gx)}{2}, \\
& \frac{d(gy_n, gy) + d(gx_{n+1}, gx_n)}{2}, \frac{d(gy_n, gy) + d(gx_{n+1}, gx)}{2}\}) + d(gx_{n+1}, gx) + d\left(gy_{n+1}, gy\right).
\end{aligned}$$

On taking limit as $n \to \infty$, we obtain that

$$d(F(x, y), gx) + d\left(F\left(y, x\right), gy\right) \leq 0, \tag{2.9}$$

that is., $F(x,y)=gx$ and $F(y,x)=gy$. Hence (x,y) is a coupled coincidence point and (gx,gy) is coupled point of coincidence of mappings F and g. □

Following example support Theorem 2.1.

Example 2.2. Let $X=[0,1]$ be an ordered set with the natural ordering of real numbers and d a usual metric on X. Let $F:X\times X\to X$, $g:X\to X$ and $\phi:[0,\infty)\to[0,\infty)$ be defined by

$$F(x,y)=\frac{2x+y+1}{18},g(x)=\frac{3x}{4}\text{ for all }x,y\in X, \tag{2.10}$$

and $\phi(t)=\frac{8}{9}t$, for $t\in[0,\infty)$. Note that $F(X\times X)\subseteq g(X)$ and ϕ is nondecreasing, continuous with $\phi(t)<t$ for all $t>0$.
Now for $g(x)\preceq g(u)$ and $g(y)\preceq g(v)$, we obtain

$$\begin{aligned}
&d(F(x,y),F(u,v))+d(F(y,x),F(v,u))\\
=\;&\frac{1}{18}\left|2x+y-2u-v\right|+\frac{1}{18}\left|2y+x-2v-u\right|\\
\le\;&\frac{1}{18}\left|2(x-u)+(y-v)\right|+\frac{1}{18}\left|2(y-v)+(x-u)\right|\\
\le\;&\frac{6}{18}(|x-u|+|y-v|)\\
=\;&\frac{2}{3}\frac{\left|\frac{3}{4}x-\frac{3}{4}u\right|+\left|\frac{3}{4}y-\frac{3}{4}v\right|}{2}\cdot\frac{4}{3}=\frac{8}{9}\frac{\left|\frac{3}{4}x-\frac{3}{4}u\right|+\left|\frac{3}{4}y-\frac{3}{4}v\right|}{2}\\
=\;&\phi(\frac{d(gx,gu)+d(gy,gv)}{2})\\
\le\;&\phi(\max\{\frac{d(gx,gu)+d(gy,gv)}{2},\frac{d(F(x,y),gx)+d(F(x,y),gu)}{2},\\
&\frac{d(gy,gv)+d(F(x,y),gx)}{2},\frac{d(gy,gv)+d(F(x,y),gu)}{2}\}).
\end{aligned}$$

Thus (2.1) is satisfied and F and g have coupled coincidence points. Here, $\left(\frac{2}{21},\frac{2}{21}\right)$ is a coupled coincidence point and $\left(g\left(\frac{2}{21}\right),g\left(\frac{2}{21}\right)\right)=\left(\frac{1}{14},\frac{1}{14}\right)$ is coupled point of coincidence of mappings F and g. □

Remark 2.3. Since F has not a mixed monotone property with respect to g, it follows that a coupled coincidence point $\left(\frac{2}{21},\frac{2}{21}\right)$ cannot be obtained by Theorem 2.1. from [2].

Corollary 2.4. *Let $(X,d,\preceq)$ be a partially ordered metric space. Suppose that a mapping $F:X\times X\to X$ is a monotone with respect to $g:X\to X$ and*

$$\begin{aligned}
&d(F(x,y),F(u,v))+d(F(y,x),F(v,u))\\
\le\;&k\max\{d(gx,gu)+d(gy,gv),d(F(x,y),gx)+d(F(x,y),gu),\\
&d(gy,gv)+d(F(x,y),gx),d(gy,gv)+d(F(x,y),gu)\}
\end{aligned} \tag{2.11}$$

for all $x,y,u,v\in X$, for which $g(x)\preceq g(u)$ and $g(y)\preceq g(v)$, where $k\in[0,\frac{1}{2})$. If $F(X\times X)$ is contained in a complete set $g(X)$, $(X,d\preceq)$ is a regular and if there exist $x_0,y_0\in X$ such that

$$g(x_0)\preceq F(x_0,y_0)\text{ and }g(y_0)\preceq F(y_0,x_0),$$

then there exist $x,y\in X$ such that $g(x)=F(x,y)$ and $g(y)=F(y,x)$.

Proof. Taking $\phi(t)=kt$ with $k\in[0,\frac{1}{2})$ in Theorem 2.1, we obtain Corollary 2.1. □

Corollary 2.5. Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that a mapping $F : X \times X \to X$ is a monotone and

$$\begin{aligned}
& d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\
\leq \;& \phi(\max\{\frac{d(x,u)+d(y,v)}{2}, \frac{d(F(x,y),x)+d(F(x,y),u)}{2}, \\
& \frac{d(y,v)+d(F(x,y),x)}{2}, \frac{d(y,v)+d(F(x,y),u)}{2}\}) \qquad (2.12)
\end{aligned}$$

for all $x, y, u, v \in X$, for which $x \preceq u$ and $y \preceq v$. If $(X, d, \preceq)$ is a complete and regular and if there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \preceq F(y_0, x_0),$$

then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.
Proof. The result follows by taking $g = I$ (identity mapping) in Theorem 2.1. $\square$
Corollary 2.6. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that a mapping $F : X \times X \to X$ is a monotone with respect to $g : X \to X$ and*

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \leq \phi(\frac{d(gx,gu)+d(gy,gv)}{2}) \qquad (2.13)$$

for all $x, y, u, v \in X$, for which $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function such that $\phi(t) < t$ for all $t > 0$. If $F(X \times X)$ is contained in a complete set $g(X)$, $(X, d \preceq)$ is a regular and if there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0) \text{ and } g(y_0) \preceq F(y_0, x_0),$$

then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

As $\phi(\max\{a, b\}) = \max\{\phi(a), \phi(b)\}$ for all $a, b \in [0, \infty)$ if $\phi : [0, \infty) \to [0, \infty)$ is nondecreasing map, then we obtain following equivalent form of Theorem 2.1.
Theorem 2.7. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that a mapping $F : X \times X \to X$ is a monotone with respect to $g : X \to X$ and*

$$\begin{aligned}
& d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\
\leq \;& \max\{\phi(\frac{d(gx,gu)+d(gy,gv)}{2}), \phi(\frac{d(F(x,y),gx)+d(F(x,y),gu)}{2}), \\
& \phi(\frac{d(gy,gv)+d(F(x,y),gx)}{2}), \phi(\frac{d(gy,gv)+d(F(x,y),gu)}{2})\} \qquad (2.14)
\end{aligned}$$

for all $x, y, u, v \in X$, for which $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function such that $\phi(t) < t$ for all $t > 0$. If $F(X \times X)$ is contained in a complete set $g(X)$, $(X, d \preceq)$ is a regular and if there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0) \text{ and } g(y_0) \preceq F(y_0, x_0),$$

then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.
Theorem 2.8. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that a mapping $F : X \times X \to X$ is a monotone with respect to $g : X \to X$ and*

$$d(F(x,y), F(u,v)) \leq \frac{\phi(d(F(x,y),gx)) + \phi(d(F(u,v),gu))}{2} \qquad (2.15)$$

for all $x, y, u, v \in X$, for which $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function such that $\phi(t) < t$ for all $t > 0$. If

$F(X\times X)$ is contained in a complete set $g(X)$, $(X,d\preceq)$ is a regular and if there exist $x_0,y_0\in X$ such that

$$g(x_0)\preceq F(x_0,y_0) \text{ and } g(y_0)\preceq F(y_0,x_0),$$

then there exist $x,y\in X$ such that $g(x)=F(x,y)$ and $g(y)=F(y,x)$.

Proof. Let $x_0,y_0\in X$ be such that $g(x_0)\preceq F(x_0,y_0)$ and $g(y_0)\preceq F(y_0,x_0)$. Using the similar arguments to those given in Theorem 2.1, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $g(x_{n+1})=F(x_n,y_n)$ and $g(y_{n+1})=F(y_n,x_n)$ for all $n\geq 0$ and for all $n\geq 0$,

$$\begin{aligned} gx_0 &\preceq gx_1\preceq ...\preceq gx_n\preceq gx_{n+1}\preceq ..., \text{ and}\\ gy_0 &\preceq gy_1\preceq ...\preceq gy_n\preceq gy_{n+1}\preceq .... \end{aligned}$$

Now we will suppose that $d(gx_n,gx_{n+1})>0$ and $d(gy_n,gy_{n+1})>0$ for all n, otherwise, F and g have a coupled coincidence point at (x_n,y_n), and so we have finished the proof. From (2.15),

$$\begin{aligned} & d(gx_n,gx_{n+1})=d(F(x_{n-1},y_{n-1}),F(x_n,y_n))\\ \leq\ & \frac{\phi(d(F(x_{n-1},y_{n-1}),gx_{n-1})+\phi(d(F(x_n,y_n),gx_n))}{2}\\ =\ & \frac{\phi(d(gx_n,gx_{n-1}))+\phi(d(gx_{n+1},gx_n))}{2}\\ \leq\ & \frac{\phi(d(gx_n,gx_{n-1}))+d(gx_{n+1},gx_n)}{2}, \end{aligned}$$

that is.,

$$d(gx_n,gx_{n+1})\leq\phi(d(gx_{n-1},gx_n)). \tag{2.16}$$

Similarly,

$$d(gy_n,gy_{n+1})\leq\phi(d(gy_{n-1},gy_n)). \tag{2.17}$$

From (2.16) and (2.17), we obtain

$$d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})\leq\phi(d(gx_{n-1},gx_n))+\phi(d(gy_{n-1},gy_n)). \tag{2.18}$$

Now

$$\begin{aligned} & d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})\\ \leq\ & \phi(d(gx_{n-1},gx_n))+\phi(d(gy_{n-1},gy_n))\\ \leq\ & \phi^2(d(gx_{n-2},gx_{n-1}))+\phi^2(d(gy_{n-2},gy_{n-1}))\\ \leq\ & ...\leq\phi^n(d(gx_0,gx_1))+\phi^n(d(gy_0,gy_1)). \end{aligned}$$

For a given $\varepsilon>0$, since $\lim_{n\to\infty}[\phi^n(d(gx_0,gx_1))+\phi^n(d(gy_0,gy_1))]=0$ and $\phi(\varepsilon)<\varepsilon$, there is a positive integer n_0 such that for all $n\geq n_0$,

$$\phi^n(d(gx_0,gx_1))+\phi^n(d(gy_0,gy_1))<\varepsilon-\phi(\varepsilon). \tag{2.19}$$

Hence

$$d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})<\varepsilon-\phi(\varepsilon),$$

that is.,

$$d(gx_n,gx_{n+1})<\varepsilon-\phi(\varepsilon) \text{ and } d(gy_n,gy_{n+1})<\varepsilon-\phi(\varepsilon). \tag{2.20}$$

Now, for any $m,n\in N$ with $m>n$, we claim that

$$d(gx_n,gx_m)<\varepsilon \tag{2.21}$$

and

$$d(gy_n, gy_m) < \varepsilon. \tag{2.22}$$

We prove the inequalities (2.21) by induction on m. The inequalities (2.21) holds for $m = n+1$ by using (2.20). Assume that (2.21) holds for $m = k$. Since $gx_n \preceq gx_k$ and $gy_n \preceq gy_k$, so that for $m = k + 1$, we have

$$\begin{aligned}
& d(gx_n, gx_m) = d(gx_n, gx_{k+1}) \leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{k+1}) \\
\leq\ & \varepsilon - \phi(\varepsilon) + d(gx_{n+1}, gx_{k+1}) = \varepsilon - \phi(\varepsilon) + d(F(x_n, y_n), F(x_k, y_k)) \\
\leq\ & \varepsilon - \phi(\varepsilon) + \frac{\phi(d(F(x_n, y_n), gx_n)) + \phi(d(F(x_k, y_k), gx_k))}{2} \\
=\ & \varepsilon - \phi(\varepsilon) + \frac{\phi(d(gx_{n+1}, gx_n)) + \phi(d(gx_{k+1}, gx_k))}{2} \\
\leq\ & \varepsilon - \phi(\varepsilon) + \frac{\phi(\varepsilon - \phi(\varepsilon)) + \phi(\varepsilon - \phi(\varepsilon))}{2} \\
=\ & \varepsilon - \phi(\varepsilon) + \phi(\varepsilon - \phi(\varepsilon)) \leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon.
\end{aligned}$$

Similarly, we obtain

$$d(gy_n, gy_m) < \varepsilon.$$

By induction on m, we conclude that (2.21) and (2.22) holds for $m > n \geq n_0$. Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$, so there exists x and y in X such that $\{gx_n\}$ and $\{gy_n\}$ converges to gx and gy respectively. Now, we prove that $F(x, y) = gx$ and $F(y, x) = gy$.

Since $gx_n \preceq gx$ and $gy_n \preceq gy$ for all $n \geq 0$, so that we have

$$\begin{aligned}
& d(F(x, y), gx) \leq d(F(x, y), gx_{n+1}) + d(gx_{n+1}, gx) \\
=\ & d(F(x_n, y_n), F(x, y)) + d(gx_{n+1}, gx) \\
\leq\ & \frac{\phi(d(F(x_n, y_n), gx_n)) + \phi(d(F(x, y), gx))}{2} + d(gx_{n+1}, gx) \\
=\ & \frac{\phi(d(gx_{n+1}, gx_n)) + \phi(d(gx, gx))}{2} + d(gx_{n+1}, gx).
\end{aligned}$$

On taking limit as $n \to \infty$, we obtain

$$d(F(x, y), gx) \leq \phi(0) = 0, \tag{2.23}$$

and $F(x, y) = gx$. Similarly, it can be shown that $F(y, x) = gy$. Hence (x, y) is a coupled coincidence point and (gx, gy) is coupled point of coincidence of mappings F and g. $\square$

Corollary 2.9. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that a mapping $F : X \times X \to X$ is a monotone and*

$$d(F(x, y), F(u, v)) \leq \frac{\phi(d(F(x, y), x) + d(F(u, v), u))}{2}, \tag{2.24}$$

for all $x, y, u, v \in X$, for which $x \preceq u$ and $y \preceq v$. If $(X, d, \preceq)$ is a complete and regular and if there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \preceq F(y_0, x_0),$$

then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof. The results follows by taking $g = I$ (identity mapping) in Theorem 2.7. $\square$

Theorem 2.10. *Let $(X,d,\preceq)$ be a partially ordered metric space. Suppose that a mapping $F:X\times X\to X$ is a monotone with respect to $g:X\to X$ and*

$$\begin{aligned} & d(F(x,y),F(u,v))+d\left(F\left(y,x\right),F\left(v,u\right)\right)\\ \leq\ & a_1d(gx,gu)+a_2d(gy,gv)+a_3d(F(x,y),gx)\end{aligned}$$

$$+a_4d(F(u,v),gu)+a_5d(F(x,y),gu) \tag{2.25}$$

for all $x,y,u,v\in X$, for which $g(x)\preceq g(u)$ and $g(y)\preceq g(v)$, with nonnegative real numbers a_i, $i=1,2,...,5$ and $\sum_{i=1}^{5}a_i<1$. If $F(X\times X)$ is contained in a complete set $g(X)$, $(X,d\preceq)$ is a regular and if there exist $x_0,y_0\in X$ such that

$$g(x_0)\preceq F(x_0,y_0)\text{ and }g(y_0)\preceq F(y_0,x_0),$$

then there exist $x,y\in X$ such that $g\left(x\right)=F\left(x,y\right)$ and $g\left(y\right)=F\left(y,x\right)$.

Proof. Let $x_0,y_0\in X$ be such that $g(x_0)\preceq F(x_0,y_0)$ and $g(y_0)\preceq F(y_0,x_0)$. Using the similar arguments to those given in Theorem 2.1, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1})=F(x_n,y_n)\text{ and }g(y_{n+1})=F(y_n,x_n)\text{ for all }n\geq 0,$$

and for all $n\geq 0$,

$$\begin{aligned} gx_0 &\preceq gx_1\preceq ...\preceq gx_n\preceq gx_{n+1}\preceq ...,\text{ and}\\ gy_0 &\preceq gy_1\preceq ...\preceq gy_n\preceq gy_{n+1}\preceq ....\end{aligned}$$

Now we will suppose that $d(gx_n,gx_{n+1})>0$ and $d(gy_n,gy_{n+1})>0$ for all n, otherwise, F and g have a coupled coincidence point at (x_n,y_n), and so we have finished the proof. From (2.25), we have

$$\begin{aligned} & d(gx_n,gx_{n+1})+d\left(gy_n.gy_{n+1}\right)\\ =\ & d(F(x_{n-1},y_{n-1}),F(x_n,y_n))+d\left(F\left(y_{n-1},x_{n-1}\right),F\left(y_n,x_n\right)\right)\\ \leq\ & a_1d(gx_{n-1},gx_n)+a_2d(gy_{n-1},gy_n)+a_3d(F(x_{n-1},y_{n-1}),gx_{n-1})\\ & +a_4d(F(x_n,y_n),gx_n)+a_5d(F(x_{n-1},y_{n-1}),gx_n)\\ =\ & a_1d(gx_{n-1},gx_n)+a_2d(gy_{n-1},gy_n)+a_3d(gx_n,gx_{n-1})\\ & +a_4d(gx_{n+1},gx_n)+a_5d(gx_n,gx_n)\\ =\ & (a_1+a_3)d(gx_{n-1},gx_n)+a_2d(gy_{n-1},gy_n)+a_4d(gx_{n+1},gx_n),\end{aligned}$$

from which it follows

$$d(gx_n,gx_{n+1})+d\left(gy_n.gy_{n+1}\right)\leq\frac{1}{1-a_4}[(a_1+a_3)\,d(gx_{n-1},gx_n)+a_2d(gy_{n-1},gy_n)]. \tag{2.26}$$

From (2.26), we obtain

$$\begin{aligned} & d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})\\ \leq\ & \frac{a_1+a_3}{1-a_4}[d(gy_{n-1},gy_n)+d(gx_{n-1},gx_n)],\end{aligned}$$

that is.,

$$d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})\leq\lambda[d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)], \tag{2.27}$$

where $\lambda = \dfrac{a_1 + a_3}{1 - a_4}$. Obviously, $0 \le \lambda < 1$. Now

$$\begin{aligned} d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) &\le \lambda[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \\ &\le \lambda^2[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})] \\ &\le ... \\ &\le \lambda^n[d(gx_0, gx_1) + d(gy_0, gy_1)]. \end{aligned}$$

Then, for all $n, m \in N$, $m > n$, we have

$$\begin{aligned} d(gx_n, gx_m) + d(gy_n, gy_m) &\le d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_{n+1}, x_{x+2}) \\ &\quad +d(y_{n+1}, y_{x+2}) + ... + d(x_{m-1}, x_m) + d(y_{m-1}, y_m) \\ &\le \frac{\lambda^n}{1-\lambda}[d(gx_0, gx_1) + d(gy_0, gy_1)], \end{aligned}$$

which implies that $d(gx_n, gx_m)+d(gy_n, gy_m) \to 0$, as $n, m \to \infty$, that is., $d(gx_n, gx_m) \to 0$ and $d(gy_n, gy_m) \to 0$ as $n, m \to \infty$. Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$, so there exists x and y in X such that $\{gx_n\}$ and $\{gy_n\}$ converges to gx and gy respectively. Now, we prove that $F(x, y) = gx$ and $F(y, x) = gy$.
Since $gx_n \preceq gx$ and $gy_n \preceq gy$ for all $n \ge 0$, so that we have

$$\begin{aligned} & d(F(x, y), gx) + d(F(y, x), gy) \\ \le\ & d(F(x, y), gx_{n+1}) + d(F(y, x), gy_{n+1}) + d(gx_{n+1}, gx) + d(gy_{n+1}, gy) \\ =\ & d(F(x_n, y_n), F(x, y)) + d(F(y_n, x_n), F(y, x)) + d(gx_{n+1}, gx) + d(gy_{n+1}, gy) \\ \le\ & a_1 d(gx_n, gx_n) + a_2 d(gy_n, gy_n) + a_3 d(F(x_n, y_n), gx_n) + a_4 d(F(x, y), gx) \\ & a_5 d(F(x_n, y_n), gx) + d(gx_{n+1}, gx) + d(gy_{n+1}, gy) \\ =\ & a_3 d(gx_{n+1}, gx_n) + a_4 d(F(x, y), gx) + a_5 d(gx_{n+1}, gx) + d(gx_{n+1}, gx) + d(gy_{n+1}, gy) \end{aligned}$$

On taking the limit as $n \to \infty$, we obtain that

$$d(F(x, y), gx) + d(F(y, x), gy) \le a_4 d(F(x, y), gx).$$

Since $a_4 < 1$, so that $F(x, y) = gx$ and $F(y, x) = gy$. Hence (x, y) is a coupled coincidence point and (gx, gy) is coupled point of coincidence of mappings F and g. □

Corollary 2.11. *Let $(X, d, \preceq)$ be a partially ordered set and d a metric on X. Suppose that a mapping $F : X \times X \to X$ is a monotone with respect to $g : X \to X$ and*

(2.28) $d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \le kd(F(x, y), gx) + ld(F(u, v), gu)]$

for all $x, y, u, v \in X$, for which $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$ and $k, l \ge 0$ with $k + l < 1$. If $F(X \times X)$ is contained in a complete set $g(X)$, $(X, d \preceq)$ is a regular and if there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0) \text{ and } g(y_0) \preceq F(y_0, x_0),$$

then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Remarks 2.12. Also, almost all known results from several papers on partially ordered metric spaces can be considered with monotone mappings instead with mappings which have a mixed monotone property. We note that the concept of coupled coincidence point for monotone mappings is essentially different of the corresponding one for mixed monotone mappings (for tripled case see [12]).

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